Statistics for HEP

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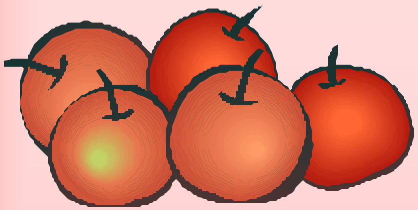
Lecture 4: Confidence Intervals

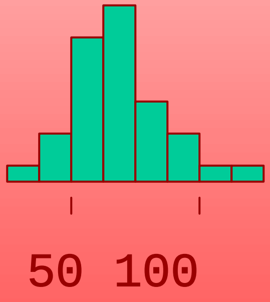
The Straightforward Example



Apples of different weights

Need to describe the distribution

μ = 68g σ = 17 g



All weights between 24 and 167 g (Tolerance)

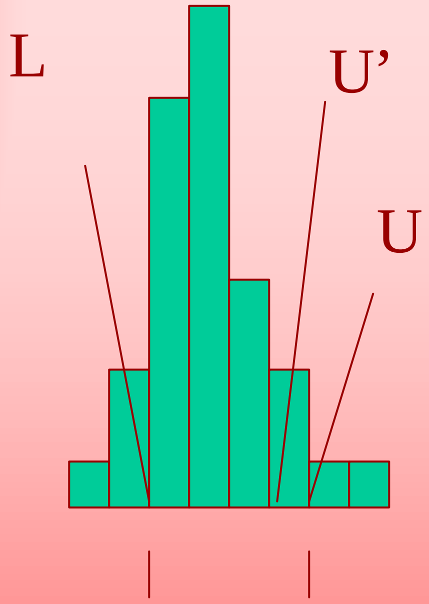
90% lie between 50 and 100 g

94% are less than 100 g

96% are more than 50 g

} Confidence level statements

Confidence Levels



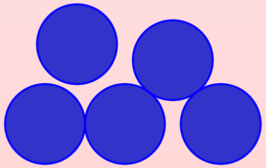
- Can quote at any level

  (68%, 95%, 99%…)
- Upper or lower or twosided

  (x<U   x<L   L<x<U)
- Two-sided has further choice

  (central, shortest…)

The Frequentist Twist



Particles of the same weight

Distribution spread by measurement errors

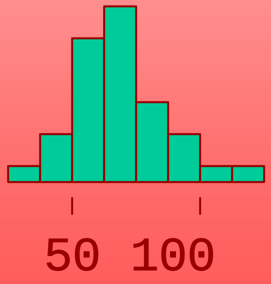
What can we say about M?

$\mu = 68$ $\sigma = 17$

“M<90” or “M>55” or “60<M<85” @90% CL

These are each always true or always false

Solution: Refer to ensemble of statements



Frequentist CL in detail

You have a meter: no bias,

Gaussian error 0.1.

