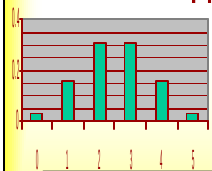
Statistics for HEP

Roger Barlow
Manchester University

Lecture 2: Distributions

The Binomial



n trials r successes
Individual success probability p

$$P(r;n,p)=\frac{n!}{r!(n-r)!}p^r(1-p)^{n-r}$$

Mean

$\mu=\langle r\rangle=\Sigma rP(r)$

$= np$

Variance

$V\equiv\sigma^2=\langle(r-\mu)^2\rangle=\langle r^2\rangle-\langle r\rangle^2$

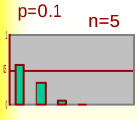
$=np(1-p)$

$1-p \equiv \bar{p} \equiv q$

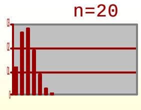
Met with in Efficiency/Acceptance calculations

Binomial Examples

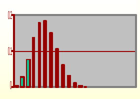
p=0.1 n=5



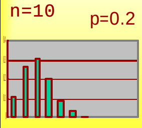
n=20



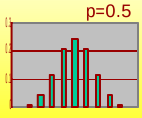
n=50



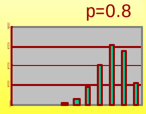
n=10 p=0.2



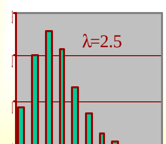
p=0.5



p=0.8



Poisson



'Events in a continuum'
e.g. Geiger Counter clicks
Mean rate λ in time interval
Gives number of events in data

$$P(r;\lambda)=e^{-\lambda}\frac{\lambda^r}{r!}$$

Mean

$\mu=\langle r\rangle=\Sigma rP(r)$

$=\lambda$

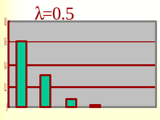
Variance

$V\equiv\sigma^2=\langle(r-\mu)^2\rangle=\langle r^2\rangle-\langle r\rangle^2$

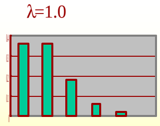
$=\lambda$

Poisson Examples

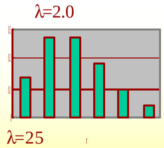
λ=0.5



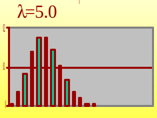
λ=1.0



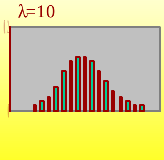
λ=2.0



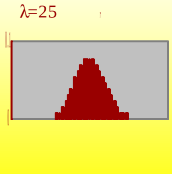
λ=5.0



λ=10



λ=25



Binomial and Poisson

From an exam paper

A student is standing by the road, hoping to hitch a lift. Cars pass according to a Poisson distribution with a mean frequency of 1 per minute. The probability of an individual car giving a lift is 1%. Calculate the probability that the student is still waiting for a lift

(a) After 60 cars have passed

(b) After 1 hour

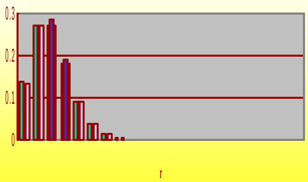
a) 0.99^{60}=0.5472 b) $e^{-0.6}$=0.5488

Poisson as approximate binomial

Poisson mean 2

Binomial: p=0.1, 20 tries
Binomial: p=0.01, 200 tries

Use: MC simulation (Binomial) of Real data (Poisson)



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Two Poissons

2 Poisson sources, means λ_1 and λ_2

Combine samples

e.g. leptonic and hadronic decays of W
Forward and backward muon pairs
Tracks that trigger and tracks that don't

What you get is a *Convolution*

$$P(r) = \sum P(r'; \lambda_1)\, P(r-r'; \lambda_2)$$

Turns out this is also a Poisson with mean $\lambda_1+\lambda_2$

Avoids lots of worry

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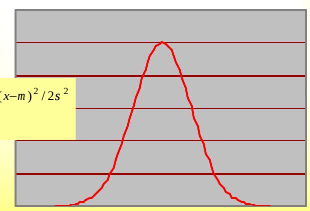
The Gaussian

Probability Density

$$P(x; m, s) = \frac{1}{s\sqrt{2p}} e^{-(x-m)^2/2s^2}$$



Mean

$\mu = \langle x \rangle = \int x P(x)\, dx$

$= \mu$

Variance

$V \equiv \sigma^2 = \langle (x-\mu)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$

$= \sigma^2$

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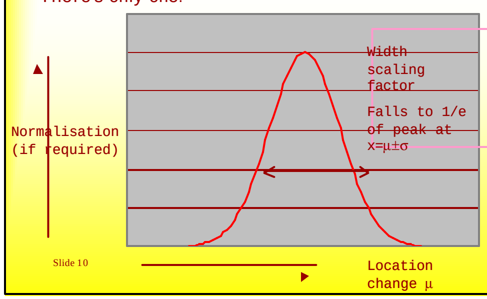
Different Gaussians

There's only one!



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Probability Contents

68.27% within 1σ
95.45% within 2σ
99.73% within 3σ

90% within $1.645\,\sigma$
95% within $1.960\,\sigma$
99% within $2.576\,\sigma$
99.9% within 3.290σ

These numbers apply to Gaussians and only Gaussians

Other distributions have equivalent values which you could use of you wanted

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Central Limit Theorem

Or: why is the Gaussian Normal?

