Onset of mesonic condensation phenomena from higher order susceptibilities

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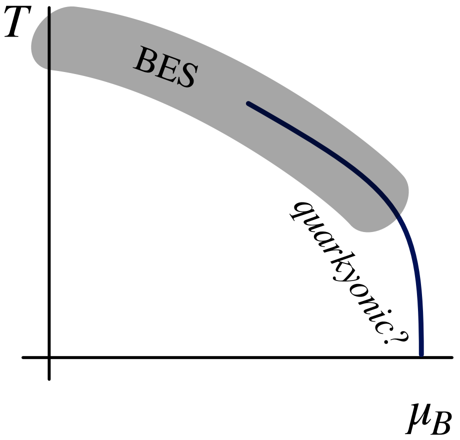
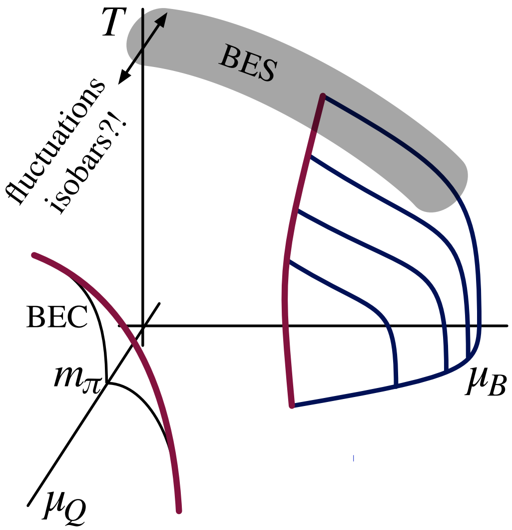
Swagato Mukherjee and V.S. work in progress

Higher order cumulants of conserved charge fluctuations are sensitive probes.

Proper measurement is challenging.



BES II provides unique opportunity to learn about $T - \mu_B$ phase-diagram, but…



- What happens at non-zero μ_Q, μ_I or μ_S?
- Naively $\mu_Q \neq 0$ can be studied with isobars & BES:
  fixed $A = 96, Z = 40, 44$.
  Baryon stopping and charge transfer?!
- Complimentary approach: we can probe those regions with fluctuations.
- $\mu_I \neq 0$ can be studied on lattice owing to absence of sign problem

PUNCHLINE

Thermodynamic partition function is meromorphic function of T and μ.

This allows to study otherwise **unaccessible** region of phase diagram: analytical continuation.

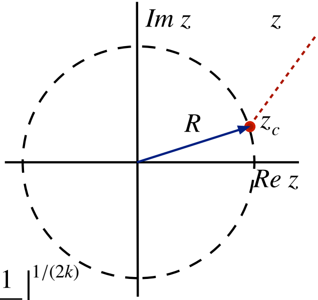
In theory, we can probe high charge and high strange chemical potential, including Bose-Einstein condensation of pions and kaons by studying fluctuation at zero n_Q and n_S.

In practice, we can measure in experiment/compute on lattice **finite** number of cumulants; this limits precision of analysis. Need for refined methods to extract information.

Radius of convergence

- Lets consider even function

$$f(z) = \sum_k c_{2k} z^{2k} \quad \text{with} \quad c_{2k} = \frac{1}{(2k)!} \frac{\partial^{2k} f}{\partial z^{2k}}$$



- Closest singularity (z_c) to expansion point ($z = 0$) defines radius of convergence R of series
- Definition of **ratio** and root test:

$$R = \lim_{k\to\infty} \inf R_{2k}, \quad R_{2k} = \left|\frac{c_{2k-2}}{c_{2k}}\right|^{1/2} \quad \text{or} \quad R_k = \left|\frac{1}{c_{2k}}\right|^{1/(2k)}$$

- According to Darboux theorem (1878):
  late terms in the development of a function as a Taylor series are determined by the behaviour of the function in the vicinity of singular points on the circle of convergence
  In other words: one can determine not only location by type of singularity (critical exponent).

