Solitons in AdS space

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Introduction

The objective of this study was to study solitons (kinks) in scalar field theories in two-dimensional Anti-de-Sitter space. Such solitons have been studied extensively in Minkowski space. Some concrete goals of the study were:

- Construction of simple scalar field theories in AdS_{1+1} that feature analytic soliton solutions.
- Calculation of the classical soliton mass.
- Study the relation between the soliton's mass, BPS bound, and topological charge
- Determination of the soliton excitation spectrum.
- Calculation of the one-loop quantum corrections to the soliton's mass

AdS space

AdS_{1+1} space can be viewed as a $SO(2,1)$ invariant hyperboloidal hypersurface

$$(X^0)^2 + (X^1)^2 - (X^2)^2 = m^2$$

embedded in a three-dimensional pseudo-Euclidean space with invariant interval

$$ds^2 = (dX^0)^2 + (dX^1)^2 - (dX^2)^2.$$

Here m parameterizes the inverse length scale in AdS_{1+1} space. Global coordinates can be defined as

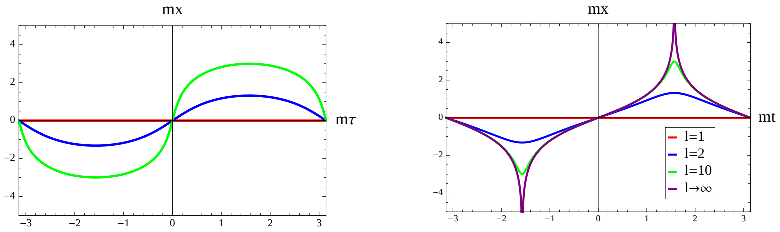
$$\begin{aligned} X^0 &= m\cos(mt)\cosh(mx) \\ X^1 &= m\sin(mt)\cosh(mx) \\ X^2 &= m\sinh(mx). \end{aligned}$$

The resulting induced metric in these coordinates is

$$ds^2 = \cosh^2(mx)dt^2 - dx^2.$$

- The hyperboloid surface is covered once by $mt \in [-\pi, \pi]$ and $mx \in [-\infty, \infty]$. However, in order to avoid closed time curves the covering space is considered by removing the restriction on the range of the time coordinate.
- A boundary is located at $x \to \pm\infty$.
- In order to have well defined time evolution, consistent boundary conditions must be specified apart from initial conditions.

Projections of geodesic trajectories



In terms of the proper time:

$$\sinh(mx) \quad = \quad \pm\sqrt{l^2-1}\,\sin(m(\tau-\tau_0)).$$

In terms of the coordinate time:

$$\sinh(mx) \quad = \quad \pm\frac{\sqrt{l^2-1}\,\sin[m(t-t_0)]}{\sqrt{l^2\cos^2[m(t-t_0)]+\sin^2[m(t-t_0)]}}.$$

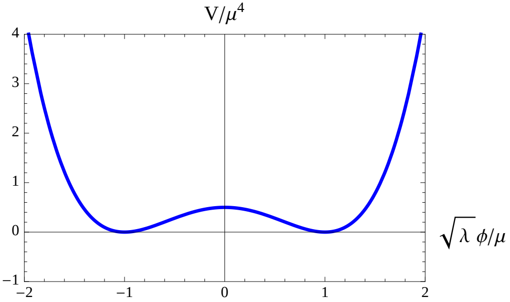
Here l and t_0 are integration constants. Null geodesics are obtained for $l\to\infty$.

Scalar field theory in AdS_{1+1}

- Two degenerate discrete vacua need to occur in order for kink solutions to exist.
- This can be incorporated in a model by realizing a spontaneously broken Z_2 symmetry.
- For simplicity, and to keep the connection with well studied solitons in Minkoski space, we consider a "phi to the fourth" scalar potential with a negative mass term.

The $SO(2,1)$ invariant Lagrangian density takes the form

$$\mathcal{L} = \frac{1}{\cosh^2(mx)}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}(-\mu^2 + \lambda\phi^2)^2.$$



The two degenerate vacua at $\phi = \pm \frac{\mu}{\sqrt{\lambda}}$ each spontaneously break the Z_2 symmetry, but leave the SO(2,1) symmetry unbroken.

