COULD LEPTONS, QUARKS OR BOTH BE HIGHLY RELATIVISTIC, BOUND STATES OF A MINIMALLY INTERACTING FERMION AND SCALAR?

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MOTIVATION:
COULD LEPTONS, QUARKS OR BOTH BE COMPOSITE?

INDIRECT EXPERIMENTAL EVIDENCE

FAMILIES	EXPLANATION
18 families of elements	Atoms are composite.
SU(3) multiplets of mesons and baryons	Mesons and baryons are composite.
4 families of leptons and quarks	??

DIRECT EXPERIMENTAL EVIDENCE - ALMOST NONEXISTENT

muon g-2: E821 [PRL **92** 1618102 (2004)
arXiv:0809.3085 (2008)
arXiv:0906.5443v1 (2009)]

3.8 σ discrepancy with standard model calculation

CONSTRAINTS ON COMPOSITE MODEL OF LEPTONS, QUARKS OR BOTH

BOUND-STATE SYSTEM MUST BE HIGHLY RELATIVISTIC

$m_e << m_\mu << m_\tau$
$m_u << m_c << m_t$
$m_d << m_s << m_b$

POSSIBLE CONSTITUENTS

MODEL #1: spin-0 boson, spin-1/2 fermion (possible mechanism for existence of only j=1/2 bound states)

MODEL #2: three spin-1/2 fermions

INTERACTION

electromagnetism

POSSIBLE COMPOSITE MODEL OF LEPTONS AND QUARKS

If there are two constituent fermions and two constituent bosons, they can combine in four ways to create four bound states (or four families).

FAMILY	CONSTITUENT FERMION CHARGE	CONSTITUENT BOSON CHARGE	CONSTRAINT
Electron	q_{f1}	Q_{S1}	$q_{f1} + Q_{S1} = -1$
Neutrino	q_{f2}	Q_{S1}	$q_{f2} + Q_{S1} = 0$
Negative quarks	q_{f1}	Q_{S2}	$q_{f1} + Q_{S2} = -\frac{1}{3}$
Positive quarks	q_{f2}	Q_{S2}	$q_{f2} + Q_{S2} = \frac{2}{3}$

There are three independent equations and four unknown charges. Possible charges ≤ 2 that are multiples of $\pm\frac{1}{3}$ are as follows:

POSSIBLE CHARGES OF CONSTITUENT FERMIONS AND BOSONS

Constituent Fermion Charges		Constituent Boson Charges	
q_{f1}	q_{f2}	Q_{S1}	Q_{S2}
1	2	-2	$-\frac{4}{3}$
$\frac{2}{3}$	$\frac{5}{3}$	$-\frac{5}{3}$	-1
$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{2}{3}$
$-\frac{4}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1
$-\frac{5}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
-2	-1	1	$\frac{5}{3}$

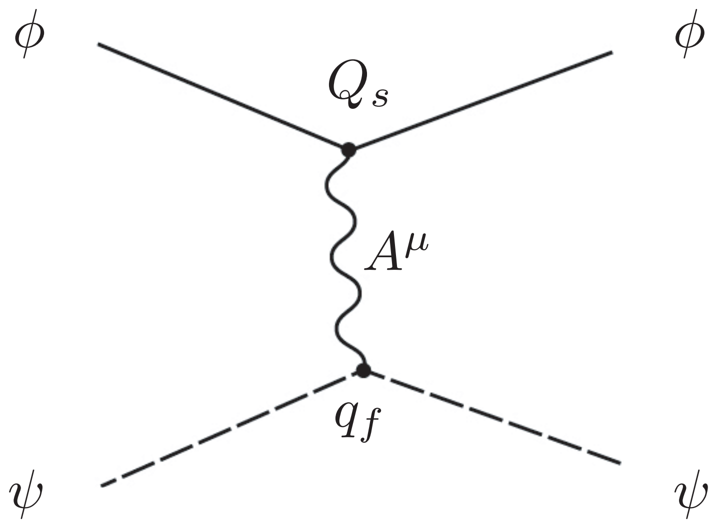
FERMION-SCALAR QED

$$L_{\rm int} = :-q_f A^\mu \bar{\psi}\gamma_\mu \psi + \imath Q_s\, A_\mu \left[\phi\left(\partial^\mu \phi^\dagger\right) - \left(\partial^\mu \phi\right)\phi^\dagger\right]$$
$$+\, Q_s^2 A^\mu A_\mu \phi\, \phi^\dagger :$$

