Masses and non-gauge interactions

It can be shown that the most general renormalizable supersymmetric term has the form (in Weyl notation)

$$\mathcal{L}_{\rm ng} = -\frac{1}{2}\, W_{ij}\, \psi_i \psi_j + W_i\, F_i + \text{h.c.}$$

where

$$W_i = \frac{\partial W}{\partial \phi_i} \qquad\qquad W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}$$

and W is the superpotential:

$$W = \frac{1}{2} M_{ij}\, \phi_i \phi_j + \frac{1}{6} y_{ijk}\, \phi_i \phi_j \phi_k,$$

The constants M_{ij} and y_{ijk} must be totally symmetric in their indices.

The proof is not difficult. The interaction term is strongly restricted by renormalizability: operators in the lagrangian must have a mass dimension ≤ 4.

For example: since

$$[\psi] = m^{3/2} \quad [\phi] = m \quad [F] = m^2$$

it follows that W_{ij} can be at most linear in ϕ.

The other constraints come from invariance under supersymmetry transformations.

An important feature: the superpotential W must be an analytic function of the scalar fields. It is a function of ϕ, not of ϕ and ϕ^*.

A complex function $f(z, z^*) = u(x,y) + iv(x,y)$ of a complex variable $z = x + iy$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This implies

$$\begin{aligned}\frac{\partial f}{\partial z^*} &= \frac{\partial(u+iv)}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial(u+iv)}{\partial y}\frac{\partial y}{\partial z^*} \\ &= \frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = 0\end{aligned}$$

The equations of motion for the auxiliary fields F_i are now

$$F_i^* = -W_i = M_{ij}\,\phi_j + \frac{1}{2} y_{ijk}\,\phi_j\phi_k$$

and therefore

$$\begin{aligned}\mathcal{L}_{\rm ng} + \mathcal{L}_a &= -\frac{1}{2}\,W_{ij}\,\psi_i\psi_j - \frac{1}{2}\,W^*_{ij}\,\psi_i^\dagger\psi_j^\dagger + W_i F_i + W_i^* F_i^* - F_i\,F_i^* \\ &= -\frac{1}{2}\,W_{ij}\,\psi_i\psi_j - \frac{1}{2}\,W^*_{ij}\,\psi_i^\dagger\psi_j^\dagger - W_i\,W_i^*\end{aligned}$$

The scalar potential is entirely determined by the superpotential:

$$\begin{aligned}V(\phi) &= W_i\,W_i^* = F_i\,F_i^* \\ &= M_{ij}M^*_{il}\,\phi_j\phi_l^* \qquad \text{scalar masses} \\ &\quad + \frac{1}{2}\,M_{ij}y^*_{ilm}\,\phi_j\phi_l^*\phi_m^* + \frac{1}{2}\,M^*_{il}y_{ijk}\,\phi_j\phi_k\phi_l^* \qquad \text{trilinear couplings} \\ &\quad + \frac{1}{4}\,y_{ijk}y^*_{ilm}\,\phi_j\phi_k\phi_l^*\phi_m^* \qquad \text{quartic couplings}\end{aligned}$$

The remaining terms contain mass terms for the fermions and Yukawa couplings:

$$\begin{aligned}
-\frac{1}{2}\,W_{ij}\,\psi_i\psi_j - \frac{1}{2}\,W^*_{ij}\,\psi^\dagger_i\psi^\dagger_j \quad &= \quad -\frac{1}{2}\,M_{ij}\,\psi_i\psi_j - \frac{1}{2}\,M^*_{ij}\,\psi^\dagger_i\psi^\dagger_j \\
&\quad\; -\frac{1}{2}\,y_{ijk}\,\phi_i\psi_j\psi_k - \frac{1}{2}\,y^*_{ijk}\,\phi^*_i\psi^\dagger_j\psi^\dagger_k
\end{aligned}$$

Many interaction terms, but couplings related by supersymmetry.

Some comments:

- **The scalar potential is positive semi-definite: (almost) no problems with the stability of the ground state.**
- **The quartic scalar self-coupling and the fermion-fermion-scalar Yukawa coupling are related:** $\lambda_S = |\lambda_f|^2$.
- **It is easy to check that scalars and fermions have the same squared mass matrix**

$$(M^2)_{ij} = M_{ik} M^*_{kj}$$

which is symmetric and can be diagonalized.

- **The analiticity of** W **requires the introduction of two Higgs doublets in the supersymmetric version of the Standard Model.**

In summary, we have the following

Recipe

for non-gauge interaction terms of a set of chiral supermultiplets ϕ_i, ψ_i, F_i:

1. **Give a superpotential**

$$W = \frac{1}{2} M_{ij}\, \phi_i \phi_j + \frac{1}{6} y_{ijk}\, \phi_i \phi_j \phi_k,$$

   with M_{ij} and y_{ijk} completely symmetric.

