Near and Far from Equilibrium Power-Law Statistics

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What is it about?

- Random Filling Patterns in Phase Space
- Master Equation Near and Far from Equilibrium
- $n \to \infty$: continous models
- Entropy Production Rate / Jet Fragmentation

Ideal Gas Phase Space

PS-Volume ratios

Phase space volume: Hypersphere with radius E, dimension n.

Pick up one with ω energy. Probability weight factor $\propto$ ph.sp. volume ratio:

$$p(\omega) \sim \frac{\Omega(E-\omega)}{\Omega(E)} = \left(1-\frac{\omega}{E}\right)^n. \tag{1}$$

Textbook limit

$$p(\omega) \sim \lim_{\substack{E\to\infty \\ n\to\infty \\ E/n=T}} \left(1-\frac{\omega}{E}\right)^n = e^{-n\omega/E}. \tag{2}$$

Ideal Gas Phase Space

event average PS-Volume ratios

Average of the ratio in *small systems*:

$$\langle p(\omega)\rangle = \sum_{n=0}^{\infty} P_n \left(1 - \frac{\omega}{E}\right)^n . \quad (3)$$

- P_n Poisson:

$$\langle p(\omega)\rangle^{\mathrm{POI}} = \mathrm{e}^{-\langle n\rangle\,\omega/E} .$$

- P_n NBD:

$$\langle p(\omega)\rangle^{\mathrm{NBD}} = \left(1 + \frac{\langle n\rangle}{k+1}\frac{\omega}{E}\right)^{-k-1} .$$

General event distribution

Tsallis – interpretation

Compare the result with Tsallis Pareto distribution

$$p(\omega) \;=\; \left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} \tag{4}$$

Expanding up to terms quadratic in ω we obtain

☺

$$\boldsymbol{T = \frac{E}{\langle n \rangle}, \qquad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}} \tag{5}$$

$(q-1)$ **is non-Poissonity!**

General Phase Space

event average PS-Volume ratios

Compare the Einstein ratio with Tsallis Pareto distribution

$$\langle p(\omega)\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle \approx \left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} \qquad (6)$$

Expanding up to terms quadratic in ω we obtain

☺

$$\frac{1}{T} = \langle S'(E) \rangle = \beta, \qquad q = 1 - \frac{1}{C} + \frac{\Delta\beta^2}{\beta^2} \qquad (7)$$

$(q-1)$ is non-Gaussianity in β fluctutation.

Schemes of Master Equations

Balanced vs One-Sided Growth

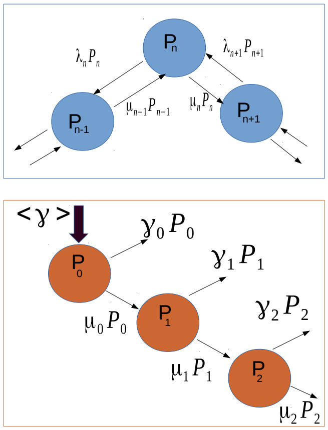
Near Equilibrium: Diffusion

$$\begin{aligned}\dot{P}_n &= [(\lambda P)_{n+1} - (\lambda P)_n] \\ &- [(\mu P)_n - (\mu P)_{n-1}] \qquad (8)\end{aligned}$$

Concurring Race: Avalanche

$\dot{P}_0 = \langle\gamma\rangle - (\gamma_0 + \mu_0) P_0$ and ☺

$$\dot{P}_n = \mu_{n-1} P_{n-1} - (\mu_n + \gamma_n)\ P_n \qquad (9)$$



Avalanche dynamics:
stationary distribution

Of special interest: "Mean Aging Model" **MAM.**
For $\forall n: \ \gamma_n = \gamma$ we have $\langle \gamma \rangle = \gamma$.

Stationary limit: $P_n(t) \to Q_n$, from $\dot{Q}_n = 0$ one obtains

$Q_0 = \gamma/(\gamma + \mu_0)$ and stationary ☺

$$Q_n = \frac{\mu_{n-1}}{\mu_n + \gamma} Q_{n-1} = \cdots = Q_0 \prod_{j=1}^{n} \frac{\mu_{j-1}}{\mu_j + \gamma}. \tag{10}$$

Constant rates

$\rightarrow$ exponential distribution

Assume $\mu_j = \sigma$, attachment rate independent of number of links.

$$Q_n = Q_0 \prod_{j=1}^{n} \frac{\sigma}{\sigma+\gamma} = Q_0 \left(1+\gamma/\sigma\right)^{-n}. \qquad (11)$$