$\psi =$ spin-1/2 constituent field, charge q_f, mass m_f

$\phi =$ spin-0 constituent field, charge Q_s, mass m_s

$A^\mu =$ electromagnetic field



MAJOR DIFFERENCES BETWEEN THE BOUND-STATE SCHRÖDINGER AND BETHE-SALPETER EQUATIONS

SCHRÖDINGER EQUATION

$$H(x)\Psi(x) = E\Psi(x)$$

BETHE-SALPETER EQUATION

$$[(p^0 + \xi E)\gamma_0 + p^i\gamma_i - m_f]\{[p^0 + (\xi - 1)E]^2 - \mathbf{p}^2 - m_s^2)\chi(p)$$
$$= \frac{iq_f Q_s}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4q}{(p-q)^2 + i\varepsilon}\{\gamma^0[p_0 + q_0 + 2(\xi - 1)E] + \gamma^i(p_i + q_i)\})\chi(q)\,.$$

1. Hamiltonian does not exist

2. Eigenvalue equation for the coupling constant $q_f Q_s/(4\pi)$ instead of the energy

MAJOR DIFFERENCES BETWEEN THE BOUND-STATE SCHRÖDINGER AND BETHE-SALPETER EQUATIONS

$$[(p^0 + \xi E)\gamma_0 + p^i\gamma_i - m_f]\{[p^0 + (\xi - 1)E]^2 - \mathbf{p}^2 - m_s^2)\chi(p)$$
$$= \frac{iq_f Q_s}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4 q}{(p-q)^2 + i\varepsilon}\{\gamma^0[p_0 + q_0 + 2(\xi - 1)E] + \gamma^i(p_i + q_i)\})\chi(q)\,.$$

3. Interaction is covariant - no action at a distance

4. Bethe-Salpeter is an integral equation (boundary conditions are incorporated in the equation)

5. Bethe-Salpeter is separable when energy $E = 0$ and is generally nonseparable when $E \neq 0$

METHOD FOR SOLVING THE BOUND-STATE BETHE-SALPETER EQUATION

Solutions are calculated in five steps:

Step #1. Eliminate singularities in the Bethe-Salpeter equation.

The kernel of the equation and the solutions are singular as a result of propagators in Minkowski space.

$$[(p^0 + \xi E)\gamma_0 + p^i\gamma_i - m_f]\{[p^0 + (\xi - 1)E]^2 - \mathbf{p}^2 - m_s^2)\chi(p)$$
$$= \frac{iq_f Q_s}{(2\pi)^4}\int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(p-q)^2 + i\varepsilon}\{\gamma^0[p_0 + q_0 + 2(\xi - 1)E] + \gamma^i(p_i + q_i)\})\chi(q)\,.$$

The singularities can be eliminated by rewriting the equation in Euclidean space [Wick, 1954].

METHOD FOR SOLVING THE BETHE-SALPETER EQUATION

Step #2. Calculate zero-energy solutions.

Writing the separated, zero-energy Bethe-Salpeter equation in the form

$$A(|p|)\chi(|p|) = \frac{q_f Q_s}{4\pi} \int_0^\infty \mathrm{d}|q| \, B(|p|, |q|)\chi(|q|),$$

$|p| =$ magnitude of the Euclidean four-momentum ,

the equation is discretized by expanding the solution in terms of a finite set of basis functions $\mathcal{B}_j(|p|)$:

$$\chi(|p|) = \sum_{j=1}^{N} c_j \mathcal{B}_j(|p|)\,.$$

METHOD FOR SOLVING THE BETHE-SALPETER EQUATION

Substituting the expansion for $\chi(|p|)$ into the Bethe-Salpeter equation,

$$\sum_{j=1}^{N} A(|p|)\mathcal{B}_j(|p|)c_j = \frac{q_f Q_s}{4\pi}\sum_{j=1}^{N}\int_0^\infty \mathrm{d}|q|\, B(|p|,|q|)\mathcal{B}_j(|q|)c_j.$$

Multiplying both sides of the above equation by $\mathcal{B}_i^\dagger(|p|)$ and integrating over $|p|$,

$$\sum_{j=1}^{N}\int_0^\infty \mathrm{d}|p|\mathcal{B}_i^\dagger(|p|)A(|p|)\mathcal{B}_j(|p|)c_j$$

$$= \frac{q_f Q_s}{4\pi}\sum_{j=1}^{N}\int_0^\infty \mathrm{d}|p|\int_0^\infty \mathrm{d}|q|\, \mathcal{B}_i^\dagger(|p|)B(|p|,|q|)\mathcal{B}_j(|q|)c_j.$$

