Improving the precision of light quark mass determinations

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Introduction

- Light quark masses (up-, down- and strange-quark masses) can be determined non-perturbatively with lattice simulations in QCD
- Result from RBC/UKQCD Coll., domain-wall fermions:

$$m_{ud}^{\overline{\rm MS}}(2\ \mathrm{GeV}) = 3.72(0.16)_{\rm stat}(0.18)_{\rm syst}(0.33)_{\rm ren}\ \mathrm{MeV}$$
$$m_{s}^{\overline{\rm MS}}(2\ \mathrm{GeV}) = 107.3(4.4)_{\rm stat}(4.9)_{\rm syst}(9.7)_{\rm ren}\ \mathrm{MeV}$$

C. Allton et al.

- Error (11%) from renormalization dominates (>60% of tot.)
- Pert. calculations are performed in dim. reg.
  $\rightsquigarrow$ not directly amenable to lattice calculations
- Direct calculation of bare quantity with lattice spacing acting as ultra-violet cutoff in some particular discretization of QCD instead of space-time dimension $d \neq 4$
- Minimal subtraction(MS) à la dim. reg. not directly possible

Introduction

Regularization invariant momentum subtraction schemes

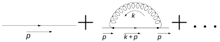
- Use for renormalization regularization invariant(RI) scheme, which removes ultraviolet divergences at a certain momentum point(subtraction point) $\rightsquigarrow$ RI/MOM-scheme

  Martinelli et al. '93-'95

- Determine QCD parameters: $m_R = Z_m m_B$, $\Psi_R = Z_q^{1/2} \Psi_B$,..

  $\rightsquigarrow$ fix renormalization constants, define scheme in PT:

$$S_R^{-1} = Z_q^{-1} S_B^{-1} \propto \not{p} \Sigma_R^V(p^2) - m_R \Sigma_R^S(p^2) \quad \Leftrightarrow$$



$$\lim_{m_R \to 0} \frac{1}{12 m_R} \mathrm{Tr}[S_R^{-1}(p)]\Big|_{p^2=-\mu^2} = 1 \qquad \lim_{m_R \to 0} \frac{1}{48} \mathrm{Tr}\left[\gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu}\right]\Big|_{p^2=-\mu^2} = -1 \quad \text{RI/MOM}$$

$$\lim_{m_R \to 0} \frac{1}{12 m_R} \mathrm{Tr}[S_R^{-1}(p)]\Big|_{p^2=-\mu^2} = 1 \qquad \lim_{m_R \to 0} \frac{1}{12 p^2} \mathrm{Tr}[S_R^{-1}(p)\not{p}]\Big|_{p^2 \to -\mu^2} = -1 \quad \text{RI'/MOM}$$

Introduction

Ward-Takahashi identities

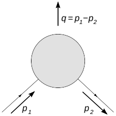
- Ward-Takahashi identities(WI)

$$q_\mu \Lambda^\mu_{V,B}(p_1,p_2) = S_B^{-1}(p_2) - S_B^{-1}(p_1)$$

$$-iq_\mu \Lambda^\mu_{A,B}(p_1,p_2) = 2m_B \Lambda_{P,B}(p_1,p_2) - i\gamma_5 S_B^{-1}(p_1) - S_B^{-1}(p_2) i\gamma_5$$

- WI valid for renorm. quantities: $O_R = Z_O O$, $\Lambda_{O,R} = \frac{Z_O}{Z_q}\Lambda_{O,B}$
- Renormalization condition on $S \Leftrightarrow$ condition on Λ_O

$$\frac{1}{N}\mathrm{Tr}\left[\Lambda_{O,R}(p_1,p_2)P_O\right]\Big|_{mom.conf.} = 1$$



$\rightsquigarrow$ study quark bilinear operators with vector(γ^μ), axial-vector($\gamma_5\gamma^\mu$), pseudo-scalar(γ_5) and scalar(**1**) operators

- Renormalization constants related:
  $Z_A = 1 = Z_V$, $Z_P = Z_S$, $Z_P = 1/Z_m$

