Sensitivity of SHiP to Violations of Lepton Universality in Nu-tau Scattering

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Outlook

- **Introduction**
- **Formalism**
  - Scalar and Tensor Interactions
  - Explicit Leptoquark Model
  - $V \pm A$ Interactions
- **Constraints on NP couplings**
  - $\tau^- \rightarrow \nu_\tau + \pi^-$
  - $\tau^- \rightarrow \pi^- + \pi^0 + \nu_\tau$
- **Numerical analysis**
- **Conclusion**

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Introduction

A key property of the SM gauge interactions

Lepton flavor universality

Evidence for violation of this property

Sign of new physics (NP) beyond the SM

Note: for NP, the second and third generation quarks and leptons could be special because they are comparatively heavier and are expected to be relatively more sensitive to NP.
Recently hints of lepton flavor non-universality

Some reports on non-universality in the lepton sector from BaBar and LHCb experiments.

**BaBar:**

\[
R(D) \equiv \frac{\mathcal{B}(B \rightarrow D^{+}\tau^{-}\bar{\nu}_{\tau})}{\mathcal{B}(B \rightarrow D^{+}\ell^{-}\bar{\nu}_{\ell})} = 0.440 \pm 0.058 \pm 0.042 ,
\]

\[
R(D^{*}) \equiv \frac{\mathcal{B}(B \rightarrow D^{*+}\tau^{-}\bar{\nu}_{\tau})}{\mathcal{B}(B \rightarrow D^{*+}\ell^{-}\bar{\nu}_{\ell})} = 0.332 \pm 0.024 \pm 0.018 ,
\]

(\text{In SM: } R(D) = 0.297 \pm 0.017 \text{ and } R(D^{*}) = 0.252 \pm 0.003)

- The BaBar Collaboration reported a 3.4σ deviation from SM.

**LHCb:** ratio of decay rates for \( B^+ \rightarrow K^+ l^+ l^- \) (\( l = e, \mu \))

in the dilepton invariant mass-squared range \( 1 \text{ GeV}^2 \leq q^2 \leq 6 \text{ GeV}^2 \)

\[
R_K \equiv \frac{\mathcal{B}(B^+ \rightarrow K^+ \mu^+ \mu^-)}{\mathcal{B}(B^+ \rightarrow K^+ e^+ e^-)} = 0.745^{+0.090}_{-0.074} \text{ (stat)} \pm 0.036 \text{ (syst)}
\]

- A 2.6σ difference from the SM prediction.
Recently hints of lepton flavor non-universality

\[
\begin{align*}
\text{LHCb: Branching fraction ratio } & \bar{B}_0 \rightarrow D^{*+}\tau^- \bar{\nu}_\tau/\bar{B}_0 \rightarrow D^{*+}\mu^- \bar{\nu}_\mu = 0.336 \pm 0.027 \pm 0.030
\end{align*}
\]
SHiP can Probe the New Physics

“In this paper we focus on observables that may be measured at a $\nu_\tau$ scattering experiment. There is a proposed Search for Hidden Particles (SHiP) experiment at CERN [13] which is expected to have a large sample of tau neutrinos which could be used to probe new physics in $\nu_\tau$ scattering ... “
**Formalism**

In the presence of NP, the effective Hamiltonian for the scattering process $\nu_\tau + N \rightarrow \tau + X$

$$\mathcal{H}_{eff} = \frac{4G_F V_{ud}}{\sqrt{2}} \left[ (1 + V_L) [\bar{u}\gamma_\mu P_L d] [\bar{l}\gamma^\mu P_L \nu_l] + V_R [\bar{u}\gamma_\mu P_R d] [\bar{l}\gamma_\mu P_L \nu_l] 

+ S_L [\bar{u} P_L d] [\bar{l} P_L \nu_l] + S_R [\bar{u} P_R d] [\bar{l} P_L \nu_l] + T_L [\bar{u}\sigma^{\mu\nu} P_L d] [\bar{l}\sigma_{\mu\nu} P_L \nu_l] \right]$$