Geometrical sum for normalization. We obtain

exponential ☺

$$\boldsymbol{Q_n = \frac{1}{1+\sigma/\gamma}\, e^{-n \cdot \ln(1+\gamma/\sigma)}.} \qquad (12)$$

Linear preference rates

$\rightarrow$ Waring distribution

Linear preference in attachment: $\mu_j = \sigma(j+b) \quad (b>0)$.

$$Q_n \;=\; Q_0 \prod_{j=1}^{n} \frac{j-1+b}{j+b+\gamma/\sigma} \;=\; Q_0\, \frac{(b)_n}{(c)_n}. \tag{13}$$

with $c = b + 1 + \gamma/\sigma$. Norm:
$\sum_n Q_n = Q_0\,(c-1)/(c-1-b) = 1$.

Pochhammer ratio (Waring) ☺

$$Q_n \;=\; \frac{c-1-b}{c-1}\,\frac{(b)_n}{(c)_n} \tag{14}$$

Linear preference rates: tail of Waring

$\rightarrow$ power-law tailed distribution

The above result in the $n \to \infty$ limit:
Since

$$\lim_{n\to\infty} n^{c-b} \frac{\Gamma(n+b)}{\Gamma(n+c)} = 1, \tag{15}$$

we obtain

Pochhammer in $n \to \infty$ limit: power-law! ☺

$$Q_n \to \frac{\gamma}{\gamma + b\sigma} \frac{\Gamma(c)}{\Gamma(b)} n^{-1-\gamma/\sigma}. \tag{16}$$

NBD from avalanche

T. Osada et al. Prog.Theor.Phys. **98** 1289 (1997)

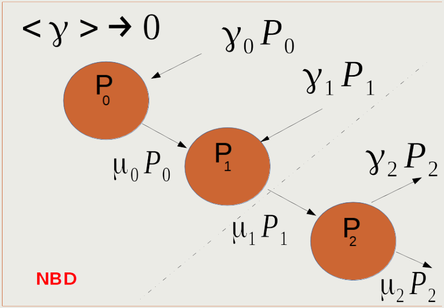
With the special design:

$$\begin{aligned} \gamma_n &= \sigma(n-kf) \\ \mu_n &= \sigma f(n+k) \\ \gamma_n + \mu_n &= \sigma(1+f)\,n \quad (17) \end{aligned}$$

The chain is fed for all $n < kf$ and decays for all $n > kf$; $\langle \gamma_n \rangle \to 0$.

The stationary distribution is NBD

$$Q_n = \binom{n+k-1}{n} f^n (1+f)^{-n-k}. \quad (18)$$



Avalanche dynamics in the **large n limit!**

continuous variable: $x = n \cdot \Delta x$

- $P_n(t) = \Delta x \cdot P(n \cdot \Delta x, t)$ ensures $\sum_{n=0}^{\infty} P_n(t) = \int_0^{\infty} P(x,t) dx$.
- $\mu_n = \frac{1}{\Delta x} \cdot \mu(n \cdot \Delta x)$ and $\gamma_n = \gamma(n \cdot \Delta x)$ lead to

Continuum Master: ☺

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} (\mu(x) P(x,t)) - \gamma(x) P(x,t). \qquad (19)$$

with the stationary distribution

$$Q(x) = \frac{K}{\mu(x)} e^{-\int_0^x \frac{\gamma(u)}{\mu(u)} du}. \qquad (20)$$

Particular continuous stationary distributions

with constant $\gamma(x) = \gamma$.

For constant rate $\mu(x) = \sigma$ **exponential**:

$$Q(x) = \frac{\gamma}{\sigma} e^{-\frac{\gamma}{\sigma} x}. \tag{21}$$

For linear preference $\mu(x) = \sigma(x + b)$ **Tsallis–Pareto**:

$$Q(x) = \frac{\gamma}{\sigma b} \left(1 + \frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{22}$$

For exponential dispreference $\mu(x) = \sigma e^{-ax}$ **Gompertz**

$$Q(x) = \frac{\gamma}{\sigma} e^{ax + \frac{\gamma}{a\sigma}(1 - e^{ax})}. \tag{23}$$

Rate, Survival, Hazard

Connection to failure probability

Fluctuation-Dissipation vs. rate reconstruction vs. hazard

Cumulative hazard	$H(x)$
hazard (rate)	$h(x) = H'(x)$
PDF	$Q(x) = h(x)\, e^{-H(x)}$
Survival (rate)	$R(x) = \int_x^\infty Q(u)\, du = e^{-H(x)}$

