Non-binomial efficiency corrections, plus comments on multi-particle correlations

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Efficiency correction is important

STAR (thanks to X. Luo)

Au+Au : 0-5%
Net-proton
0.4<p_T< 2 (GeV/c),|y|<0.5
Eff. Corrected
Eff. Uncorrected

Colliding Energy √s_{NN} (GeV)
If efficiency is driven by binomial with $p$ (or $\epsilon$)

\[
\begin{align*}
\left\langle \frac{N!}{(N-i)!} \right\rangle &= \frac{1}{p^i} \left\langle \frac{n!}{(n-i)!} \right\rangle \\
F_i &= \frac{1}{p^i} f_i
\end{align*}
\]

So we express true cumulants through factorial moments $F_i$, which are known from the above equality ($f_i$ is measured, $p$ is known)

AB, V. Koch, 
PRC 86 (2012) 044904; PRC 91 (2015) 027901
Binomial efficiency

\[ \rho_2(y_1, y_2) = \rho(y_1) \rho(y_2) [1 + c_2(y_1, y_2)] \]

two-particle density

two-particle density

single-particle densities

Binomial efficiency modifies only single particle distribution, it does not influence physics

Binomial efficiency

\[ B(n, N) = \frac{N!}{n! (N - n)!} [p]^n [1 - p]^{N-n} \]
Non-binomial efficiency

\[
\rho_2(y_1, y_2) = \rho(y_1)\rho(y_2)[1 + c_2(y_1, y_2)]
\]

is not efficiency independent (unless we have binomial)

\[
\rho_3(y_1, y_2, y_3) = \rho(y_1)\rho(y_2)\rho(y_3)[1 + c_2(y_1, y_2) + \cdots + c_3(y_1, y_2, y_3)]
\]
STAR efficiencies at 19.6 GeV and 7.7 GeV

We see non-binomial efficiency.

If $\epsilon$ depends on $N$ the whole method brakes down.

$$f_i = \sum_N \epsilon^i(N) P(N) \frac{N!}{(N - i)!}$$

Let’s test it. Suppose that

$$P(N) = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle},$$

$$\epsilon(N) = \epsilon_0 + \epsilon'(N - \langle N \rangle)$$

with $\langle N \rangle = 40$, $\epsilon_0 = 0.65$ and plot $K_n/K_2$ as a function of $\epsilon'$. We calculate exact $f_i$ and correct using constant efficiency $F_i = f_i/\epsilon_0^i$. 

R. Holzmann, talk at HIC for FAIR
We obtain

Large corrections for small $\epsilon'$
Another example of non-binomial distribution
Beta-binomial distribution (we return 2 balls)

\[
B(n, N=40)
\]

<table>
<thead>
<tr>
<th>Beta-binomial</th>
<th>( \alpha = 30 )</th>
<th>( \alpha = 60 )</th>
<th>( \alpha = 150 )</th>
<th>( \alpha = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_3/K_2 )</td>
<td>1.28</td>
<td>1.24</td>
<td>1.13</td>
<td>1.02</td>
</tr>
<tr>
<td>( K_4/K_2 )</td>
<td>0.82</td>
<td>1.45</td>
<td>1.35</td>
<td>1.07</td>
</tr>
<tr>
<td>( K_5/K_2 )</td>
<td>-1.11</td>
<td>1.15</td>
<td>1.63</td>
<td>1.16</td>
</tr>
<tr>
<td>( K_6/K_2 )</td>
<td>5.71</td>
<td>-0.44</td>
<td>1.80</td>
<td>1.32</td>
</tr>
</tbody>
</table>

AB, R.Holzmann, V.Koch
arXiv:1603.09057
We need to do proper unfolding
For example:

\[ p(n) = \sum_{N=n}^{\infty} P(N) \frac{N!}{n!(N-n)!} \epsilon^n (1 - \epsilon)^{N-n} \]

\[
\begin{pmatrix}
  p(0) \\
  p(1) \\
  p(2) \\
  p(3) \\
  p(4)
\end{pmatrix}
= 
\begin{pmatrix}
  1 & 1 - \epsilon & (1 - \epsilon)^2 & (1 - \epsilon)^3 & (1 - \epsilon)^4 \\
  0 & \epsilon & 2\epsilon(1 - \epsilon) & 3\epsilon(1 - \epsilon)^2 & 4\epsilon(1 - \epsilon)^3 \\
  0 & 0 & \epsilon^2 & 3\epsilon^2(1 - \epsilon) & 6\epsilon^2(1 - \epsilon)^2 \\
  0 & 0 & 0 & \epsilon^3 & 4\epsilon^3(1 - \epsilon) \\
  0 & 0 & 0 & 0 & \epsilon^4
\end{pmatrix}
\begin{pmatrix}
  P(0) \\
  P(1) \\
  P(2) \\
  P(3) \\
  P(4)
\end{pmatrix}
\]

matrix is pseudo-singular

We can easily use \( \epsilon(N) \), matrix is much more complicated but it is not a big deal.