For a value X_T it gives a value X_M according to a Gaussian Distribution

X_M is within 0.1 of X_T 68% of the time

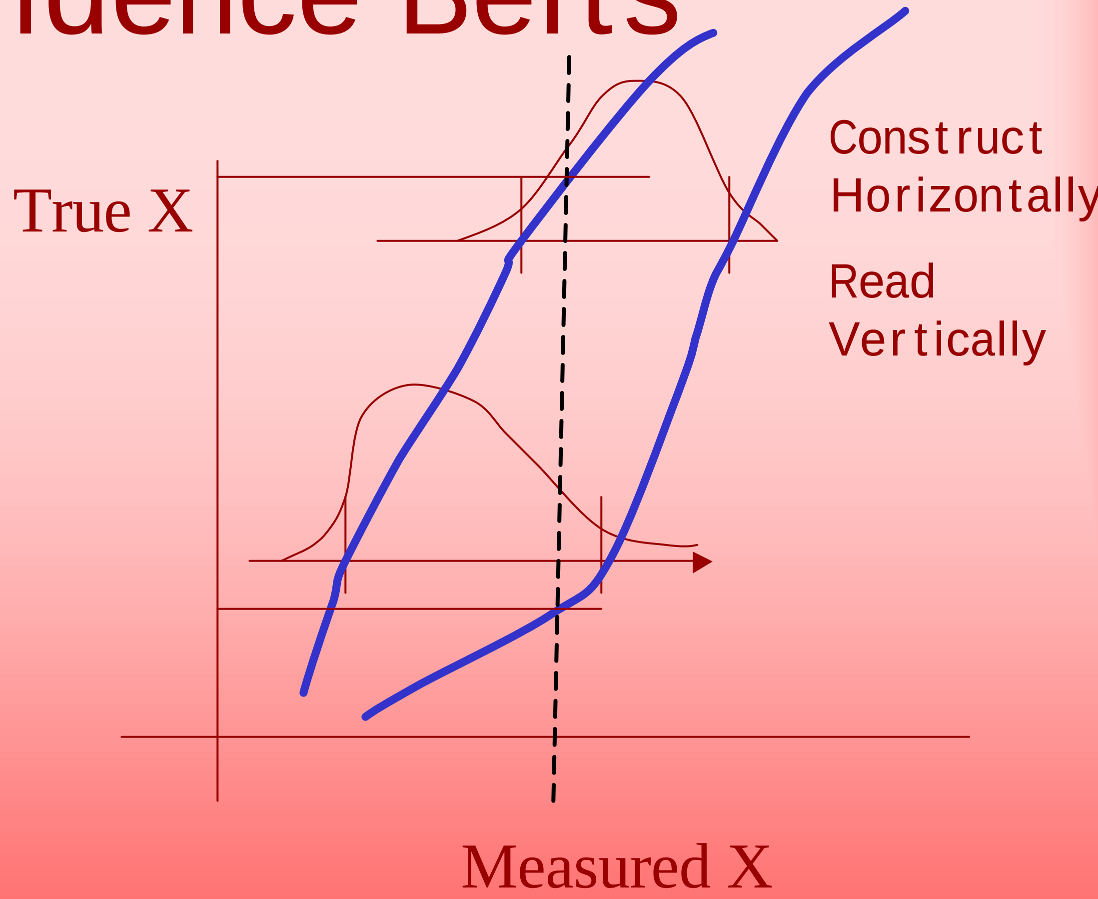
X_T is within 0.1 of X_M 68% of the time

Can state $X_M - 0.1 < X_T < X_M + 0.1$ @68% CL

Confidence Belts

For more complicated distributions it isn't quite so easy

But the principle is the same



Coverage

“L<x<U” @ 95% confidence

(or “x>L” or “x<U”)

This statement belongs to an ensemble of similar statements of which at least* 95% are true

95% is the *coverage*

This is a statement about U and L, not about x.

*Maybe more. Overcoverage EG composite hypotheses. Meter with resolution ≤ 0.1

Think about: the difference between a 90% upper limit and the upper limit of a 90% central interval.

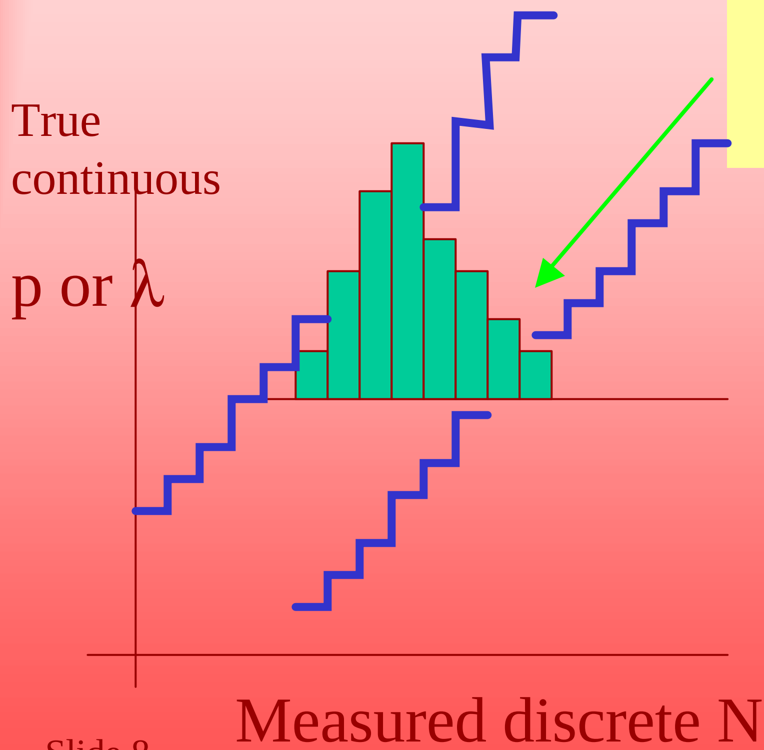
Discrete Distributions

CL belt edges become steps

May be unable to select (say) 5% region

Play safe.