Radius of convergence: relation to cumulants

- Pressure can be expanded to

$$p/T^4 = \sum_k \frac{\chi_k}{k!} \cdot \left(\frac{\mu - \mu_0}{T}\right)^k$$

where χ_k is k-th order cumulant computed at μ_0/T.

- Phase transitions are singularities in (complex) μ plane.
- Radius of convergence defines location of closest singularity.
- Cumulants are related to fluctuations of corresponding charge, i.e.

$$\chi_4 = \frac{\langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2}{VT^3}$$

Ratio test for radius of convergence is particularly convenient because VT^3 cancels

SINGULARITIES ON THE COMPLEX PLANE ARE RELATED TO

- critical point of second-order phase transition. Singularity lies on real μ axis.
- crossover transition. Singularity is at some complex μ.
  See P. C. Hemmer and E. H. Hauge, Phys. Rev. **133**, A1010 (1964); C. Itzykson, R. B. Pearson and J. B. Zuber, Nucl. Phys. B **220**, 415 (1983); M. A. Stephanov, Phys. Rev. D **73**, 094508 (2006).
- spinodal lines for first-order phase transition. Singularities are either at real or complex values of μ. Some model calculations: essential singularity.

  See M. A. Stephanov, Phys. Rev. D **73**, 094508 (2006);
- "thermal singularities" associated with zeros of inverse Fermi-Dirac function. In QCD, owing to Z(3) symmetry the singularities are located at quark chemical potential $\mathrm{Im}\,\mu_q = \pm(\pi T + 2\pi nkT)/3$.

  See F. Karbstein and M. Thies, Phys. Rev. D **75**, 025003 (2007); V.S., K. Morita, B. Friman Phys.Rev. D83 (2011) 071502

FREE PIONS

Pressure:

$$\frac{p_{\pi^+} + p_{\pi^-}}{T^4} = \frac{m_\pi^2}{\pi^2 T^2} \sum_{l=1}^{\infty} K_2(l m_\pi/T) \cosh(l \mu_Q/T)$$

Cumulants at zero μ_Q:

$$\chi_n^Q = \frac{m_\pi^2}{\pi^2 T^2} \sum_{l=1}^{\infty} l^{n-2} K_2(l m_\pi/T)$$

The coefficients of expansion

$$c_n = \frac{\chi_n^Q}{n!}$$

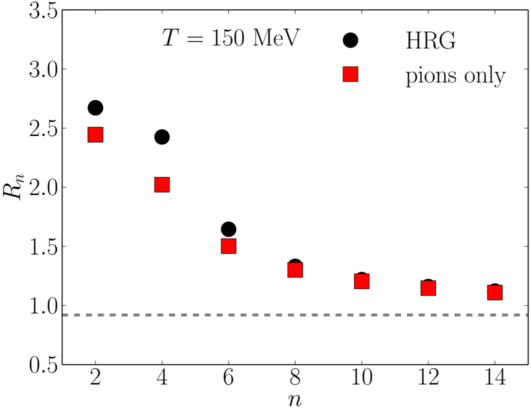
Hadron Resonance Gas

n-th cumulant is given by

$$\chi_n^Q = \sum_{i \in \text{ hadrons}} \chi_{ni}^Q; \quad \chi_{in}^Q = \frac{(2s_i - 1)m_i^2 Q_i^n}{\pi^2 T^2} \sum_{l=1}^{\infty} (\pm 1)^{n-1} l^{n-2} K_2(lm_i/T)$$

upper/lower sign corresponds to bosons/fermions

Conclusion from this exercise

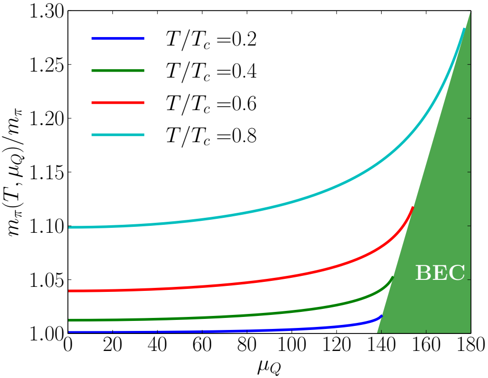


- at moderate n - significant contribution from doubly charge baryons (see e.g. P. Braun-Munzinger, B. Friman, F. Karsch, K. Redlich and V.S.,Nucl. Phys. A **880**, 48 (2012)

To describe location of BEC transition, higher order cumulants are required

Can we improve our methods?