In general

\[ p(n) = \sum_{N=n}^{\infty} P(N)B(n; N) \]
The method works for $\epsilon(N)$

$\langle N \rangle = 40, \epsilon_0 = 0.7, \epsilon' = -0.0005$

It works very well, statistical errors are under control
Take-home message

- Non-binomial efficiency is (most likely) important

- Technique based on correcting factorial moments is not good enough

- Proper unfolding is warranted – see R. Holzmann’s talk
Multi-particle correlation functions
\[
\rho_2(y_1, y_2) = \rho(y_1)\rho(y_2)[1 + c_2(y_1, y_2)]
\]

\[
\langle N(N - 1) \rangle = \langle N \rangle^2 + \langle N \rangle^2 c_2
\]

\[
c_2 = \frac{\int \rho(y_1)\rho(y_2)c_2(y_1, y_2) dy_1 dy_2}{\int \rho(y_1)\rho(y_2) dy_1 dy_2}
\]

coupling

and the second order cumulant

\[
K_2 = \langle N \rangle + \langle N \rangle^2 c_2
\]
In the same way

\[ \rho_3(y_1, y_2, y_3) = \rho(y_1)\rho(y_2)\rho(y_3)[1 + c_2(y_1, y_2) + \cdots + c_3(y_1, y_2, y_3)] \]

\[ F_3 = \langle N(N - 1)(N - 2) \rangle = \langle N \rangle^3 + 3\langle N \rangle^3 c_2 + \langle N \rangle^3 c_3 \]

\[ c_3 = \frac{\int \rho(y_1)\rho(y_2)\rho(y_3)c_3(y_1, y_2, y_3)dy_1dy_2dy_3}{\int \rho(y_1)\rho(y_2)\rho(y_3)dy_1dy_2dy_3} \]

coupling

and the third order cumulant

\[ K_3 = \langle N \rangle + 3\langle N \rangle^2 c_2 + \langle N \rangle^3 c_3 \]
Finally we obtain

\[ K_2 = \langle N \rangle + \langle N \rangle^2 c_2 \]

\[ K_3 = \langle N \rangle + 3 \langle N \rangle^2 c_2 + \langle N \rangle^3 c_3 \]

\[ K_4 = \langle N \rangle + 7 \langle N \rangle^2 c_2 + 6 \langle N \rangle^3 c_3 + \langle N \rangle^4 c_4 \]

Let’s fix the rapidity bin:

- \( K_i \) scales like \( \langle N \rangle^i \) if correlation function do not depend on \( \langle N \rangle \).

- Comparing different centralities is tricky. Physics \( (c_2, 3, 4) \) could be the same but \( K_4/K_2 \) can be different because of changing \( \langle N \rangle \).

- Same with energy!

- If \( \langle N \rangle < 1 \) (anti-protons at low energy) higher order correlation functions are getting less visible, \( K_i \sim \langle N \rangle \), and the ratios \( \to 1 \).
Anti-protons and protons are different at low energies

\[ \langle N_{\text{proton}} \rangle \sim 40, \quad \langle N_{\text{anti-proton}} \rangle \sim 0.25 \]
\( K_2 = \langle N \rangle + \langle N \rangle^2 c_2 \)

\( K_3 = \langle N \rangle + 3 \langle N \rangle^2 c_2 + \langle N \rangle^3 c_3 \)

\( K_4 = \langle N \rangle + 7 \langle N \rangle^2 c_2 + 6 \langle N \rangle^3 c_3 + \langle N \rangle^4 c_4 \)

\[ c_2 = \frac{\int \rho(y_1) \rho(y_2) c_2(y_1, y_2) dy_1 dy_2}{\int \rho(y_1) \rho(y_2) dy_1 dy_2} \]

Now let’s change the rapidity bin, \( \Delta y \).
Assume that \( c_n(y_1, \ldots, y_n) \) changes slowly within measured rapidity bin.
In this case \( c_n \) does not depend on \( \Delta y \).

\( \langle N \rangle \sim \Delta y \) and \( K_2 \sim \Delta y^2, K_4 \sim \Delta y^4 \).

Consequently \( K_4/K_2 \sim \Delta y^2 \)
Any initial state effect (i.e., correlation long-range in rapidity, e.g., volume fluctuation) can give this.

This plot only proves that correlation between particle is not very short-range in rapidity.
We extracted multi-particle correlation couplings

Physics rather flat as a function of \( N_{\text{part}} \)

\[ c_2 > c_3 > c_4 \] (what do we expect for the critical point?)
... and for smaller $N_{\text{part}}$

![Graph showing non-trivial dependence for smaller $N_{\text{part}}$. Something changes around $N_{\text{part}} = 100$.](image)

Non-trivial dependence for smaller $N_{\text{part}}$. Something changes around $N_{\text{part}} = 100$. 

![Graph showing non-trivial dependence for smaller $N_{\text{part}}$. Something changes around $N_{\text{part}} = 100$.](image)
Take-home message

- Cumulants ratios are tricky to interpret, they mix correlations function of different order and ratios can depend on average number of particles.

- Multi-particle correlation couplings are much cleaner.

- In data $c_2 > c_3 > c_4$ at 7.7 GeV. What do we expect for the critical point?
Conclusions

Non-binomial efficiency is a serious problem. We need to do proper unfolding.

Studying multi-particle correlation functions is much cleaner.