Gives overcoverage



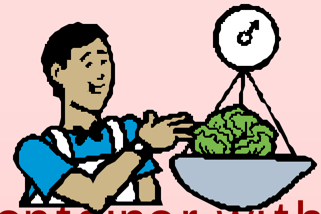
Binomial: see tables

Poisson

	Upper			Lower		
	90	95	99	90	95	99
0	2.3	3	4.61			
1	3.89	4.74	6.64	0.11	0.05	0.01
2	5.32	6.3	8.41	0.53	0.36	0.15
3	6.68	7.75	10.05	1.1	0.82	0.44

Given 2 events, if the true mean is 6.3 (or more) then the chance of getting a fluctuation this low (or lower) is only 5% (or less)

Problems for Frequentists



Weigh object+container with some Gaussian precision

Get reading R

$R-\sigma < M+C < R+\sigma$ @68%

$R-C-\sigma < M < R-C+\sigma$ @68%

E.g. C=50, R=141, σ=10

81<M<101 @68%

E.g. C=50, R=55, σ=10

-5 < M < 5 @68%

E.g. C=50, R=31, σ=10

-29 < M < -9 @68%

Poisson: Signal + Background

Background mean 2.50

Detect 3 events:

Total < 6.68 @ 95%

Signal<4.18@95%

Detect 0 events

Total < 2.30 @ 95%

Signal < -0.20 @ 95%

These statements are OK.
We are allowed to get 32% / 5% wrong.

But they are stupid

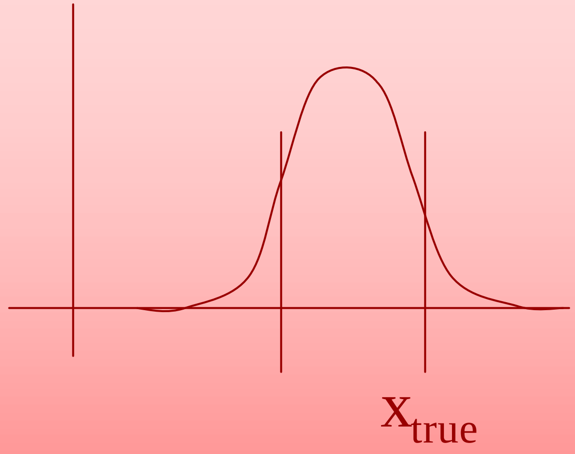
Bayes to the rescue

$$P(Theory\,|\,Data) = \frac{P(Data\,|\,Theory)}{P(Data)}\,P(Theory)$$

Standard (Gaussian) measurement

- No prior knowledge of true value
- No prior knowledge of measurement result
- P(Data|Theory) is Gaussian
- P(Theory|Data) is Gaussian

Interpret this with Probability statements in any way you please



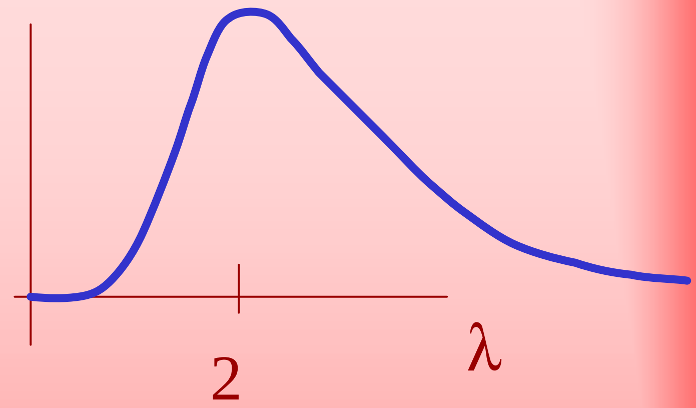
Gives same limits as Frequentist method for simple Gaussian