- Only in free theory BE condensation occurs at vacuum pion mass.
- Functional Renormalization Group approach to take into account fluctuations (Intro: previous talk by Gergely)
- Pion mass dependence on μ_Q for different T:

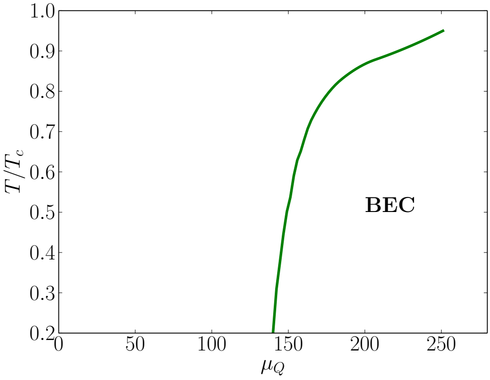


- Details of model: R. Pisarski and V.S. 1604.00022
- Details of FRG solution: V.S., B. Friman, K. Redlich 1108.3231

- Increase of pion mass with T and μ_Q

Non-perturbative chiral matrix model

- The phase diagram in $T - \mu_Q$ plane:



- Details of model:
  R. Pisarski and V.S. 1604.00022
- Details of FRG solution:
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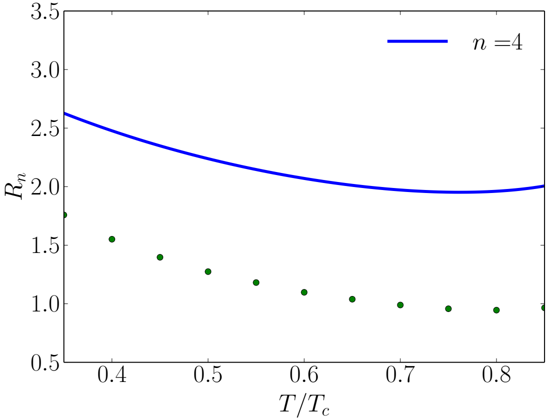
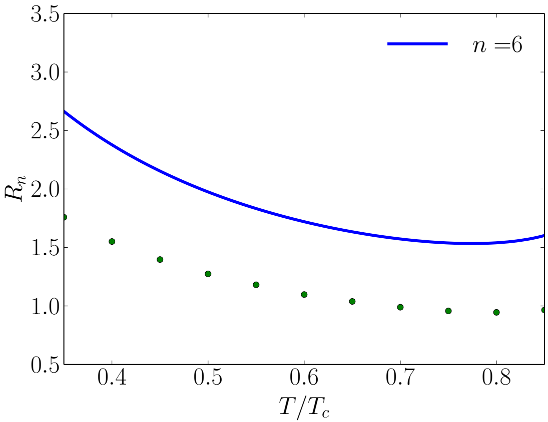
- Non-trivial dependence of T_{BEC} on μ_Q.
- **Can it be reproduced with cumulant analysis? What order is required?!**

- At low T transition is at about $\mu_Q = 140$ MeV as expected.