In which $G_F$ is the Fermi coupling constant

$V_{qq'}$ is Cabibbo-Kobayashi-Maskawa (CKM) matrix element

$P_{L,R} = (1 \mp \gamma_5)/2, \sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$

- To introduce non-universality the NP couplings are different for different lepton flavors.
- In this paper, it is assumed the NP effect is mainly through the $\tau$ lepton.
- The effective Hamiltonian involves the quarks of the first generations only.
- It’s not assume any connection between NP for the different generations of quarks.
- The SM effective Hamiltonian corresponds to $g_L = g_R = g_S = g_P = 0$. 
Formalism

The Hamiltonian in the presence of only scalar and tensor operators:

\[ \mathcal{H}_{\text{eff}} = \frac{G_F V_{qq'}}{\sqrt{2}} \left[ \bar{l}(1 - \gamma_5) \nu_l \bar{q'}(A + B \gamma_5)q + T_L \bar{\sigma}_{\mu\nu}(1 - \gamma_5)\nu_l \bar{q'}\sigma^{\mu\nu}(1 - \gamma_5)q \right] \]

A = \( S_R + S_L \) and B = \( S_R - S_L \),
S\(_L\) and S\(_R\) are the left and right handed scalar couplings,
T\(_L\) is the tensor coupling.

- In the following, first a model independent approach is employed and treat the scalar and tensors coupling one at a time.
- Considering of an explicit LQ model where both the scalar and tensor couplings are present.
- The Hamiltonian in the presence of only V ± A operators.
Formalism: Scalar and Tensor Interactions

\[ \nu_\tau + N \rightarrow \tau + X, \ \nu_\mu + N \rightarrow \mu + X \]

The total differential cross section:

\[ \frac{d\sigma_{\text{tot}}}{dxdy} = \frac{d\sigma_{\text{SM}}}{dxdy} + \frac{d\sigma_{\text{LQS}}}{dxdy} + \frac{d\sigma_{\text{LQT}}}{dxdy} + \frac{d\sigma_{\text{SM,LQ}}}{dxdy} + \frac{d\sigma_{\text{LQS,LQT}}}{dxdy} \]

The total differential cross section in terms of the cross section amplitude:

\[ \frac{d\sigma}{dxdy} = \frac{1}{32\pi ME_\nu} \int \frac{d\xi}{\xi} f(\xi) |\mathcal{M}(\xi)|^2 \delta(\xi - x) \]

- the incoming neutrino energy
- momentum fraction of target nucleon
- parton distribution function (PDF) inside a nucleon
Formalism: Scalar and Tensor Interactions

Differential cross section could be also written with respect to the scaling variables.

\[ x = \frac{q^2}{2\nu}, \quad y = \frac{\nu}{ME_\nu} \]

the Bjorken variable

the four-momentum transfer of the leptonic probe

\[ \nu = -p \cdot q = M(E_\nu - E_\ell) \]

the inelasticity with q

Therefore the differential cross sections could be written in terms of \((t, \nu)\) variables:

\[ \frac{d\sigma}{dx dy} = 2ME_\nu \nu \frac{d\sigma}{dq^2 d\nu} \]

We can also measure the differential cross section if we could have more information given by \(x\).

*Note:* the related formulas can be found inside the paper.

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LQ Model
Formalism: Explicit LQ Model

- Tensor operators in the effective Lagrangian get contributions only from scalar LQs.
- So only scalar LQs is considered.
- Considering the case that the LQ is a weak doublet or a weak singlet.
- The weak doublet leptoquark, $R_2$ has the quantum numbers $(3, 2, 7/6)$ under SU$(3)_c \times SU(2)_L \times U(1)_Y$ while the singlet leptoquark $S_1$ has the quantum numbers $(\bar{3}, 1, 1/3)$.
- Assuming the quark mixing matrices to be hierarchical, and considering only the leading contribution we can ignore the effect of mixing.
- The general Wilson coefficients at the leptoquark mass scale contributing to the $\nu_\tau + N \rightarrow \tau + X$ process:

$$S_L = \frac{1}{2\sqrt{2}G_F V_{ud}} \left[ -\frac{g_{1L}^1 g_{1R}^{13*}}{2M^2_{S_1}} - \frac{g_{2L}^1 g_{2R}^{13*}}{2M^2_{R_2}} \right],$$

$$T_L = \frac{1}{2\sqrt{2}G_F V_{ud}} \left[ \frac{g_{1L}^1 g_{1R}^{13*}}{8M^2_{S_1}} - \frac{g_{2L}^1 g_{2R}^{13*}}{8M^2_{R_2}} \right].$$

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V ± A Model
Formalism: $V \pm A$ Interactions

- The DIS differential cross section in the presence of $V \pm A$ operators ($\text{Vector-Axial Vector}$) in terms of the momentum transfer:

\[
\frac{d\sigma_{\text{SM}+(V\pm A)}}{dq^2 d\nu} = \frac{G_F^2}{8\pi M E_\nu^2} (((|a'|^2 + |b'|^2)) (m_\ell^2 + q^2) W_1 \\
+ \frac{1}{2M} (|a'|^2 + |b'|^2) (4E_\nu^2 M - 4E_\nu \nu - M(m_\ell^2 + q^2)) W_2 \\
+ \frac{1}{M^2} \text{Re}[a'b^*](2E_\nu Mq^2 - \nu(m_\ell^2 + q^2)) W_3 - \frac{1}{M} (|a'|^2 + |b'|^2) m_\ell^2 E_\nu W_5)
\]

- Where:
  
  \[
a' = 1 + \gamma^\rho, \\
b' = 1 + \gamma^\kappa, \\
\gamma^\rho = V_L + V_R, \\
\gamma^\kappa = V_L - V_R.
\]
Constraints on NP couplings

• The scalar couplings $S_L$ and $S_R$ can be constrained by the tau decay channel
  $\tau^{-}(k_1) \rightarrow \nu_\tau (k_2) + \pi^-(q)$.

• The tensor coupling $T_L$ can be constrained by the decay channel
  $\tau^{-}(p) \rightarrow \pi^-(p1) + \pi^0 (p_2) + \nu_\tau (p_3)$.
PdG Numbers

\[
\frac{\Gamma(\tau^- \rightarrow \pi^-\nu_\tau)}{\Gamma_{\text{total}}} = 10.828 \pm 0.070 \pm 0.078 \% \\
\frac{\Gamma(\tau^- \rightarrow \pi^-\pi^0\nu_\tau)}{\Gamma_{\text{total}}} = 25.471 \pm 0.097 \pm 0.085 \%
\]

ALEPH 1991-1995 LEP runs

- The BR was measured at LEP.
- If the BR(\(\tau^- \rightarrow \pi^-\nu_\tau\), \(\tau^- \rightarrow \pi^-\pi^0\nu_\tau\)) could be measured in a better way - in the future! -, it will improve the constrained region but by how much?
- Plausibly we need FCC-ee to do a better measurement...
- This analysis refer to the \(\tau^-\nu_\tau\) ubar-d vertex and therefore \(D_s \rightarrow \tau\nu_\tau\) (cbar-s) does not affect on this limits.
Constraints on NP couplings
\[ \tau^- (k_1) \rightarrow \nu_\tau (k_2) + \pi^- (q) \]

- The constraints on the scalar couplings \( S_L \) and \( S_R \).
- In right plot, it’s assumed both couplings are present and take the couplings to be real.
- In the plot below, it’s assumed the couplings are complex and take one coupling at a time.
- The colored region is allowed.
- It is measured within the 2\( \sigma \) level.

- Left: \( S_R = 0 \) and treat \( S_L \) as a complex coupling.
- Right: \( S_L = 0 \) and treat \( S_R \) as a complex coupling.
Constraints on NP couplings
\( \tau^- (p) \rightarrow \pi^- (p_1) + \pi^0 (p_2) + \nu_\tau (p_3) \)

- If the tensor coupling be complex.
- The plot shows the allowed region for the real and imaginary components of the complex LQ coupling \( T_L \).
- The constraint on \( T_L \) is from \( \tau^- \rightarrow \pi^- \pi^0 \nu_\tau \).
- \( 0.04 < |T_L| < 0.104 \) within the 2\( \sigma \) level.