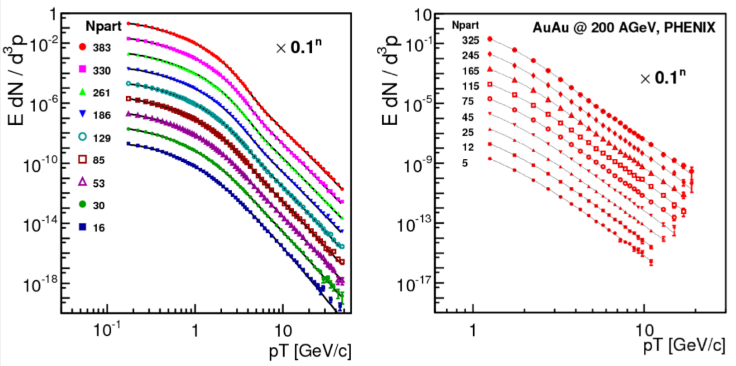
For $\gamma(x) = \gamma$ constant ☺

$$\mu(x) = \gamma \frac{R(x)}{Q(x)} = \frac{\gamma}{h(x)}. \tag{24}$$

p_T spectra

muliplicity selection

ALICE PLB 720 (2013) 52 **PHENIX PRL 101 (2008) 232301**

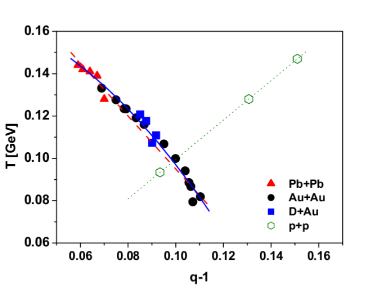


Measured parameters: T vs $(q-1)$

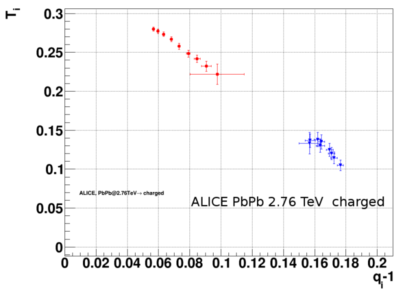
Theoretical background = ?

$T_{AA} = 0.22 - 1.25(q-1)$ GeV;
$T_{pp} = (q-1)$ GeV

$\Delta T/T$ =const; $T = E(\sigma^2 - (q-1))$
$\langle n \rangle / k$ =const; $T = (E/f)(q-1)$.



by Grzegorz Wilk

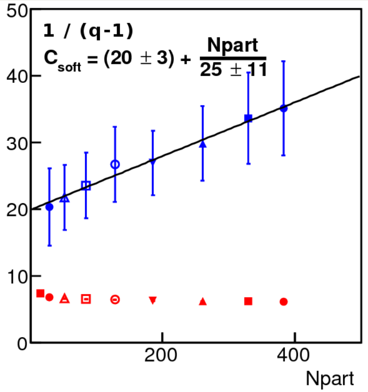


by Gábor Bíró, Ke-Ming Shen

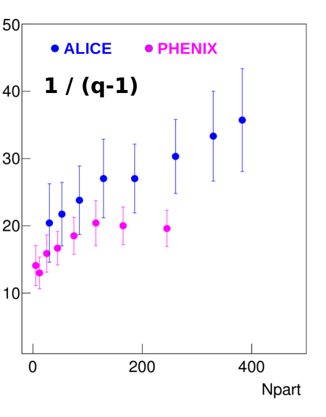
Parameter trends: $1/(q-1)$ vs N_{part}

by K. Ürmössy

soft vs hard



soft only



PHENIX AuAu NBD parameters

J.Mitchell, Nukleonika 51, S89 (2006)

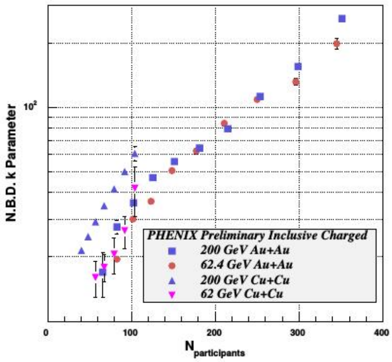


Figure 4. Inclusive charged particle multiplicity fluctuations in terms of the k parameter from a negative binomial distribution fit to the data as a function of centrality for $\sqrt{s_{NN}}$