The Euler-Lagrange equation of motion is

$$\partial_\mu \left(\frac{\partial(\mathcal{L}\sqrt{g})}{\partial(\partial_\mu \phi)} \right) - \frac{\partial(\mathcal{L}\sqrt{g})}{\partial \phi} = 0.$$

It is convenient to introduce dimensionless variables

$$\begin{aligned} s &\equiv xm, \quad \tau \equiv tm \\ \sigma &\equiv \sqrt{\lambda}\frac{\phi}{\mu}, \quad \alpha \equiv \frac{2\lambda\mu^2}{m^2}. \end{aligned}$$

The equation of motion then takes the form

$$\frac{1}{\cosh^2(s)} \frac{\partial^2 \sigma}{\partial \tau^2} - \frac{\partial^2 \sigma}{\partial s^2} - \tanh(s) \frac{\partial \sigma}{\partial s} + \alpha\sigma(-1 + \sigma^2) = 0.$$

Excitation spectrum in the trivial sector

In order to find the excitation spectrum, the scalar field is expanded around one of the equivalent degenerate vacua:

$$\sigma(s,\tau) \quad = \quad 1 + \eta(s,\tau).$$

For small fluctuations only linear terms need to be considered in the equation of motion:

$$\frac{1}{\cosh^2(s)}\frac{\partial^2\eta}{\partial\tau^2} - \frac{\partial^2\eta}{\partial s^2} - \tanh(s)\frac{\partial\eta}{\partial s} + 2\alpha\eta = 0.$$

This equation is solved by separation of variables:

$$\eta(s,\tau) \quad = \quad e^{i\hat{\omega}\tau}X(s).$$

The equation describing the space dependent part of the normal modes reads

$$-\cosh^2(s)\frac{d^2X}{ds^2} - \sinh(s)\cosh(s)\frac{dX}{ds} + 2\alpha\cosh^2(s)X \quad = \quad \hat{\omega}^2X.$$

It has the form of a time-independent Schrödinger equation, and we can therefore use familiar tools to find the normalizable eigenfunctions and eigenvalues.

Supersymmetry of the equivalent quantum mechanics problem

The normal mode equation can be suggestively written as

$$H_1 X \quad = \quad (\hat{\omega}^2 - g_2^2) X = E^{(1)} X.$$

This Hamiltonian can be factorized:

$$H_1 \quad = \quad A^\dagger A,$$

with the differential operators A and $A^\dagger$ defined by

$$A^\dagger \quad = \quad -\cosh(s)\frac{d}{ds} + g_2 \sinh(s), \qquad A = +\cosh(s)\frac{d}{ds} + g_2 \sinh(s),$$

while the parameter g_2 takes the value

$$g_2 \quad = \quad \frac{1}{2} + \sqrt{\frac{1}{4} + 2\alpha}.$$

The Hamiltonian H_1 forms a supersymmetric system together with $H_2 = AA^\dagger$.

The ground state energy of Hamiltonian H_1 vanishes, and the ground state wavefunction of H_1 is annihilated by the operator A:

$$E_0^{(1)} = 0 \qquad\qquad AX_0^{(1)} = 0$$

Moreover, supersymmetry relates the eigenfunctions and eigenvalues of the Hamiltonians H_1 and H_2:

$$E_n^{(2)} = E_{n+1}^{(1)} \qquad\qquad \begin{aligned} X_n^{(2)} &= AX_{n+1}^{(1)} \\ X_{n+1}^{(1)} &= A^\dagger X_n^{(2)} \end{aligned}$$

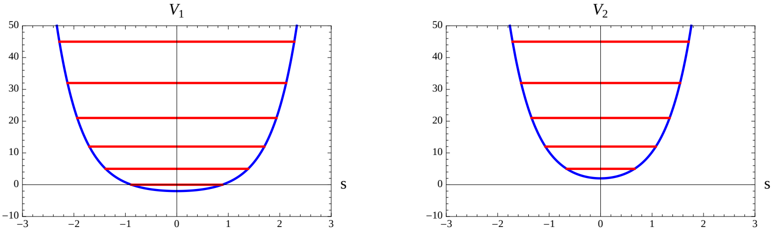
Shape invariance of the equivalent quantum mechanics problem