Bayesian Confidence Intervals (contd)

Observe (say) 2 events

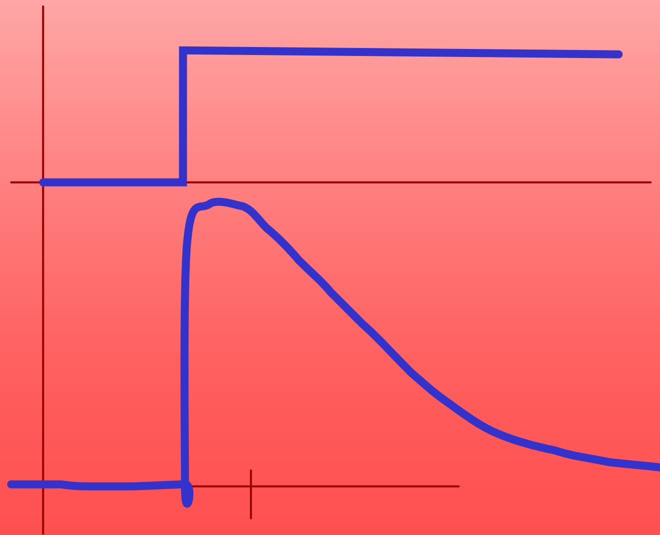
$P(\lambda;2) \propto P(2;\lambda) = e^{-\lambda}\lambda^{2}$

Normalise and interpret



If you know background mean is 1.7, then you know $\lambda > 1.7$

Multiply, normalise and interpret



Bayes: words of caution

Taking prior in λ as flat is not justified

Can argue for prior flat in $\ln \lambda$ or $1/\sqrt{\lambda}$ or whatever

Good practice to try a couple of priors to see if it matters

Feldman-Cousins Unified Method

Physicists are human

Ideal Physicist

1. Choose Strategy
2. Examine data
3. Quote result

Real Physicist

1. Examine data
2. Choose Strategy
3. Quote Result

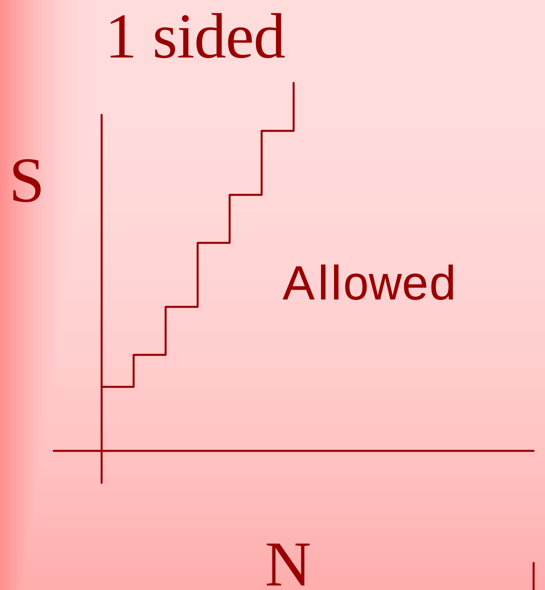
Example:

