Near and Far from Equilibrium Power-Law Statistics

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Abstract. We analyze the connection between p_T and multiplicity distributions in a statistical framework. We connect the Tsallis parameters, T and q, to physical properties like average energy per particle and the second scaled factorial moment, $F_2 = \langle n(n-1) \rangle / \langle n \rangle^2$, measured in multiplicity distributions. Near and far from equilibrium scenarios with master equations for the probability of having n particles, P_n, are reviewed based on hadronization transition rates, μ_n, from n to $n+1$ particles.

In this paper we approach the hadronization problem in high energy physics from statistical viewpoint. In a balanced version of decay and growth processes a simple master equation arrives at a final state including the Poisson, Bernoulli, negative binomial and Pólya distribution [1]. Such decay and growth rates incorporate a symmetry between the observed subsystem and the rest of a total system as a rule. For particle physics problems P_n is the probability of having n particles (or other quanta). In an avalanche type process the simplest assumptions about elementary rates in the master equation result in the exponential distribution with constant rates and in the power-law tailed Waring [2] distribution with linear preference rates. In this short paper we review relevant random filling patterns in phase space and treat thermal parameters as averages. We present master equations classified for describing dynamical stochastic processes near and far from equilibrium, and in particular analyze stationary distributions for large n. In this limit a set of coupled ordinary differential equations is replaced by a partial differential equation.

Following Einstein, we assume that the available phase-space volume at a given total energy, E, is filled evenly in statistical equilibrium or in a stationary state. Then the blind chance to find a part of the phase space, a.o. a single particle with energy $\omega \approx m_T - m$, will be given by $\Omega(\omega)\Omega(E-\omega)/\Omega(E)$. Beyond the one-particle phase space factor, $\Omega(\omega)$, the rest occurs as an environmental weight factor

$$p(\omega) \;=\; \frac{\Omega(E-\omega)}{\Omega(E)}. \tag{1}$$

In the simplest, idealized case the phase space volume is just a hypersphere with radius E, in dimension n: $\Omega(E) \propto E^n$. The above weight factor for picking out a single particle with energy ω becomes:

$$p(\omega) \;=\; \left(1 - \frac{\omega}{E}\right)^n. \tag{2}$$

The often cited thermodynamical limit then leads to a Boltzmann–Gibbs factor

$$p(\omega) \;\sim\; \lim_{\substack{E\to\infty, n\to\infty \\ E/n=T}} \left(1 - \frac{\omega}{E}\right)^n \;=\; \mathrm{e}^{-\omega/T}, \tag{3}$$

interpreting the kinetic temperature as $T = E/n$.

However, experimentally studied physical systems are often far from the thermodynamical limit. In *small systems* fluctuations can be large, and an averaging over several millions of events is done for the histogram bins in obtaining p_T spectra. In thermal models ω is a monotonic rising function of p_T. Such an average of the above ratio,

$$\langle p(\omega) \rangle \; = \; \sum_{n=0}^{\infty} P_n \left(1 - \frac{\omega}{E} \right)^n , \tag{4}$$

is interesting to be inspected for famous n-distributions. For the Poisson we obtain

$$\langle p(\omega) \rangle^{\text{POISSON}} = \mathrm{e}^{-\langle n \rangle \, \omega / E}, \tag{5}$$

and for the negative binomial distribution (NBD)

$$\langle p(\omega) \rangle^{\text{NBD}} \; = \; \left(1 + \frac{\langle n \rangle}{k+1} \frac{\omega}{E} \right)^{-k-1} . \tag{6}$$

For a general event distribution we expand eq.(4) for $\omega \ll E$ and compare the result with the Tsallis–Pareto distribution

$$p(\omega) \; = \; \left(1 + (q-1) \frac{\omega}{T} \right)^{-\frac{1}{q-1}} . \tag{7}$$

Expanding both expressions (4,7) up to terms quadratic in ω we obtain

$$T \; = \; \frac{E}{\langle n \rangle}, \qquad \text{and} \qquad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}. \tag{8}$$