The Hamiltonians H_1 and H_2 take very similar forms:

$$\begin{aligned} H_1 &= A^\dagger A \\ &= \cosh^2(s)\left[-\frac{d^2}{ds^2} - \tanh(s)\frac{d}{ds}\right] + (g_2^2 - g_2)\cosh^2(s) - g_2^2 \\ H_2 &= AA^\dagger \\ &= \cosh^2(s)\left[-\frac{d^2}{ds^2} - \tanh(s)\frac{dX}{ds}\right] + (g_2^2 + g_2)\cosh^2(s) - g_2^2. \end{aligned}$$

In fact, the two Hamiltonians are related by shape invariance, a combination of a shift in the parameter g_2 and an additive constant:

$$H_2(s; g_2) = H_1(s; g_2 + 1) + 2g_2 + 1.$$



The spectra of H_1 and H_2 are now algebraically determined by the combination of supersymmetry and shape invariance:

$$
\begin{aligned}
E_0^{(1)}(g_1) &= 0 & & \\
E_1^{(1)}(g_2) &= 2g_2+1 & \qquad E_0^{(2)}(g_2) &= 2g_2+1 \\
E_2^{(1)}(g_2) &= 4g_2+4 & \qquad E_1^{(2)}(g_2) &= 4g_2+4 \\
& \dots & & \dots
\end{aligned}
$$

The general expression for the energy spectrum of H_1 is:

$$E_n^{(1)} = 2ng_2 + n^2$$

The excitation spectrum of the scalar field theory in the trivial vacuum is thus

$$\hat{\omega}_n = \sqrt{E_n^{(1)} + g_2^2} = n + g_2 = n + \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + 2\alpha}.$$

This equally spaced spectrum is consistent with the unbroken $SO(2,1)$ symmetry. The physical “meson ”mass is defined as the lowest frequency:

$$\frac{m_{\mathrm{phys}}}{m} = \hat{\omega}_0 = \left[\frac{1}{2} + \sqrt{\frac{1}{4} + 2\alpha}\right].$$

For the particular value $\alpha = 1$ this relation becomes:

$$m_{\mathrm{phys}} = 2m.$$

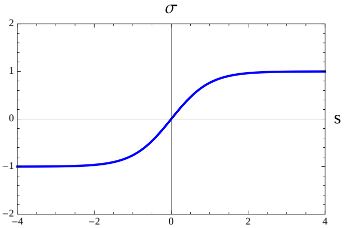
Soliton solution

A soliton is a static solution to the equation of motion

$$\frac{1}{\cosh^2(s)}\frac{\partial^2\sigma}{\partial\tau^2} - \frac{\partial^2\sigma}{\partial s^2} - \tanh(s)\frac{\partial\sigma}{\partial s} + \alpha\sigma(-1+\sigma^2) = 0$$

that interpolates between the two vacua at $\sigma_0 = \pm 1$. An analytic solution of this type was found for $\alpha = 1$:

$$\sigma_{\text{sol}}(s) \quad = \quad \tanh(s).$$



In what follows we will further consider this particular value of α.

Classical soliton mass

The covariant energy momentum tensor for the model is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}.$$

The classical energy functional is

$$E[\phi] = \int_{-\infty}^{\infty} dx \sqrt{-g} \mathcal{H} = \int_{-\infty}^{\infty} dx \sqrt{-g} \, T^0_{\ 0} \, .$$

The classical mass of a static soliton is

$$M_{sol} = \int_{-\infty}^{\infty} dx \sqrt{-g} \left[\mathcal{H}_{\rm sol} - \mathcal{H}_0\right] = \int_{-\infty}^{\infty} \cosh(mx) dx \left[\frac{1}{2} (\frac{d\phi_{\rm sol}}{dx})^2 + V(\phi_{\rm sol})\right] .$$