- The allowed regions of the LQ running couplings \( S_L(m_\tau) \) and \( T_L(m_\tau) \).
- \( S_L(m_{\text{LQ}}) = \pm 4 \ T_L(m_{\text{LQ}}) \)
- \( m_{\text{LQ}} = 1 \) TeV.
- The constraint on
  - \( S_L(m_\tau) \) is from \( \tau^- \rightarrow \pi^- \nu_\tau \)
  - \( T_L(m_\tau) \) is from \( \tau^- \rightarrow \pi^- \pi^0 \nu_\tau \).
For $V \pm A \rightarrow$ the couplings can be constrained by both decays.

- Right plot: if the couplings assumed to be real.
- The allowed regions for the real and imaginary parts are shown in the plots below.

- Left: $V_R = 0$ and $V_L$ treated as a complex coupling.
- Right: $V_L = 0$ and $V_R$ treated as a complex coupling.
Numerical analysis
Scalar and Tensor Model

- Leptoquark: The total cross section of $\nu_\tau + N \rightarrow \tau + X$.
- The green solid line corresponds to the standard model prediction $S_R = S_L = T_L = 0$.
- The blue dashed corresponds (−0.19,0.68,0.072)
- black dotted corresponds (1.98,0.04,−0.079)
- red dot dashed corresponds (−1.87,0.32,0.077)

**Scalar–Tensor Model**

**Courtesy of A. Datta**
The sensitivity of the neutrino cross-section for SHiP experiment due to the scalar and tensor interactions.

$$\langle S_R, S_L, T_L \rangle = (1.98, 0.04, -0.079)$$
Systematic and Statistical Errors

\[ S = \frac{N_{\text{Err}} - S_{\text{MErr}}}{N_{\text{Err}}} \]

- **20% SysErr**
  - \( S = 0.77 \)

- **10% SysErr**
  - \( S = 1.5 \)

- **5% SysErr**
  - \( S = 2.96 \)

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Numerical analysis
Scalar and Tensor Model

- LQ: \( \frac{d\sigma}{dt} \) of \( \nu_t + N \rightarrow \tau + X \) in the Scalar-Tensor model.
- The green lines correspond to the standard model predictions \((0,0,0)\)
- blue dashed: \((S_R, S_L, T_L) = (-1.45, 1.89, -0.056)\)
- black dotted: \((1.1, -1.8, 0.078)\)
- The physical regions of the momentum transfer taken to be \( Q^2 - (W_{\text{cut}}) \leq Q^2 \leq Q^2 + (W_{\text{cut}}) \).

With a measurement of \( Q^2 \) we can improve the sensitivity.

\[ E_\nu = 10 \]

\[ E_\nu = 20 \]
Numerical analysis

V ± A model

- V ±A model: The total cross section of $\nu_\tau + N \rightarrow \tau + X$.
- The green solid line corresponds to the SM prediction $V_L = V_R = 0$.
- blue dashed corresponds $(V_L, V_R) = (0.4596, 0.4504)$
- black dotted corresponds $(0.8189, 0.8082)$
- red dot dashed corresponds $(0.9836, 0.9731)$
Numerical analysis
V ± A model

SHiP sensitivity to V ± A Model

(V_L, V_R) = (0.9836, 0.9731)
Systematic and Statistical Errors

\[ S = \frac{NP_{Err} - SM_{Err}}{NP_{Err}} \]

- **20\% SysErr**
  - \( S = 3.67 \)

- **10\% SysErr**
  - \( S = 7.33 \)

- **5\% SysErr**
  - \( S = 14.5 \)

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Can we improve on the 20% error on the $\nu_\tau$ flux?