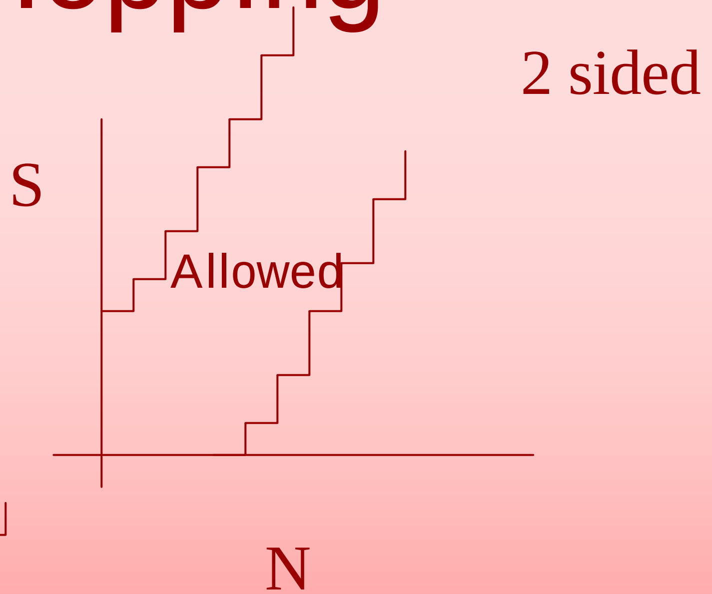
You have a background of 3.2

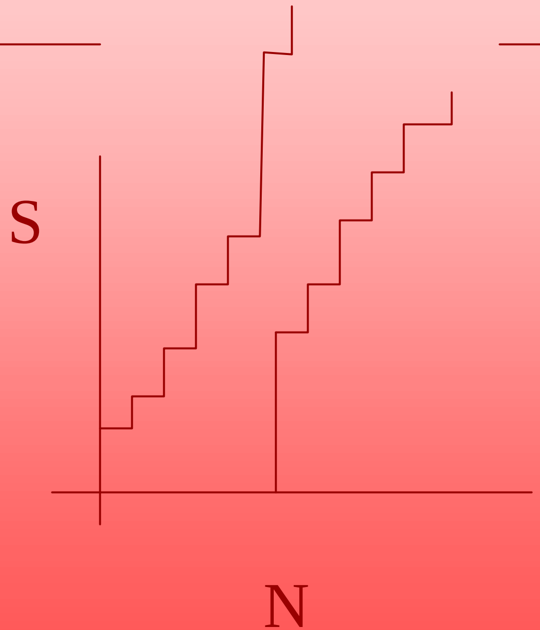
Observe 5 events? Quote one-sided upper limit (9.27-3.2 =6.07@90%)

Observe 25 events? Quote two-sided limits

"Flip-Flopping"

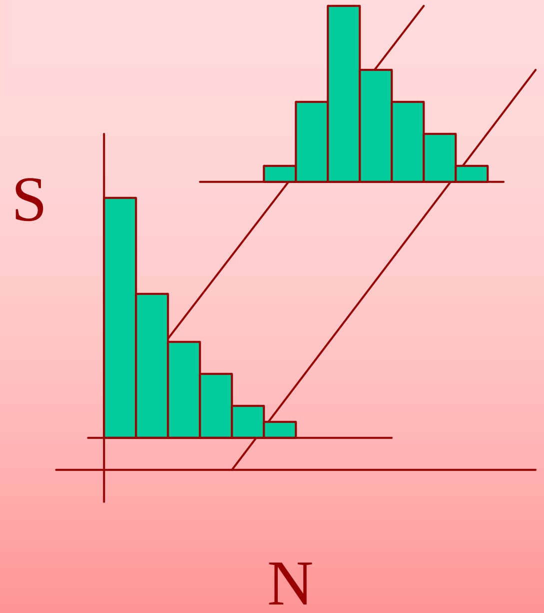






This is not a true confidence belt! Coverage varies.

Solution: Construct belt that does the flip-flopping



For 90% CL

For every S select set of N-values in belt

Total probability must sum to 90% (or more): there are many strategies for doing this

Crow & Gardner strategy (almost right):

Select N-values with highest probability $\rightarrow$ shortest interval

Better Strategy

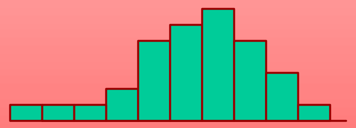
N is Poisson from S+B

B known, may be large

E.g. B=9.2,S=0 and N=1

P=.1% - not in C-G band

But any S>0 will be worse



Fair comparison of P is with best P for this N

Either at S=N-B or S=0

To construct band for a given S:

For all N:

Find P(N;S+B) and

P_{best}=P(N;N) if (N>B) else P(N;B)

Rank on P/P_{best}

Accept N into band until Σ P(N;S+B) $\geq$90%

Feldman and Cousins Summary

- Makes us more honest (a bit)
- Avoids forbidden regions in a Frequentist way

- Not easy to calculate
- Has to be done separately for each value of B
- Can lead to 2-tailed limits where you don't want to claim a discovery
- Weird effects for N=0; larger B gives lower (=better) upper limit