The resulting soliton mass in the special case $\alpha = 1$ is determined to be

$$M_{\rm sol} = \frac{3\pi}{16} \frac{m^3}{\lambda^2} .$$

Soliton excitation spectrum

In order to find the soliton excitation spectrum the scalar field is expanded around the soliton solution:

$$\sigma(s,\tau) \quad = \quad \tanh(s) + \eta(s,\tau).$$

The linearized equation of motion then takes the form

$$\frac{1}{\cosh^2(s)}\frac{\partial^2\eta}{\partial\tau^2} - \frac{\partial^2\eta}{\partial s^2} - \tanh(s)\frac{\partial\eta}{\partial s} + 3\tanh^2(s)\eta - \eta \quad = \quad 0.$$

This equation is again solved by separation of variables:

$$\eta(s,\tau) \quad = \quad e^{i\hat{\omega}\tau}X(s).$$

The space dependent part of the normal mode equation reads:

$$-\cosh^2(s)\frac{d^2X}{ds^2} - \sinh(s)\cosh(s)\frac{dX}{ds} + 2\cosh^2(s)X - 3X \quad = \quad \hat{\omega}^2 X.$$

This looks very similar to the normal mode equation in the trivial vacuum for $\alpha = 1$ which was discussed earlier. The normalizable modes and the eigenvalues can again be found by employing a combination of supersymmetry and shape invariance.

It is curious that for $\alpha = 1$ the analogous time-independent Schrödinger equation takes the form

$$H_1 X \quad = \quad (\hat{\omega}^2 - 1)X = E^{(1)}X,$$

with the Hamiltonian H_1 exactly as before in the trivial sector, and $g_2 = 2$, its value for $\alpha = 1$. Just the relation between the energy eigenvalues and the excitation frequencies is modified.

In summary, for $\alpha = 1$ the normal mode functions in the trivial and one-soliton sectors are identical, but the frequency spectra are distinct!
The excitation spectrum in the trivial sector is

$$\hat{\omega}_n = n + 2, \qquad n \in \mathbb{N},$$

while in the one-soliton sector

$$\hat{\omega}_n^{\rm sol} = \sqrt{(n+2)^2 - 3}, \qquad n \in \mathbb{N}.$$

Note that the lowest frequency in the one-soliton sector is $\omega_0^{\rm sol} = m$. This mode is obtained by acting on the soliton solution with a spontaneously broken $SO(2,1)$ symmetry generator. It is equivalent to the zero mode (Nambu-Goldstone) of solitons in Minkowski space.

The normal mode functions for $\alpha = 1$ can be written in terms of the Jacobi polynomials as

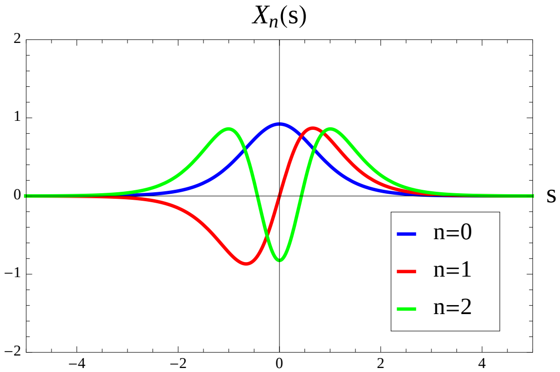
$$X_n(s) = B_n \frac{1}{\cosh^2(s)} P_n^{(3/2,3/2)}(\tanh(s)).$$

The normalization constant is given by

$$B_n = \sqrt{\frac{(n+2)}{8} \frac{\Gamma(n+1)\Gamma(n+4)}{\Gamma(n+5/2)^2}}.$$

The normal mode functions then satisfy the orthonormality condition

$$\int_{-\infty}^{\infty} \frac{1}{\cosh s} X_n^\dagger(s) X_m(s) ds = \delta_{nm}.$$



The first three normal mode functions for $\alpha = 1$. Note the similarity of the both the spectrum and mode functions with the energy spectrum and energy eigenfunctions of the regular quantum mechanical harmonic oscillator.