ALICE pp NBD parameters

V. Zaccolo PhD thesis 2015

$\sqrt{s}$ (TeV)	η Range	λ	α	$\langle n\rangle_1$	k_1	$\langle n\rangle_2$	k_2	χ^2/DOF
	$\lvert\eta\rvert < 1.5$	0.91±0.01	0.87±0.03	10.17±0.35	2.20±0.08	26.24±1.11	7.92±0.73	28.41 / 104
0.9	$\lvert\eta\rvert < 2.4$	0.97±0.01	0.80±0.09	16.57±1.55	2.66±0.23	39.61±3.63	6.57±1.05	11.77 / 127
	$-3.4 < \eta < +5.1$	0.96±0.01	0.81±0.04	27.80±1.29	3.59±0.27	64.62±3.03	9.57±0.93	31.60 / 164
2.76	$\lvert\eta\rvert < 1.5$	1.01±0.01	0.76±0.05	9.19±0.54	2.45±0.20	26.72±1.88	4.92±0.59	12.41 / 123
	$\lvert\eta\rvert < 1.5$	0.93±0.01	0.51±0.06	8.82±0.77	2.28±0.18	30.41±2.13	3.26±0.46	5.39 / 124
7	$\lvert\eta\rvert < 2.4$	0.98±0.01	0.57±0.05	16.18±1.03	2.36±0.18	52.96±2.42	4.03±0.34	24.11 / 219
	$-3.4 < \eta < +5.1$	0.95±0.01	0.68±0.04	32.44±2.06	2.35±0.18	96.32±4.21	5.76±0.50	47.75 / 289
	$\lvert\eta\rvert < 1.5$	0.97±0.01	0.66±0.07	12.02±1.21	1.91±0.17	37.68±2.84	4.48±0.60	6.33 / 159
8	$\lvert\eta\rvert < 2.4$	1.00±0.01	0.59±0.05	18.15±1.26	2.40±0.15	56.84±2.74	4.19±0.37	15.49 / 232
	$-3.4 < \eta < +5.1$	0.96±0.01	0.58±0.05	30.56±1.92	2.91±0.22	91.42±3.98	4.70±0.43	40.56 / 267

Table 9.1: Double NBD fit parameters for multiplicity distributions, NSD events.

Summary

- QCD: $1/(q-1) = k > 4$ independent of N_{part}
- soft statistics: $k = \langle n \rangle / f \propto N_{part}$ size effect
- Statistical: $k = 1/(q-1)$ connects multiplicity and 1-ptl energy distribution
- pp: two NBD, semi-hard component?

B A C K U P

Continuous avalanche

Rate reconstruction

Knowing / observing $Q(x)$ and $\gamma(x)$ one obtains

$$\mu(x) = \frac{1}{Q(x)} \int_x^\infty \gamma(u) Q(u)\, du = \langle \gamma \rangle_{\rm cut}. \qquad (25)$$

Analogy: multiplicative noise

Langevin: $\dot{p} + (\gamma p - \xi) = 0$; stochastic properties: $\langle \gamma p - \xi \rangle = K_1(p)$ and $\langle (\gamma p - \xi)(\gamma p - \xi)' \rangle = K_2(p)$.

Then the Fokker-Planck, $\frac{\partial f}{\partial t} = \frac{\partial}{\partial p}(K_1 f) + \frac{\partial^2}{\partial p^2}(K_2 f) = 0$ has the detailed balance distribution

$$Q(p) = \frac{K}{K_2(p)}\, e^{-\int_0^p \frac{K_1(q)}{K_2(q)} dq}. \qquad (26)$$

The **Fluctuation–dissipation** theorem has the form

$$K_2(p) = \frac{1}{Q(p)} \int_p^\infty K_1(q) Q(q)\, dq. \qquad (27)$$

Summary of Rates and PDF-s

$\mu(x) = \gamma / h(x)$ at constant aging γ

$\mu_n,\ \mu(x)$	$Q_n,\ Q(x)$
constant	geometrical $\to$ exponential
linear	Waring $\to$ Tsallis/Pareto
sublinear power	stretched exponential
higher polynomial	$\to$ inverse leading power
quadratic polynomial	Pearson
exponentially decaying	Gompertz
inverse power	Weibull

Deviation shrinks and moves as a soliton: $\dot{x}_c = \mu(x_c)$!

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