1) measure $D_s$ production $\rightarrow$ the ultimate systematic uncertainty on $\nu_\tau$ comes from $D_s \rightarrow \tau \nu_\tau$ BR.

$$\frac{\Gamma(D_s^+ \rightarrow \tau^+ \nu_\tau)}{\Gamma_{\text{total}}} = 5.70 \pm 0.21^{+0.31}_{-0.30} \%$$

BELL $e^+e^-$ at $\gamma(4S), \gamma(5S)$

<table>
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<th>Source</th>
<th>$K^- K^+ \pi^+$ [%]</th>
<th>$K^0 K^+$ [%]</th>
<th>$\eta \pi^+$ [%]</th>
<th>$e^+\nu_e$ [%]</th>
<th>$\mu^+\nu_\mu$ [%]</th>
<th>$\tau^+\nu_\tau$ [%]</th>
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<td>Tag bias</td>
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<td>$\pm 1.4$</td>
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<tr>
<td>Tracking</td>
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<tr>
<td>Particle ID</td>
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<td>$\pm 1.9$</td>
<td>$\pm 2.0$</td>
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<td>$\pm 1.4$</td>
<td>$\pm 4.3$</td>
<td>$\pm 1.8$</td>
<td>$\pm 0.8$</td>
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<td>Dalitz model</td>
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<td>$-$</td>
<td>$-$</td>
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<td>$-$</td>
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<tr>
<td>Fit model</td>
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<td>$\pm 0.8$</td>
<td>$\pm 2.2$</td>
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<td>$\pm 0.2$</td>
<td>$+3.3$</td>
</tr>
<tr>
<td>$D_s^+$ background</td>
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<td>$\pm 0.6$</td>
<td>$\pm 0.7$</td>
<td>$-$</td>
<td>$\pm 0.8$</td>
<td>$\pm 2.8$</td>
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<tr>
<td>$\tau$ cross-feed</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$\pm 0.9$</td>
</tr>
<tr>
<td>$B(\tau \rightarrow X)$</td>
<td>$-$</td>
<td>$-$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$\pm 0.2$</td>
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<td>Total syst.</td>
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<td>$\pm 2.9$</td>
<td>$\pm 3.9$</td>
<td>$\pm 5.4$</td>
<td>$\pm 3.8$</td>
<td>$\pm 5.4$</td>
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</table>


2) measure the $\nu_\tau$ flux directly $\rightarrow$ to be studied

- If this uncertainty can be reduced up to 1-2%, the sensitivity will improve especially in the case of LQ model.
Conclusion

• In this work we studied the sensitivity in the presence of NP and we got some promising results!
• In the case of $V \pm A$ interactions, having the 20% systematic uncertainty will not affect the result.
• In order to be sensitive the LQ model we need to shrink the uncertainty to be less then 5% and for that we need to provide a dedicated experiment to measure the flux of tau neutrinos.
• We are in touch with Datta and other theorist (Isidori, D’Ambrosio) to study more the theoretical calculation.
Backup
Formalism: Scalar and Tensor Interactions

\[
\frac{d\sigma_{SM}}{dxdy} = \frac{G_F^2 ME_\nu}{\pi} \left( y(xy + \frac{m_\ell^2}{2ME_\nu})F_1 + (1 - y - \frac{Mxy}{2E_\nu} - \frac{m_\ell^2}{4E_\nu^2})F_2 \right) \\
+ (xy(1 - \frac{y}{2}) - y\frac{m_\ell^2}{4ME_\nu})F_3 - \frac{m_\ell^2}{2ME_\nu}F_5 \right),
\]

\[
\frac{d\sigma_{LQS}}{dxdy} = \frac{G_F^2 ME_\nu}{4\pi} (A_S^2 + B_S^2) y(xy + \frac{m_\ell^2}{2ME_\nu})F_1,
\]

\[
\frac{d\sigma_{LQT}}{dxdy} = \frac{8G_F^2 ME_\nu}{\pi} T_L^2 \left( y(xy + \frac{m_\ell^2}{2ME_\nu})F_1 + 2(1 - y - \frac{Mxy}{4E_\nu} - \frac{m_\ell^2}{8E_\nu^2})F_2 - \frac{m_\ell^2}{ME_\nu}F_5 \right)
\]