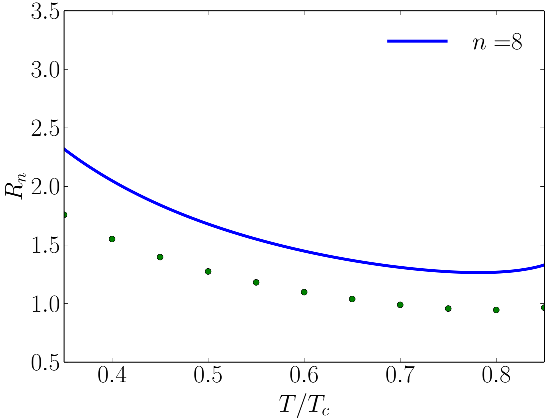


Green dots: exact results we want to reproduce. $R = m_\pi(T)/T$

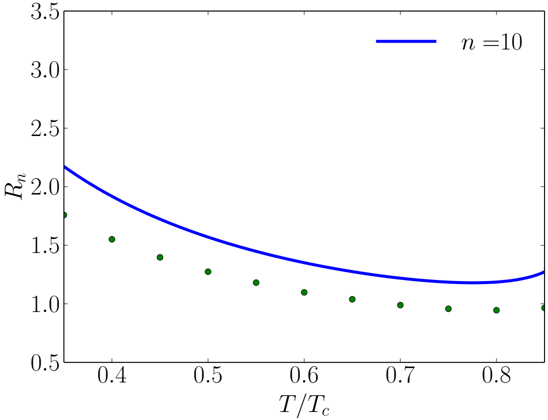
Non-perturbative chiral matrix model



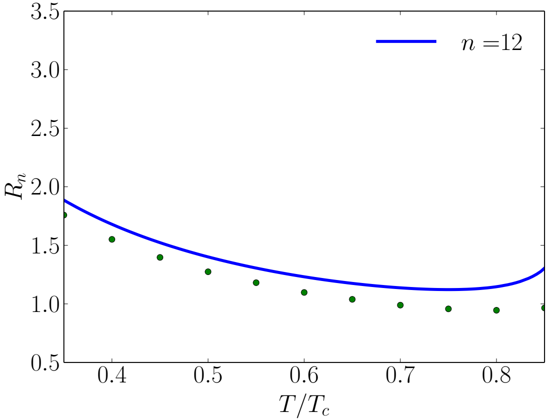
Non-perturbative chiral matrix model



Non-perturbative chiral matrix model



Non-perturbative chiral matrix model



Hopeless?!

Way to improve convergence

Conformal mapping

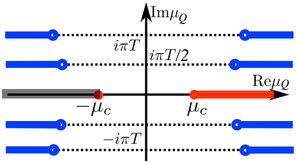
Conformal mapping is transformation $\xi = \xi(z)$ that preserves local angles.
The main idea is to extend radius of convergence and to enhance sensitivity to properties of critical point by non-linear transformation of original series.
E.g. using conformal mapping one can move physical singularities closer to expansion point, while keeping non-physical singularities as far away as possible.

Thermal singularity

In terms of quark chemical potential:

$$\text{Im}\,\mu_q = \pm(\pi T + 2\pi n T)/3, \quad n \in \mathbb{Z}$$

Thus closest singularities are located at



$$\text{Im}\,\mu_Q = \pm\pi T \text{ (related to d – quark or single charged baryons)}$$

and

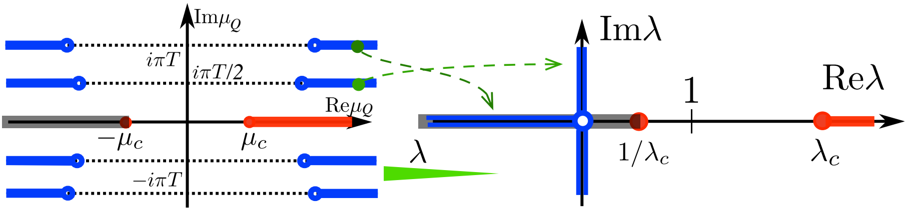
$$\text{Im}\,\mu_Q = \pm\pi T/2 \text{ (related to u – quark or doubly charged baryons)}$$