If a Variable x is produced by the convolution of variables $x_1, x_2 .. x_N$

I) $\langle x \rangle = \mu_1 + \mu_2 + .. \mu_N$

II) $V(x) = V_1 + V_2 + .. V_N$

III) P(x) becomes Gaussian for large N

There were hints in the Binomial and Poisson examples

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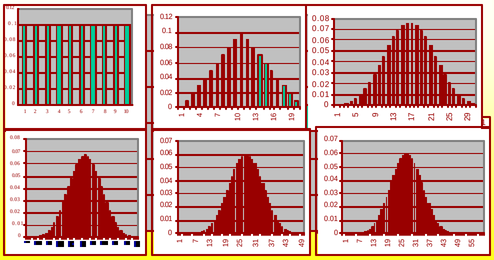
CLT demonstration

Convolute Uniform distribution with itself



CLT Proof (I) Characteristic functions

Given P(x) consider

$$\left\langle e^{ikx}\right\rangle = \int e^{ikx}P(x)dx = \tilde{P}(k) = \Phi(k)$$

The Characteristic Function

For convolutions, CFs multiply

If $f(x) = g(x)\otimes h(x)$ then $\tilde{f}(k) = \tilde{g}(k)\tilde{h}(k)$

Logs of CFs Add

CLT proof (2) Cumulants

CF is a power series in k

$\langle 1\rangle+\langle ikx\rangle+\langle (ikx)^2/2!\rangle+\langle (ikx)^3/3!\rangle+\ldots$

$1+ik\langle x\rangle-k^2\langle x^2\rangle/2!-ik^3\langle x^3\rangle/3!+\ldots$

Ln CF can then be expanded as a series

$ikK_1 + (ik)^2K_2/2! + (ik)^3K_3/3!\ldots$

K_r : the "semi-invariant cumulants of Thiele"

Total power of x^r

If $x\to x+a$ then only $K_1\to K_1+a$

If $x\to bx$ then each $K_r\to b^rK_r$

CLT proof (3)

- The FT of a Gaussian is a Gaussian

$$e^{-x^2/2\sigma^2} \to e^{-k^2\sigma^2/2}$$

Taking logs gives power series up to k^2

$K_r=0$ for $r>2$ defines a Gaussian

- Selfconvolute anything n times: $K_r'=n\,K_r$

Need to normalise – divide by n

$K_r''=n^{-r}K_r'=n^{1-r}K_r$

- Higher Cumulants die away faster

If the distributions are not identical but similar the same argument applies

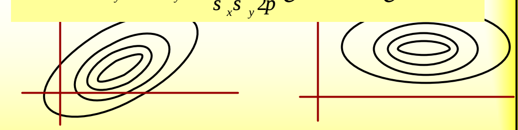
CLT in real life

Examples
- Height
- Simple Measurements
- Student final marks

Counterexamples
- Weight
- Wealth
- Student entry grades

Multidimensional Gaussian

$$P(x,y;\mu_x,\mu_y,\sigma_x,\sigma_y) = \frac{1}{\sigma_x\sigma_y 2\pi}e^{-(x-\mu_x)^2/2\sigma_x^2}e^{-(y-\mu_y)^2/2\sigma_y^2}$$



$$P(x,y;\mu_x,\mu_y,\sigma_x,\sigma_y,\rho) = \frac{1}{\sigma_x\sigma_y 2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left((x-\mu_x)^2/\sigma_x^2+(y-\mu_y)^2/\sigma_y^2-2\rho(x-\mu_x)(y-\mu_y)/\sigma_x\sigma_y\right)}$$

Chi squared

$$c^2 = \sum_{i=1}^{n} \left(\frac{x_i - m_i}{s_i} \right)^2$$

Sum of squared discrepancies, scaled by expected error

Integrate all but 1-D of multi-D Gaussian

$$P(c^2;n) = \frac{2^{-n/2}}{\Gamma(n/2)} c^{n-2} e^{-c^2/2}$$

Mean n

Variance 2n

CLT slow to operate

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Generating Distributions

Given `int rand()` in `stdlib.h`

```
float Random()
                    {return ((float)rand())/RAND_MAX;}
float uniform(float lo, float hi)
                          {return lo+Random()*(hi-lo);}
float life(float tau)
                           {return -tau*log(Random());}
float ugauss()        // really crude. Do not use
     {float x=0; for(int i=0;i<12;i++) x+=Random();
                                         return x-6;}
float gauss(float mu, float sigma)
                            {return mu+sigma*ugauss();}
```

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A better Gaussian Generator

```
float ugauss(){
   static bool igot=false;
   static float got;
   if(igot){igot=false; return got;}
   float phi=uniform(0.0F,2*M_PI);
   float r=life(1.0f);
   igot=true;
   got=r*cos(phi);
   return r*sin(phi);}
```

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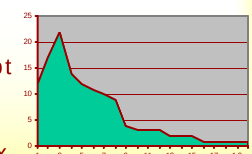
More complicated functions



Find P_0=max[P(x)].

Overestimate if in doubt

Repeat :

  Repeat:

    Generate random x

    Find P(x)

    Generate random P in range 0-P_0

  till P(x)>P

till you have enough data

Slide 22 If necessary make x non-random and compensate

Other distributions

Uniform(top hat)

σ=width/$\sqrt{12}$

Breit Wigner (Cauchy)

  Has no variance – useful for wide tails

Landau

  Has no variance or mean

  Not given by $e^{e^{-l}}$ .Use CERNLIB

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Functions you need to know

- Gaussian/Chi squared
- Poisson
- Binomial
- Everything else

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