Quantization, regularization, and renormalization

Canonical quantization:

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\frac{\partial\phi}{\partial x}^2 + V(\phi).$$

Expand the field and the canonical momentum in terms of the normal modes:

$$\phi = \sum_{n=0}^{N} \frac{a_n}{\sqrt{2\omega_n}} X_n(x) e^{i\omega_n t} + \frac{a_n^\dagger}{\sqrt{2\omega_n}} X_n^\dagger(x) e^{-i\omega_n t}.$$

$$\pi = \sum_{n=0}^{N} -i a_n \sqrt{\frac{\omega_n}{2}} \frac{X_n(x)}{\cosh(mx)} e^{i\omega_n t} + i a_n^\dagger \sqrt{\frac{\omega_n}{2}} \frac{X_n^\dagger(x)}{\cosh(mx)} e^{-i\omega_n t}.$$

Impose the quantization condition:

$$\left[a_n, a_m^\dagger\right] = \delta_{nm}.$$

The normalization condition is implied by the normal ordering procedure:

$$:H: \quad = \quad H - \int_{-\infty}^{\infty} \cosh(mx) dx \left[\frac{1}{2} \delta m^2 \phi^2 + D \right].$$

The theory is regulated by cutting off the mode sums at large mode number N. The mass counter term is logarithmically divergent:

$$\delta m^2 \quad = \quad 6\lambda^2 \sum_{i=0}^{N} \frac{1}{2\omega_i} X_i(x) X_i^\dagger(x) = \frac{3}{\pi} \lambda^2 \left[\ln(N) + \gamma - 1 + \ln(2) \right].$$

The vacuum energy counter term has a quadratic divergence:

$$D \quad = \quad \sum_{i=0}^{N} \frac{1}{2} \omega_i = \frac{1}{4} m(N+4)(N+1).$$

Quantum corrections to the soliton mass

To one loop order there are two contributions to the quantum corrections of the soliton mass.
Due to the mass renormalization:

$$\begin{aligned}\Delta M_1 &= -\int_{-\infty}^{\infty} cosh(mx)dx\frac{1}{2}\delta m^2\left[\phi_{\rm sol}^2-\phi_0^2\right] \\ &= \frac{3}{2}m\left[\ln(N)+\gamma-1+\ln(2)\right].\end{aligned}$$

Due to the vacuum energy renormalization:

$$\begin{aligned}\Delta M_2 &= \sum_{i=0}^{N}\omega_i^{\rm sol}-D \\ &= -\frac{3}{2}m\left[\ln(N)+\gamma-1\right]+\frac{1}{2}mC_S\end{aligned}$$

Here Schroeder's number is defined as

$$C_S = \sum_{i=0}^{\infty}\left[\sqrt{(i+2)^2-3}-(i+2)+\frac{3}{2}\frac{1}{(i+2)}\right]\approx -0.3485.$$

The divergent parts of the two contributions cancel against eachother. The finite, physical mass of the soliton including its one-loop quantum corrections is thus

$$M_{\rm sol} \quad = \quad M_{\rm clas} + M_1 + M_2 = \frac{3\pi}{16}\frac{m^3}{\lambda^2} + \frac{1}{2}m\left[3\ln 2 + C_S\right].$$

Discussion

- A zero mode is generically obtained in the excitation spectrum of solitons in Minkowski space due to the spontaneously broken translation symmetry. In Anti-de-Sitter space, the corresponding spontaneously broken symmetry generator gives generically rise to a mode with $\omega = m$.
- It turns out that the supersymmetric and shape invariant quantum mechanical models we encountered are already known. They are equivalent by a coordinate transformation to the Scarf I potentials.
- The soliton solution in Anti-de-Sitter space was obtained as a static solution to the second order Euler-Lagrange equation. Due to the explicit coordinate dependence of the Hamiltonian density it is not possible to derive a BPS bound and a first order BPS equation as in Minkowski space. The mass of the soliton in Anti-de-Sitter space therefore seems not to be a topological charge.

Conclusions

- An analytic soliton (kink) solution in a scalar field theory in AdS_{1+1} space was constructed. Its classical mass was determined.
- The excitation spectrum both in the trivial sector and the soliton sector were determined analytically by using a combination of supersymmetry and shape invariance.
- The spectra were then used to calculate the quantum corrections to the soliton mass to one-loop order in the semi-classical approximation.