ONSET OF MESONIC CONDENSATION PHENOMENA FROM HIGHER ORDER SUSCEPTIBILITIES

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Swagato Mukherjee and V.S. work in progress
Higher order cumulants of conserved charge fluctuations are sensitive probes.

Proper measurement is challenging.
BES II provides unique opportunity to learn about $T - \mu_B$ phase-diagram, but...
What happens at non-zero $\mu_Q$, $\mu_I$ or $\mu_S$?

Naively $\mu_Q \neq 0$ can be studied with isobars & BES: fixed $A = 96$, $Z = 40, 44$. Baryon stopping and charge transfer?!

Complimentary approach: we can probe those regions with fluctuations.

$\mu_I \neq 0$ can be studied on lattice owing to absence of sign problem
Thermodynamic partition function is meromorphic function of $T$ and $\mu$. This allows to study otherwise **unaccessible** region of phase diagram: analytical continuation.

**In theory**, we can probe high charge and high strange chemical potential, including Bose-Einstein condensation of pions and kaons by studying fluctuation at zero $n_Q$ and $n_S$.

**In practice**, we can measure in experiment/compute on lattice **finite** number of cumulants; this limits precision of analysis. Need for refined methods to extract information.
Radius of convergence

- Let's consider even function

\[ f(z) = \sum_k c_{2k} z^{2k} \quad \text{with} \quad c_{2k} = \frac{1}{(2k)!} \frac{\partial^{2k} f}{\partial z^{2k}} \]

- Closest singularity \((z_c)\) to expansion point \((z = 0)\) defines radius of convergence \(R\) of series

- Definition of ratio and root test:

\[
R = \lim_{k \to \infty} \inf R_{2k}, \quad R_{2k} = \left| \frac{c_{2k-2}}{c_{2k}} \right|^{1/2} \quad \text{or} \quad R_k = \left| \frac{1}{c_{2k}} \right|^{1/(2k)}
\]

- According to Darboux theorem (1878):

  *late terms in the development of a function as a Taylor series are determined by the behaviour of the function in the vicinity of singular points on the circle of convergence*  
  
  In other words: one can determine not only location by type of singularity (critical exponent).
**Radius of convergence: relation to cumulants**

- Pressure can be expanded to

\[ \frac{p}{T^4} = \sum_k \frac{\chi_k}{k!} \cdot \left( \frac{\mu - \mu_0}{T} \right)^k \]

where \( \chi_k \) is \( k \)-th order cumulant computed at \( \mu_0/T \).

- Phase transitions are singularities in (complex) \( \mu \) plane.

- Radius of convergence defines location of closest singularity.

- Cumulants are related to fluctuations of corresponding charge, i.e.

\[ \chi_4 = \frac{\langle (\delta N)^4 \rangle - 3\langle (\delta N)^2 \rangle^2}{VT^3} \]

Ratio test for radius of convergence is particularly convenient because \( VT^3 \) cancels
**Singularities in QCD at finite $T$ and $\mu$**

### Singularities on the complex plane are related to

- **Critical point of second-order phase transition.** Singularity lies on real $\mu$ axis.

- **Crossover transition.** Singularity is at some complex $\mu$.
  

- **Spinodal lines for first-order phase transition.** Singularities are either at real or complex values of $\mu$. Some model calculations: essential singularity.

  See M. A. Stephanov, Phys. Rev. D 73, 094508 (2006);

- **“Thermal singularities”** associated with zeros of inverse Fermi-Dirac function. In QCD, owing to $Z(3)$ symmetry the singularities are located at quark chemical potential $\text{Im } \mu_q = \pm (\pi T + 2\pi n k T)/3$.

Free pions: radius of convergence

Free pions

Pressure:

\[ \frac{p_{\pi^+} + p_{\pi^-}}{T^4} = \frac{m_{\pi}^2}{\pi^2 T^2} \sum_{l=1}^{\infty} K_2(lm_{\pi}/T) \cosh(l\mu_Q/T) \]

Cumulants at zero \( \mu_Q \):

\[ \chi_n^Q = \frac{m_{\pi}^2}{\pi^2 T^2} \sum_{l=1}^{\infty} l^{n-2} K_2(lm_{\pi}/T) \]

The coefficients of expansion

\[ c_n = \frac{\chi_n^Q}{n!} \]
**Hadron Resonance Gas: Radius of Convergence**

**Hadron Resonance Gas**

n-th cumulant is given by

\[
\chi_n^Q = \sum_{i \in \text{hadrons}} \chi_{ni}^Q, \quad \chi_{in}^Q = \frac{(2s_i - 1)m_i^2Q_i^n}{\pi^2T^2} \sum_{l=1}^{\infty} (\pm)^{n-1} l^{n-2} K_2(lm_i/T)
\]

upper/lower sign corresponds to bosons/fermions
**Conclusion from this exercise**

To describe location of BEC transition, higher order cumulants are required. Can we improve our methods?