The parameter T is the event-averaged kinetic temperature, while $(q-1)$ is a measure of the non-Poissonity in the multiplicity distribution. These findings can easily be generalized by comparing the environmental factor (1) with the Tsallis – Pareto distribution (7):

$$\langle p(\omega) \rangle \; = \; \left\langle \mathrm{e}^{S(E-\omega) - S(E)} \right\rangle \; \approx \; \left(1 + (q-1) \frac{\omega}{T} \right)^{-\frac{1}{q-1}} . \tag{9}$$

Expanding in $\omega \ll E$ both sides up to quadratic terms we obtain

$$\frac{1}{T} \; = \; \langle S'(E) \rangle = \langle \beta \rangle \, , \qquad\qquad q = 1 - \frac{1}{C} + \frac{\Delta \beta^2}{\langle \beta \rangle^2}. \tag{10}$$

This interprets the parameter T as the inverse of the average $\beta = S'(E)$ and $(q-1)$ as the non-Gaussianity in β fluctuation. For $q = 1$ the variance $\Delta\beta$ would be $1/T\sqrt{C}$ [3]. Here $\langle S''(E) \rangle = -1/CT^2$.

We discuss the following schemes of master equations. The diffusion class describes near equilibrium stochastic evolution of the probability, P_n, with the growth rate form n to $n+1$, μ_n and the decay rate to $n-1$, λ_n:

$$\dot{P}_n \; = \; [(\lambda P)_{n+1} \; - \; (\lambda P)_n] \; - \; [(\mu P)_n \; - \; (\mu P)_{n-1}] \, . \tag{11}$$

The avalanche class describes a contesting race eventually achieving stationary branching ratios:

$$\begin{aligned} \dot{P}_0 \; &= \; \langle \gamma \rangle - (\gamma_0 + \mu_0) P_0, \\ \dot{P}_n \; &= \; \mu_{n-1} P_{n-1} \; - \; (\mu_n + \gamma_n) \; P_n \qquad \text{for } n \geq 1. \end{aligned} \tag{12}$$

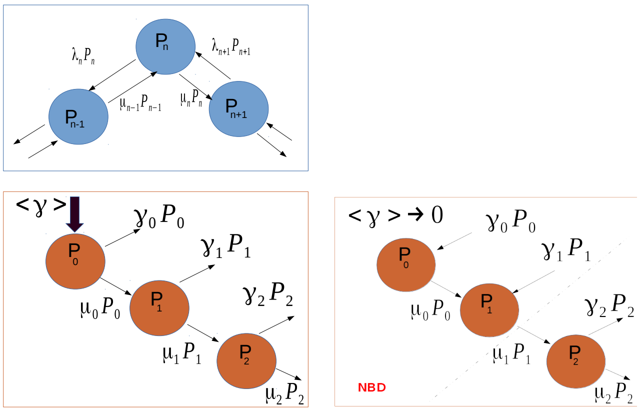

Figure 1. Comparison of near equilibrium, diffusion like and far equilibrium, avalanche like models. The graphs on the left side demonstrate the respective structures of the equations, on the right side the particular one suggested to generate the NBD distribution in Ref.[4].

Here the decay rate, γ_n, describes exit from the chain, a reduction in n is not assumed. Fig.1 depicts the difference between these classes. By studying the stationary branching in the avalanche type dynamics, we are especially interested in the mean aging model, where for $\forall n : \gamma_n = \gamma$ and therefore $\langle \gamma \rangle = \gamma$. In the stationary limit: $P_n(t) \to Q_n$, and from $\dot{Q}_n = 0$ one obtains

$$Q_n \;=\; \frac{\mu_{n-1}}{\mu_n + \gamma}\, Q_{n-1} \;=\; \cdots \;=\; \frac{\gamma}{\mu_n}\, \mathrm{e}^{-\sum\limits_{j=0}^{n} \ln(1+\gamma/\mu_j)}. \tag{13}$$

Now, constant growth rates, $\mu_j = \sigma$ lead to the exponential distribution

$$Q_n \;=\; \frac{1}{1+\sigma/\gamma}\, \mathrm{e}^{-n\cdot\ln(1+\gamma/\sigma)}. \tag{14}$$