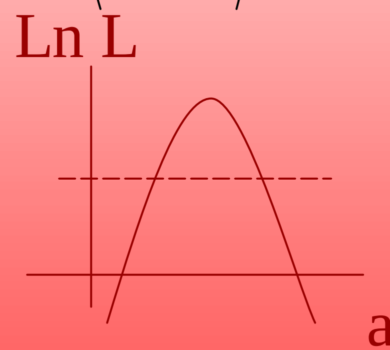
Maximum Likelihood and Confidence Levels

ML estimator (large N) has variance given by MVB

$$s_{\hat{a}}^{2} = V(\hat{a}) = \frac{-1}{\left\langle \frac{d^2 \ln L}{da^2} \right\rangle}$$

At peak $\ln L \approx L_{\max} + \frac{(a-\hat{a})^2}{2} \left. \frac{d^2 \ln L}{da^2} \right|_{a=\hat{a}}$ For large N $\left\langle \frac{d^2 \ln L}{da^2} \right\rangle = \left. \frac{d^2 \ln L}{da^2} \right|_{a=\hat{a}}$

Ln L is a parabola (L is a Gaussian)



$$\ln L = L_{\max} - \frac{(a-\hat{a})^2}{2 s_{\hat{a}}^{2}}$$

Falls by ½ at $a = \hat{a} \pm s_{\hat{a}}$

Falls by 2 at $a = \hat{a} \pm 2 s_{\hat{a}}$

Read off 68% , 95% confidence regions

MVB example

N Gaussian measurements: estimate μ

$$P(x_i; m) = \frac{1}{s\sqrt{2p}} e^{-(x_i - m)^2/2s^2}$$

Ln L given by $-\sum_i \frac{(x_i - m)^2}{2s^2} - N \ln\left(s\sqrt{2p}\right)$

Differentiate twice wrt μ $\quad -\frac{N}{s^2}$

Take expectation value – but it's a constant

Invert and negate: $\quad V(\hat{m}) = \frac{s^2}{N}$

Another MVB example

N Gaussian measurements: estimate σ

Ln L still given by $-\sum_i \frac{(x_i - m)^2}{2s^2} - N \ln\left(s\sqrt{2p}\right)$

Differentiate twice wrt σ

$$-\sum_i \frac{3(x_i - m)^2}{s^4} + \frac{N}{s^2}$$

Take expectation value $\langle (x_i - \mu)^2 \rangle = \sigma^2 \; \forall \; i$

Gives $-\frac{2N}{s^2}$

Invert and negate: $V(\hat{s}) = \frac{s^2}{2N}$

ML for small N

ln L is not a parabola

Argue: we could (invariance) transform to some a’ for which it is a parabola

We could/should then get limits on a’ using standard L_{max}-½ technique

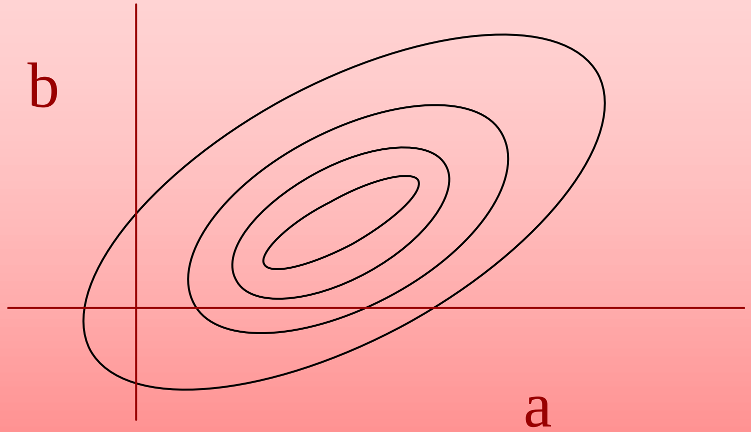
These would translate to limits on a

These limits would be at the values of a for which L= L_{max}-½

So just do it directly

Multidimensional ML

- L is multidimensional Gaussian



For 2-d 39.3% lies within 1σ
i.e. within region bounded by
$L=L_{max}-½$

For 68% need $L=L_{max}-1.15$

Construct region(s) to taste using numbers from integrated χ^2 distribution

Confidence Intervals

- Descriptive
- Frequentist
  - Feldman-Cousins technique
- Bayesian
- Maximum Likelihood
  - Standard
  - Asymmetric
  - Multidimensional