\[
\frac{d\sigma_{SM,LQ}}{dxdy} = 0,
\]

\[
\frac{d\sigma_{LQS,LQT}}{dxdy} = \frac{2G_F^2 ME_\nu}{\pi} T_L (B_S - A_S) \left( xy(1 - \frac{y}{2}) - y\frac{m_\ell^2}{4ME_\nu} \right) F_3.
\]

\[
F_1 = \sum_{q,\bar{q}} f_{q,\bar{q}}(\xi, Q^2) V_{q,q'}^2,
\]

\[
F_2 = 2 \sum_{q,\bar{q}} \xi f_{q,\bar{q}}(\xi, Q^2) V_{q,q'}^2,
\]

\[
F_3 = 2 \sum_q f_q(\xi, Q^2) V_{q,q'}^2 - 2 \sum_{\bar{q}} f_{\bar{q}}(\xi, Q^2) V_{\bar{q},\bar{q}'}^2
\]

\[
F_5 = 2 \sum_{q,\bar{q}} f_{q,\bar{q}}(\xi, Q^2) V_{q,q'}^2,
\]

- \(f_{q,\bar{q}}\) are the parton distribution functions inside a nucleon,
- \(V_{q,q'}\) is the CKM matrix element
- \(Q^2 = -q^2\).
Formalism: LQ Model

• In the most general case ➔ singlet and doublet LQs are present ➔ both scalar and tensor operators appear in the effective Hamiltonian.

• However, as there is limited experimental information, we can consider the simpler cases when only a singlet or a doublet leptoquark are present.

Case 1) only the weak doublet scalar LQ:

\[
S_L = \frac{1}{2\sqrt{2}G_F V_{cb}} \left[ -\frac{g_{2L}^{13}g_{2R}^{13\ast}}{2M_{R_2}^2} \right] \\
T_L = \frac{1}{2\sqrt{2}G_F V_{cb}} \left[ -\frac{g_{2L}^{13}\cdot g_{2R}^{13\ast}}{8M_{R_2}^2} \right]
\]

Case 2) only the singlet LQ:

\[
S_L = \frac{1}{2\sqrt{2}G_F V_{cb}} \left[ -\frac{g_{1L}^{33}g_{1R}^{23\ast}}{2M_{S_1}^2} \right] \\
T_L = \frac{1}{2\sqrt{2}G_F V_{cb}} \left[ \frac{g_{1L}^{33}\cdot g_{1R}^{23\ast}}{8M_{S_1}^2} \right]
\]

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Formalism: LQ Model

• These relations are valid at the LQ mass scale.
• So, it is needed to run them down to the τ mass scale using the scale dependence of the scalar and tensor currents:

\[
S_{L}(m_{\tau}) = \left[ \frac{\alpha_{s}(m_{t})}{\alpha_{s}(m_{\tau})} \right]^{\frac{\gamma_{S}}{2\beta_{0}(5)}} \left[ \frac{\alpha_{s}(m_{LQ})}{\alpha_{s}(m_{t})} \right]^{\frac{\gamma_{S}}{2\beta_{0}(6)}} S_{L}(m_{LQ}),
\]

\[
T_{L}(m_{\tau}) = \left[ \frac{\alpha_{s}(m_{t})}{\alpha_{s}(m_{\tau})} \right]^{\frac{\gamma_{T}}{2\beta_{0}(5)}} \left[ \frac{\alpha_{s}(m_{LQ})}{\alpha_{s}(m_{t})} \right]^{\frac{\gamma_{T}}{2\beta_{0}(6)}} T_{L}(m_{LQ}),
\]

Where \(\gamma_{S} = -6C_{F} = -8\) and \(\gamma_{T} = 2C_{F} = 8/3\)

\[\beta_{0}(f) = 11 - 2n_{f}/3\]

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