Motivation

- RI/MOM

$$\lim_{m_R \to 0} \frac{1}{48} \mathrm{Tr}\left[\Lambda^{\mu}_{V,R}(p_1,p_2)\gamma_\mu\right]\Big|_{asym} = 1, \quad \lim_{m_R \to 0} \frac{1}{48} \mathrm{Tr}\left[\Lambda^{\mu}_{A,R}(p_1,p_2)\gamma_5\gamma_\mu\right]\Big|_{asym} = 1$$

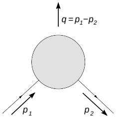
$$\lim_{m_R \to 0} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{S,R}(p_1,p_2)\mathbf{1}\right]\Big|_{asym} = 1, \quad \lim_{m_R \to 0} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{P,R}(p_1,p_2)\gamma_5\right]\Big|_{asym} = 1$$

- Asymmetric/exceptional momentum config.(MOM):

$$p_1^2 = p_2^2 = -\mu^2, \quad \mu^2 > 0, \quad p_1 = p_2, \quad q = 0$$

Symmetric/nonexceptional momentum config(SMOM):

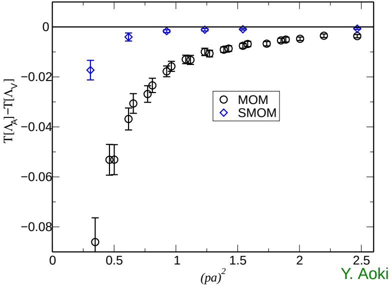
$$p_1^2 = p_2^2 = q^2 = -\mu^2, \quad \mu^2 > 0, \quad q = p_1 - p_2$$



- Renormalization constants need to be determined through simulation
- Need to introduce renormalization scale μ
  Lattice simulation: typically $a^{-1} \sim 2$ GeV

Motivation

- Symmetric subtraction point implies a lattice simulation with suppressed contamination from infrared effects
- For asymmetric subtraction point effects of chiral symmetry breaking vanish slowly like $1/p^2$ for large ext. momenta



- For SMOM infrared effects better behaved, vanishing with larger powers of p N.H. Christ

Motivation

RI-scheme $\Longrightarrow$ $\overline{\text{MS}}$-scheme

- Conversion/Matching factor:

$$m_R^{\overline{\text{MS}}} = C_m^{\text{RI/MOM}} m_R^{\text{RI/MOM}} \qquad (C_m \text{ in general gauge dependent})$$

- C_m can be computed in cont. PT, e.g. RI/MOM, RI'/MOM:

$$C_m^{\text{RI/MOM}} = 1.0 - 0.1333 - 0.0754 - 0.0495 \qquad \alpha_s(2\text{ GeV})/\pi \sim 0.1$$

$$C_m^{\text{RI'/MOM}} = 1.0 - 0.1333 - 0.0701 - 0.0458$$

G. Martinelli et al.; Franco, Lubicz; Chetyrkin, Retey; Gracey

- Matching to pert. theo.: reduce trunction error: large μ
  $\rightsquigarrow$ window problem
- Observation:
  Size of NLO, N^2LO, N^3LO contr. amount $\sim 13\%$, $\sim 7.5\%$, $\sim 5\%$
  $\rightsquigarrow$ poor convergence $\rightsquigarrow$ big error in renormalization

Concepts & Framework

RI/SMOM

- Idea: Use subtraction point with symmetric momenta

$$\lim_{m_R\to 0}\frac{1}{12q^2}\mathrm{Tr}\left[q_\mu \Lambda^\mu_{V,R}(p_1,p_2)\not{q}\right]\Big|_{sym}=1,\quad \lim_{m_R\to 0}\frac{1}{12q^2}\mathrm{Tr}\left[q_\mu \Lambda^\mu_{A,R}(p_1,p_2)\gamma_5\not{q}\right]\Big|_{sym}=1$$

$$\lim_{m_R\to 0}\frac{1}{12}\mathrm{Tr}\left[\Lambda_{S,R}(p_1,p_2)\mathbf{1}\right]\Big|_{sym}=1,\quad \lim_{m_R\to 0}\frac{1}{12i}\mathrm{Tr}\left[\Lambda_{P,R}(p_1,p_2)\gamma_5\right]\Big|_{sym}=1$$