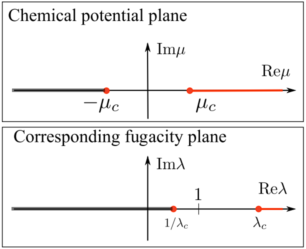
Thermal singularity

Fugacity

$$\lambda = \exp \mu_Q/T$$

- maps thermal singularities $i\pi T$ to negative real axis on fugacity plane
- maps thermal singularities $i\pi T/2$ to imaginary axis on fugacity plane





- Conjugate singularities interfere with method precision
- Mapping to fugacity plane, $\lambda = \exp\left(\frac{\mu}{T}\right) \rightsquigarrow \mu/T = \ln(\lambda)$
- To simplify notation $\tilde{\lambda} = \lambda - 1$

$$\sum_{n=0}^{n=10} \frac{1}{n!}\chi_n \left(\frac{\mu}{T}\right)^n =$$

$$\chi_0 + \frac{\tilde{\lambda}^2\chi_2}{2} - \frac{\tilde{\lambda}^3\chi_2}{2} + \tilde{\lambda}^4\left(\frac{11\chi_2}{24} + \frac{\chi_4}{24}\right) + \tilde{\lambda}^5\left(-\frac{5\chi_2}{12} - \frac{\chi_4}{12}\right)$$
$$+ \tilde{\lambda}^6\left(\frac{137\chi_2}{360} + \frac{17\chi_4}{144} + \frac{\chi_6}{720}\right) + \tilde{\lambda}^7\left(-\frac{7\chi_2}{20} - \frac{7\chi_4}{48} - \frac{\chi_6}{240}\right)$$
$$+ \tilde{\lambda}^8\left(\frac{363\chi_2}{1120} + \frac{967\chi_4}{5760} + \frac{23\chi_6}{2880} + \frac{\chi_8}{40320}\right) + \tilde{\lambda}^9\left(-\frac{761\chi_2}{2520} - \frac{89\chi_4}{480} - \frac{\chi_6}{80} - \frac{\chi_8}{10080}\right)$$
$$+ \tilde{\lambda}^{10}\left(\frac{7129\chi_2}{25200} + \frac{4523\chi_4}{22680} + \frac{3013\chi_6}{172800} + \frac{29\chi_8}{120960} + \frac{\chi_{10}}{3628800}\right) + \mathcal{O}\left(\tilde{\lambda}^{11}\right)$$

Darboux theorem I

Suppose that singularity z_c of $f(z)$ $(= \sum_n f_n z^n)$ is closest to origin, and that behaviour in its neighborhood is given by

$$f(z) = (1 - z/z_c)^{-\nu} r(z) + a(z)$$

with $r(z) = \sum_k b_k (z - z_c)^k$ and $a(z)$ being analytic in some circle with center at $z = 0$ and radius $> |z_c|$. Asymptotic expansion of $f(z)$

$$f_n \sim \sum_{k=0}^{\infty} \frac{(-1)^k b_k {z_c}^{k-n} \Gamma(n + \nu - k)}{n! \Gamma(\nu - k)}$$

To reduce corrections construct difference

$$f_n z_c - \frac{n + \nu - 1}{n} f_{n-1} \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k b_k {z_c}^{k-n} \Gamma(n + \nu - k - 1)}{n! \Gamma(\nu - k)} \sim \mathcal{O}(n^{\nu-3})$$

Consequently $z_c = \frac{f_{n-1}}{f_n} \left(1 + \frac{\nu - 1}{n}\right)$, leading term gives ratio method.

Darboux theorem II

Combine adjacent f_n to construct more accurate approximation.
This is better done by constructing recursive relation

$$\begin{aligned} X_n^0(z_c, \nu) &= f_n; \\ X_n^{m+1}(z_c, \nu) &= z_c X_n^m(z_c, \nu) - (n + \nu - 2m - 1) X_{n-1}^m(z_c, \nu)/n. \end{aligned}$$