- **at moderate $n$** - significant contribution from doubly charge baryons (see e.g. P. Braun-Munzinger, B. Friman, F. Karsch, K. Redlich and V.S., Nucl. Phys. A 880, 48 (2012)
NON-PERTURBATIVE CHIRAL MATRIX MODEL

- Only in free theory BE condensation occurs at vacuum pion mass.
- Functional Renormalization Group approach to take into account fluctuations (Intro: previous talk by Gergely)
- Pion mass dependence on $\mu_Q$ for different $T$:

![Graph showing pion mass dependence on $\mu_Q$ for different $T$.]

- Increase of pion mass with $T$ and $\mu_Q$

Details of model:
R. Pisarski and V.S. 1604.00022

Details of FRG solution:
V.S., B. Friman, K. Redlich 1108.3231
The phase diagram in $T - \mu_Q$ plane:

- Details of model: R. Pisarski and V.S. 1604.00022
- Details of FRG solution: V.S., B. Friman, K. Redlich 1108.3231

- Non-trivial dependence of $T_{\text{BEC}}$ on $\mu_Q$.
- Can it be reproduced with cumulant analysis? What order is required?!

- At low $T$ transition is at about $\mu_Q = 140$ MeV as expected.
Green dots: exact results we want to reproduce. \( R = \frac{m_\pi(T)}{T} \)
Non-perturbative chiral matrix model

\[ R_n \sim \frac{T}{T_c} \]

\( n = 6 \)

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Non-perturbative chiral matrix model

\[ \frac{T}{T_c} = 0.4, 0.5, 0.6, 0.7, 0.8 \]

\[ R_n = \text{data points} \]

\[ n = 8 \]

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Hopeless?!
**W**AY TO IMPROVE CONVERGENCE

**CONFORMAL MAPPING**

Conformal mapping is transformation $\xi = \xi(z)$ that preserves local angles. The main idea is to extend radius of convergence and to enhance sensitivity to properties of critical point by non-linear transformation of original series. E.g. using conformal mapping one can move physical singularities closer to expansion point, while keeping non-physical singularities as far away as possible.

**THERMAL SINGULARITY**

In terms of quark chemical potential:

$$\text{Im } \mu_q = \pm(\pi T + 2\pi nT)/3, \quad n \in \mathbb{Z}$$

Thus closest singularities are located at

$$\text{Im } \mu_Q = \pm \pi T \quad \text{(related to } d \text{ – quark or single charged baryons)}$$

and

$$\text{Im } \mu_Q = \pm \pi T/2 \quad \text{(related to } u \text{ – quark or doubly charged baryons)}$$
**Mapping to fugacity plane**

**Thermal singularity**

Fugacity

\[ \lambda = \exp \mu Q / T \]

- maps thermal singularities \( i\pi T \) to negative real axis on fugacity plane
- maps thermal singularities \( i\pi T / 2 \) to imaginary axis on fugacity plane

\[ \mu \quad \mu_c \quad -\mu_c \quad -i\pi T \]

\[ \lambda \quad 1 / \lambda_c \quad \lambda_c \]
Mapping to fugacity plane

Conjugate singularities interfere with method precision

Mapping to fugacity plane, \( \lambda = \exp\left(\frac{\mu}{T}\right) \sim \mu/T = \ln(\lambda) \)

To simplify notation \( \tilde{\lambda} = \lambda - 1 \)

\[
\sum_{n=0}^{n=10} \frac{1}{n!} \chi_n \left(\frac{\mu}{T}\right)^n = \\
\chi_0 + \frac{\tilde{\lambda}^2 \chi_2}{2} - \frac{\tilde{\lambda}^3 \chi_2}{2} + \tilde{\lambda}^4 \left(\frac{11 \chi_2}{24} + \frac{\chi_4}{24}\right) + \tilde{\lambda}^5 \left(-\frac{5 \chi_2}{12} - \frac{\chi_4}{12}\right) \\
+ \tilde{\lambda}^6 \left(\frac{137 \chi_2}{360} + \frac{17 \chi_4}{144} + \frac{\chi_6}{720}\right) + \tilde{\lambda}^7 \left(-\frac{7 \chi_2}{20} - \frac{7 \chi_4}{48} - \frac{\chi_6}{240}\right) \\
+ \tilde{\lambda}^8 \left(\frac{363 \chi_2}{1120} + \frac{967 \chi_4}{5760} + \frac{23 \chi_6}{2880} + \frac{\chi_8}{40320}\right) + \tilde{\lambda}^9 \left(-\frac{761 \chi_2}{2520} - \frac{89 \chi_4}{480} - \frac{\chi_6}{80} - \frac{\chi_8}{10080}\right) \\
+ \tilde{\lambda}^{10} \left(\frac{7129 \chi_2}{25200} + \frac{4523 \chi_4}{22680} + \frac{3013 \chi_6}{172800} + \frac{29 \chi_8}{120960} + \frac{\chi_{10}}{3628800}\right) + O(\tilde{\lambda}^{11})
\]
Suppose that singularity $z_c$ of $f(z) (= \sum_n f_n z^n)$ is closest to origin, and that behaviour in its neighborhood is given by