- Relations between Z-factors preserved,
  $Z_A=1=Z_V$, $Z_P=Z_S$, $Z_P=1/Z_m$
  some hold nonperturbatively, others in continuum PT
- WI: Z_q in RI/SMOM same as RI'/MOM scheme (known to 3-loop)
- Two ways to compute C_m: (follows with WI)

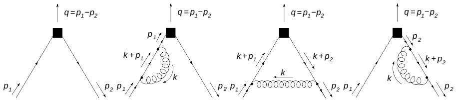
1.) $$(C_m^{\mathrm{RI/SMOM}})^{-1}=(C_m^{\mathrm{RI'/MOM}})^{-1}-\frac{1}{2}C_q^{\mathrm{RI/SMOM}}\lim_{m_R\to 0}\frac{1}{12 m_R^{\overline{\mathrm{MS}}}}\mathrm{Tr}\left[q_\mu \Lambda^{\mu,\overline{\mathrm{MS}}}_{A,R}\gamma_5\right]\Big|_{sym}$$
(massive)

2.) Via pseudo-scalar operator:

$$(C_m^{\mathrm{RI/SMOM}})^{-1}=C_P^{\mathrm{RI/SMOM}}=C_q^{\mathrm{RI/SMOM}}\lim_{m_R\to 0}\frac{1}{12i}\mathrm{Tr}\left[\Lambda^{\overline{\mathrm{MS}}}_{P,R}\gamma_5\right]\Big|_{sym}$$

RI/SMOM result of NLO calculation

- Computation straightforward:

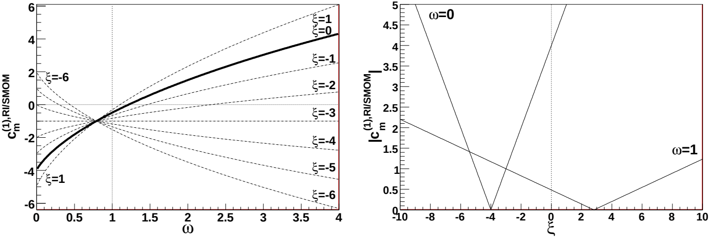


- non-singlet, anti-commuting γ_5
- $C_m^{\text{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi} C_F \left[4 + \xi - (3+\xi)\left(\frac{1}{3}\Psi'(\frac{1}{3}) - \frac{2}{9}\pi^2\right)\right]$
  $= 1 + \frac{\alpha_s}{4\pi} C_F 0.4841391\ldots + \mathcal{O}(\alpha_s^2)$ (Landau gauge)

  [C_F: color factor, ξ: gauge parameter]
- $C_m^{\text{RI,RI'}} = 1 - \frac{\alpha_s}{4\pi} C_F 4 + \mathcal{O}(\alpha_s^2)$ (Landau gauge)
  1-loop coeff. almost a factor 10 smaller compared to RI, RI'
  Size of 1-loop $\sim$ 1.6% compared to 3-loop of RI/MOM $\sim$ 5%
- Both methods 1.), 2.) give same result

RI/SMOM result of NLO calculation

- Different subtraction points and gauge parameter dependence:
  (subtraction "point" $p_1^2 = p_2^2 = -\mu^2$ and $q^2 = -\omega\mu^2$), $C_m = 1 + \frac{\alpha_s}{4\pi} C_F c_m^{(1)}$

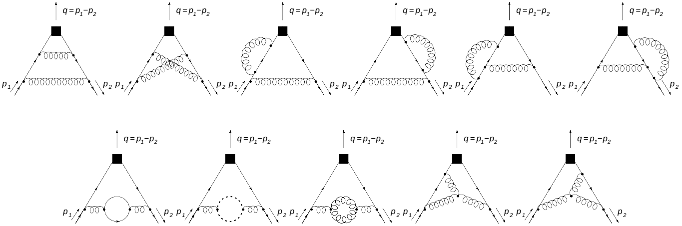


- ...if confirmed at higher orders
  ⇝ significant reduction in the calculated value of the quark mass