Easy to show that

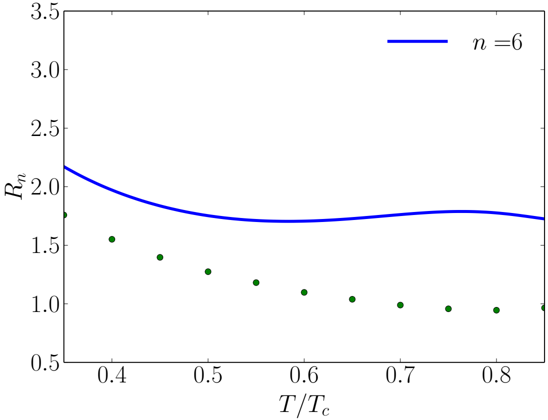
$$X_n^m \sim \mathcal{O}(n^{\nu - 2m - 1})$$

Two unknowns z_c and ν can be found by solving two equations

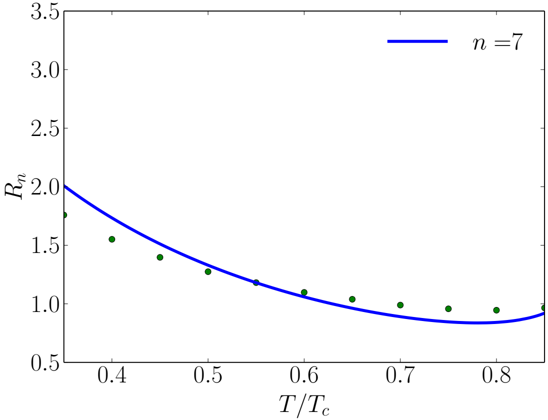
$$X_n^m(z_c, \nu) = 0, \quad X_{n-1}^m(z_c, \nu) = 0$$

For simplicity $m = 1$ and we use fugacity mapping.

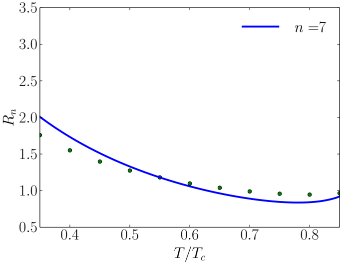
Applying Darboux theorem



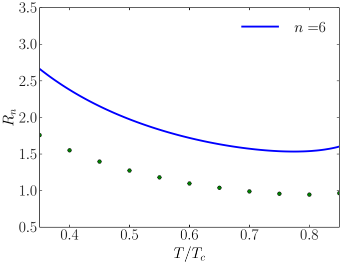
Applying Darboux theorem



Method based on Darboux theorem:



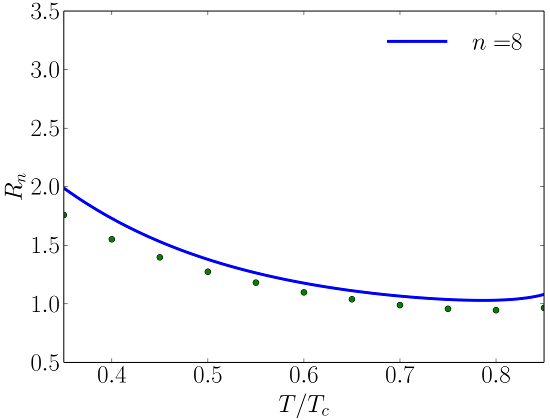
Ratio test:



Reminder: 7-th order in fugacity includes only 6-th and lower order cumulants:

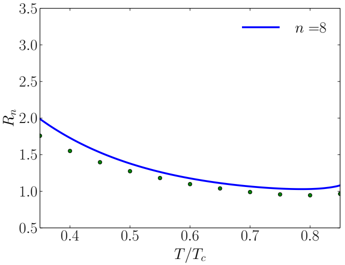
$$\tilde{\lambda}^7 \left(-\frac{7\chi_2}{20} - \frac{7\chi_4}{48} - \frac{\chi_6}{240} \right)$$