Linear preference growth rates $\mu_j = \sigma(j+b)$ $(b > 0)$ lead to the Waring distribution [2; 5; 6],

$$Q_n \;=\; \frac{\gamma}{\gamma + b\sigma}\, \frac{\Gamma(n+b)\,\Gamma(b+1+\gamma/\sigma)}{\Gamma(b)\,\Gamma(n+b+1+\gamma/\sigma)}. \tag{15}$$

This distribution has a power-law tail for large n as $\sim n^{-1-\gamma/\sigma}$. For negligible decay compared to the growth, $\gamma \ll \sigma$, one arrives at the Zipf distribution. It is interesting to note that it has been suggested by Osada et.al. [4] that the special dynamics with $\gamma_n = \sigma(n - kf)$ and $\mu_n = \sigma f(n+k)$ leads exactly to the NBD as stationary distribution,

$$Q_n \;=\; \binom{n+k-1}{n}\, f^n\, (1+f)^{-n-k}. \tag{16}$$

We note that for large n the avalanche dynamics effectively uses the variable $x = n\Delta x$ and solves $\frac{\partial}{\partial t} P(x,t) \;=\; -\frac{\partial}{\partial x}\left(\mu(x)\, P(x,t)\right) - \gamma(x) P(x,t)$. The stationary distribution is

$$Q(x) \;=\; \frac{K}{\mu(x)}\, \mathrm{e}^{-\int\limits_0^x \frac{\gamma(u)}{\mu(u)}\, du}. \tag{17}$$

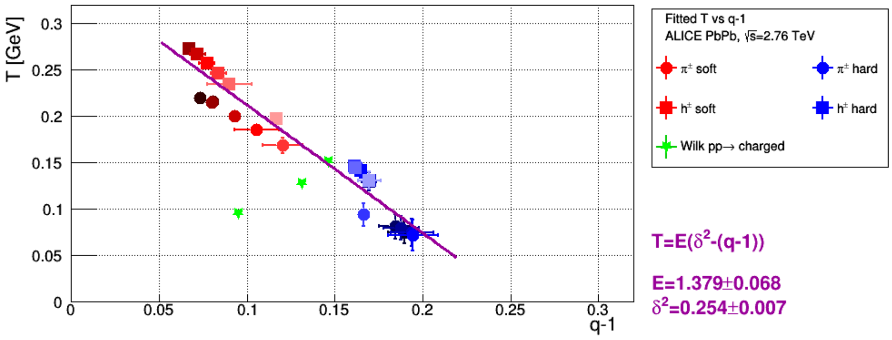

Figure 2. The Tsallis parameters T vs $q-1$ from fits to the 2.76 TeV ALICE PbPb data.

In this framework the constant rate $\mu(x) = \sigma$ leads to the exponential, and the linear rate, $\mu(x) = \sigma(x+b)$ to the Tsallis–Pareto stationary distribution [7]

$$Q(x) = \frac{\gamma}{\sigma\, b} \left(1 + \frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{18}$$

A connection between p_T and n distributions is hinted at in finding different q values for different participant numbers in experiments [8–10]. Our model form eqs.(8,10) predicts $T = E(\delta^2 - (q-1))$ with $E = CT \approx 1.4$ GeV and $\delta = \Delta\beta/\langle\beta\rangle \approx 0.5$. Fig.2 shows our fit results to the LHC ALICE PbPb data at 2.76 TeV. Darker points belong to more central collisions. We also distinguish between soft ($p_T < 5$ GeV, red data points) and hard ($p_T > 3$ GeV, blue data points) spectral parts. For fit parameters and further details see legend. The green stars are data from pp collision [11], lying on a $T = E(q-1)$ line with $E \approx 1$ GeV. The AA points seem to favor $\delta^2 = 0.25$ independent of $\langle n\rangle$, while the pp points $\delta^2 \propto 1/\langle n\rangle$, meaning a constant $f = \langle n\rangle/k$ value in the NBD distribution.

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