The NNLO calculation

RI/SMOM

- Diagrams to the NNLO calculation



Concepts of calculation

Techniques

Integration-by-parts (IBP): K.G. Chetyrkin, F.V. Tkachov

$$0 = \int [d^D\ell_1]\dots[d^D\ell_4] \quad \partial_{(\ell_j)_\mu}\left(\ell_l^\mu\, I_{\alpha\beta}\right) \quad , \qquad j, l = 1, \dots, \text{loops=4}$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \dots\}$

and scalar-product powers $\beta = \{\beta_1, \dots\}$

Laporta-Algorithm: S. Laporta, E. Remiddi

Idea:
- – IBP-identities for explicit numerical values of α, β
- – Introduction of an order among the integrals
- – Solving a linear system of equations

Solve master integrals

Alternative schemes

RI/SMOM

- Compute quantities in different schemes and convert to $\overline{\text{MS}}$
  $\rightsquigarrow$ better control over truncation + simulation uncertainties
- Alternative conditions with different projectors:
  $\lim_{m_R \to 0} \frac{1}{48} \mathrm{Tr}\left[\Lambda^{\mu}_{V,R}(p_1,p_2)\gamma_\mu\right]\Big|_{sym} = 1, \qquad \lim_{m_R \to 0} \frac{1}{48} \mathrm{Tr}\left[\Lambda^{\mu}_{A,R}(p_1,p_2)\gamma_5\gamma_\mu\right]\Big|_{sym} = 1,$
- Conversion factor:

$$C_m^{\text{RI/SMOM}_{\gamma_\mu}} = 1 - \frac{\alpha_s}{4\pi} C_F 1.4841391\ldots$$

- Two-loop anomalous dimensions:

$$\gamma_m^{\text{RI/SMOM}} = \gamma_m^{\overline{\text{MS}}} - \beta\, \partial_{a_s} \ln C_m^{\text{RI/SMOM}}$$

$$\gamma_m^{\text{RI/SMOM}} = -\frac{\alpha_s}{\pi} - \left(\frac{\alpha_s}{\pi}\right)^2 (4.6521 - 0.16579 n_f)$$

In analogy $\gamma_q^{\text{RI/SMOM}} = \gamma_q^{\text{RI'/MOM}}$

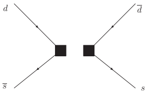
Other application

B_K parameter

- B_K parameter plays important role in flavor phenomenology
- The B_K parameter parameterizes the hadronic matrix element needed for the theory prediction of $K^0 - \overline{\mathrm{K}}^0$ mixing

$$B_K \sim < \overline{K}^0 | Q | K^0 >,$$

Q : 4-quark operator $(V - A)(V - A)$



$\rightsquigarrow$ Renormalization constant Z_{B_K}, RI/MOM $\rightsquigarrow$ RI/SMOM

- Exceptional: 2 p coming in, 2 p going out
- A choice of non-ex. mom.: $p_1^2 = p_2^2 = (p_1 - p_2)^2 = -\mu^2$
- B_K parameter did not suffer from infrared effects
- Informative to have multiple schemes

  $\rightsquigarrow$ for a better assessment of systematic errors
- Matching calculation in progress C.T. Sachrajda, et al.

Summary & Conclusion

- Framework + concepts for renorm. of quark bilinear operators in a MOM scheme with symmetric subtraction point has been discussed
- Renormalization with an exceptional momentum subtraction introduces a large systematic error in light quark mass determinations
- RI/SMOM: PT truncation is smaller at same order in PT than in RI/MOM and less sensitive to infrared effects in the latt. simulation
  - $\rightsquigarrow$ The use of non-exceptional momenta will reduce the systematic error and improve precision of light quark mass determinations from lattice simulations
- Two-loop corrections are work in progress