Applying Darboux theorem

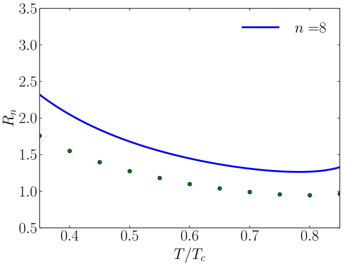


Applying Darboux theorem

Method based on Darboux theorem:



Ratio test:



Superior reconstruction of phase boundary

Conclusions and Outlook

Conclusions

- Cumulant analysis may eventually allow us to reconstruct 3D phase diagram
- In this talk, we explored μ_Q as 3-d axis. Commonly applied ratios test requires very high order cumulants and very ineffective
- Using conformal mapping to remove spurious singularities and Darboux theorem for properties of late coefficients, with reasonable number of cumulants $n = 6$ we were able to reconstruct onset of BEC.

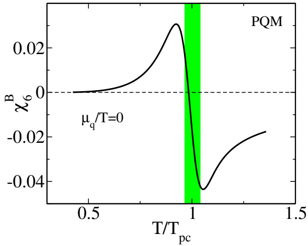
Outlook. Things I did not have time to show

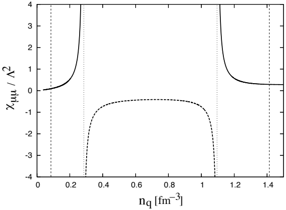
- It is sufficient to have only positive/negative pion fluctuations.
- Using similar approach it is possible to probe kaon condensation using strangeness fluctuations or kaon fluctuations.

- Static:
  - O(4) crossover: universal sign structure of higher order cumulants, χ_6
  - Divergent cumulants of baryon number fluctuations at CP

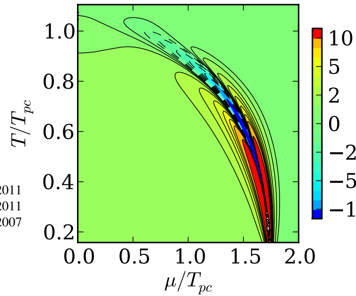
$$\chi_n \propto \xi^{3\left(\frac{n\beta\gamma}{2-\alpha}-1\right)}$$

  - In vicinity of CP: universal sign structure of cumulants.
  - Divergent baryon number susceptibility at off-equilibrium first order phase transition





B. Friman, F. Karsch, K. Redlich, V. S. 2011
M. Stephanov 2011
C. Sasaki, K. Redlich, B. Friman 2007

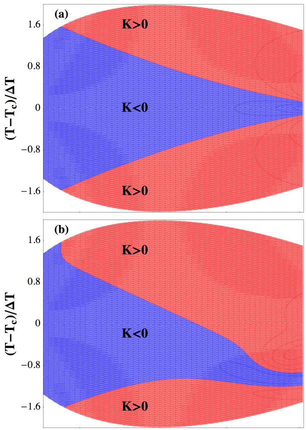


Fluctuations as a probe of phase diagram

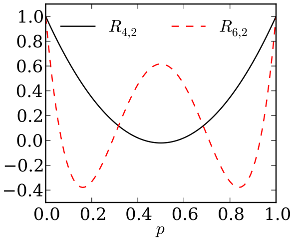
Last slide: predictions for an infinite static medium.
In HIC:

- Finite lifetime (S. Mukherjee, R. Venugopalan, Y. Yin 2015/2016)
- Finite size and anisotropy (G. Almasi and V.S. work in progress)
- Conservation laws (A. Bzdak, V. Koch, V.S. 2011)
- Fluctuations not related to interesting physics (V.S., B. Friman, K. Redlich 2012)

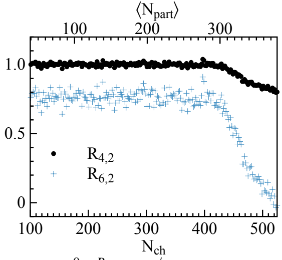
Modification due to finite life-time:



S. Mukherjee, R. Venugopalan, Y. Yin 2015



p is a fraction of measured baryons



$\mu = 0$ $\quad R_{n,m} = \chi_n/\chi_m$