METHOD FOR SOLVING THE BETHE-SALPETER EQUATION

The Bethe-Salpeter equation has been discretized by rewriting it as a generalized $N \times N$ matrix eigenvalue equation of the form,

$$Ac = \frac{q_f Q_s}{4\pi}\, Bc,$$

where

$$A_{i\,j} = \int_0^\infty \mathrm{d}|p| \mathcal{B}_i^\dagger(|p|) A(|p|) \mathcal{B}_j(|p|)\,,$$

$$B_{i\,j} = \int_0^\infty \mathrm{d}|p| \int_0^\infty \mathrm{d}|q|\, \mathcal{B}_i^\dagger(|p|) B(|p|, |q|) \mathcal{B}_j(|q|)\,.$$

METHOD FOR SOLVING THE BETHE-SALPETER EQUATION

Discretized Bethe-Salpeter Equation: $$Ac = \frac{q_f Q_s}{4\pi} Bc,$$

The eigenvalues of a generalized matrix eigenvalue equation are real if

(1) A and B are Hermitian.

(2) A, B or both are positive definite.

Both of the above conditions typically cannot be satisfied so there is usually no way to force all eigenvalues (coupling constants) to be real.

Even if an eigenvalue is real, the series expansion for the solution may not converge, so a solution is not actually obtained.

METHOD FOR SOLVING THE BETHE-SALPETER EQUATION

Solutions with real eigenvalues are almost always obtained when the basis functions $\mathcal{B}_i(|p|)$ (almost) obey the boundary conditions.

Step #3. Solutions with nonzero energy are expanded in terms of a set of basis functions.

In the zero-energy limit the expansion for finite-energy solutions must become the expansion for zero-energy solutions.

Step #4. The Bethe-Salpeter equation is rotationally invariant in three-dimensional space, allowing the separation of two angular variables.

Step #5. The partially separated Bethe-Salpeter equation is discretized and solved numerically by converting it into a generalized matrix eigenvalue equation for the coupling constant.

In the zero-energy limit the discretized equation for finite-energy solutions must become the the discretized equation for zero-energy solutions.

DO STRONGLY BOUND SOLUTIONS EXIST WHEN $\frac{q_f Q_s}{4\pi} \sim \alpha$?

Typically strongly bound solutions of the Bethe-Salpeter equation exist only when the coupling constant is on the order of or greater than unity.

$$[(p^0 + \xi E)\gamma_0 + p^i\gamma_i - m_f]\{[p^0 + (\xi - 1)E]^2 - \mathbf{p}^2 - m_s^2)\chi(p)$$
$$= \frac{iq_f Q_s}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(p-q)^2 + i\varepsilon} \{\gamma^0[p_0 + q_0 + 2(\xi - 1)E] + \gamma^i(p_i + q_i)\})\chi(q)\,.$$

If solutions exist for $\frac{q_f Q_s}{4\pi} \sim \alpha$, at least one integral must be very large.

BOUNDARY CONDITIONS THAT COULD YIELD STRONGLY BOUND STATES WHEN $\frac{q_f Q_s}{4\pi} \sim \alpha$

$$F(|p|) \xrightarrow[|p|\to\infty]{} F_\infty \, |p|^{-(k_1+\frac{7}{2})} \,,$$

$$G(|p|) \xrightarrow[|p|\to\infty]{} G_\infty \, |p|^{-(k_1+\frac{5}{2}+\epsilon_g)}$$

$k_1 = j, j+1, \ldots,$ $\quad 0 < \epsilon_g < 2$ if $k_1 = \frac{1}{2}$ $\quad 0 \le \epsilon_g < 2$ if $k_1 \ge \frac{3}{2}$

At large momenta, the zero-energy Bethe-Salpeter equation decouples and takes the form

$$G_\infty \sim -\frac{q_f Q_s}{4\pi} \int_{\mathrm{constant}}^{\infty} \frac{\mathrm{d}|q|}{|q|^{2k_1+\epsilon_g}} G_\infty$$

$$\frac{q_f Q_s}{4\pi} \sim -(2k_1 - 1 + \epsilon_g)$$

All strongly bound states of a minimally interacting spin-1/2 fermion and spin-0 boson must have spin-1/2!

SUMMARY

- It might be possible to describe leptons, quarks or both as highly relativistic bound states of a spin-0 and spin-1/2 constituent bound by minimal electrodynamics.

- Strongly bound states with coupling constants on the order of the electromagnetic fine structure constant α are allowed by the boundary conditions and likely exist.

- All such strongly bound states would have spin-1/2, as do the leptons and quarks.