$$f(z) = (1 - z/z_c)^{-\nu} r(z) + a(z)$$

with $r(z) = \sum_k b_k (z - z_c)^k$ and $a(z)$ being analytic in some circle with center at $z = 0$ and radius $> |z_c|$. Asymptotic expansion of $f(z)$

$$f_n \sim \sum_{k=0}^{\infty} \frac{(-1)^k b_k z_c^{k-n} \Gamma(n + \nu - k)}{n! \Gamma(\nu - k)}$$

To reduce corrections construct difference

$$f_n z_c - \frac{n + \nu - 1}{n} f_{n-1} \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1} kb_k z_c^{k-n} \Gamma(n + \nu - k - 1)}{n! \Gamma(\nu - k)} \sim O(n^{\nu-3})$$

Consequently $z_c = \frac{f_{n-1}}{f_n} \left( 1 + \frac{\nu - 1}{n} \right)$, leading term gives ratio method.
Combine adjacent $f_n$ to construct more accurate approximation. This is better done by constructing recursive relation

$$
X_0^n(z_c, \nu) = f_n;
$$

$$
X_{n+1}^m(z_c, \nu) = z_c X_n^m(z_c, \nu) - (n + \nu - 2m - 1)X_{n-1}^m(z_c, \nu)/n.
$$

Easy to show that

$$
X_n^m \sim O(n^{\nu-2m-1})
$$

Two unknowns $z_c$ and $\nu$ can be found by solving two equations

$$
X_n^m(z_c, \nu) = 0, \quad X_{n-1}^m(z_c, \nu) = 0
$$

For simplicity $m = 1$ and we use fugacity mapping.
Applying Darboux theorem

$\frac{T}{T_c}$

$R_n$

$n = 6$

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Applying Darboux theorem
Applying Darboux theorem

Method based on Darboux theorem:

Ratio test:

Reminder: 7-th order in fugacity includes only 6-th and lower order cumulants:

\[
\tilde{\lambda}^7 \left( -\frac{7\chi_2}{20} - \frac{7\chi_4}{48} - \frac{\chi_6}{240} \right)
\]
Applying Darboux theorem
Applying Darboux theorem

Method based on Darboux theorem:

Ratio test:

Superior reconstruction of phase boundary
**CONCLUSIONS**

- Cumulant analysis may eventually allow us to reconstruct 3D phase diagram.
- In this talk, we explored $\mu_Q$ as 3-d axis. Commonly applied ratios test requires very high order cumulants and very ineffective.
- Using conformal mapping to remove spurious singularities and Darboux theorem for properties of late coefficients, with reasonable number of cumulants $n = 6$ we were able to reconstruct onset of BEC.

**OUTLOOK. THINGS I DID NOT HAVE TIME TO SHOW**

- It is sufficient to have only positive/negative pion fluctuations.
- Using similar approach it is possible to probe kaon condensation using strangeness fluctuations or kaon fluctuations.
Signatures of phase transition and critical point

Static:
- O(4) crossover: universal sign structure of higher order cumulants, $\chi_6$
- Divergent cumulants of baryon number fluctuations at CP
  $$\chi_n \propto \xi^3 \left( \frac{\beta \gamma}{2 - \alpha} - 1 \right)$$
- In vicinity of CP: universal sign structure of cumulants.
- Divergent baryon number susceptibility at off-equilibrium first order phase transition

B. Friman, F. Karsch, K. Redlich, V. S. 2011
M. Stephanov 2011
C. Sasaki, K. Redlich, B. Friman 2007
FLUCTUATIONS AS A PROBE OF PHASE DIAGRAM

Last slide: predictions for an infinite static medium.

In HIC:

- Finite lifetime (S. Mukherjee, R. Venugopalan, Y. Yin 2015/2016)
- Finite size and anisotropy (G. Almasi and V.S. work in progress)
- Conservation laws (A. Bzdak, V. Koch, V.S. 2011)
- Fluctuations not related to interesting physics (V.S., B. Friman, K. Redlich 2012)

\[ p \text{ is a fraction of measured baryons} \]

\[ \mu = 0 \quad R_{n,m} = \frac{\chi_n}{\chi_m} \]

Modification due to finite life-time:

S. Mukherjee, R. Venugopalan, Y. Yin 2015