2. **Build the interaction lagrangian as**

$$\mathcal{L}_{\rm ng} = -\frac{1}{2} W_{ij}\, \psi_i \psi_j + W_i\, F_i + \text{h.c.}$$

   where

$$W_i = \frac{\partial W}{\partial \phi_i} \qquad W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}$$

Gauge interactions

A gauge supermultiplet contains a vector field A_μ^a and a Weyl fermion λ^a. An infinitesimal gauge symmetry transformation with parameter α^a acts on λ^a as

$$\delta_{\rm gauge}\lambda^a = g\, f^{abc}\lambda^b\alpha^c$$

$a = 1,\ldots,8$, $f^{abc} \to f^{abc}_{SU(3)}$ **and** $g \to g_s$ **for color**
$a = 1,\ldots,3$, $f^{abc} \to \epsilon^{abc}$ **and** $g \to g$ **for weak isospin**
$a = 1$, $f^{abc} \to 0$ **and** $g \to g'$ **for weak hypercharge.**

Counting fermionic and bosonic degrees of freedom, we see that we should add one real scalar auxiliary field, D^a (off shell, $n_b = 3$ from A_μ and $n_f = 4$ from λ). Then

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4} F^a_{\mu\nu}F^{a\mu\nu} + i\lambda^{a\dagger}\bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a$$

where

$$D_\mu\lambda^a = \partial_\mu\lambda^a - g f^{abc} A^b_\mu \lambda^c$$

The corresponding supersymmetry transformation laws are easily worked out, requiring that δA_μ^a be real, and δD^a proportional to the equation of motion of λ^a:

$$\delta A_\mu^a = -\frac{1}{\sqrt{2}}\left(\eta^\dagger\,\bar\sigma_\mu\,\lambda^a + \lambda^{a\,\dagger}\,\bar\sigma_\mu\,\eta\right)$$

$$\delta\lambda^a = -\frac{i}{2\sqrt{2}}\,\sigma^\mu\bar\sigma^\nu\eta\,F_{\mu\nu}^a + \frac{1}{\sqrt{2}}\,D^a\eta$$

$$\delta D^a = -\frac{i}{\sqrt{2}}\left[\eta^\dagger\,\bar\sigma^\mu D_\mu\,\lambda^a - (D_\mu\lambda^a)^\dagger\,\bar\sigma^\mu\,\eta\right]$$

With these definitions,

$$(\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1})X = -(\eta_1^\dagger\,\bar\sigma^\mu\,\eta_2 - \eta_2^\dagger\,\bar\sigma^\mu\,\eta_1)\,iD_\mu X$$

on $X = F_{\mu\nu}^a, \lambda^a, D^a$.

Back to the chiral supermultiplet $X = \phi, \psi, F$. All of them must be in the same representation of the gauge group:

$$\delta_{\text{gauge}} X_i = i\, g \alpha^a \, (T^a)_{ij} \, X_j$$

where T^a are the generators of the gauge group in the representation of X. Ordinary derivatives must be replaced by covariant derivatives everywhere:

$$\partial_\mu \to D_\mu = \partial_\mu + i\, g\, A_\mu^a \, T^a$$

Is this enough to turn our simple model into a gauge-invariant supersymmetric theory?

Not quite: **there are other possible gauge invariant and renormalizable terms that can be formed out of the fields in the gauge and chiral supermultiplets:**

$$\phi^* T^a \psi \lambda^a \qquad (\phi^* T^a \phi) D^a$$

They are not taken into account by the covariant derivatives, because they involve the superpartners of the gauge fields.

They can be added to our supersymmetric lagrangian, with appropriate coefficients, provided the transformation laws of the auxiliary field F is slightly modified:

$$\delta\phi = \eta\,\psi$$

$$\delta\psi = -i(\sigma^\mu \epsilon \eta^*)\, D_\mu \phi + F\eta$$

$$\delta F = -i\,\eta^\dagger\,\bar\sigma^\mu\, D_\mu\psi + g\sqrt{2}(T^a\phi)\lambda^{a\,\dagger}\eta^\dagger$$

Then

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{chiral}} + g\sqrt{2}\left(\phi^* T^a \psi \lambda^a + \phi T^a \lambda^{a\dagger} \psi^\dagger\right) + g(\phi^* T^a \phi) D^a$$

is supersymmetric, provided the superpotential is gauge-invariant:

$$\delta_{\text{gauge}} W = W_i (T^a)_{ij} \phi_j = 0$$

The equations of motion for D^a are now

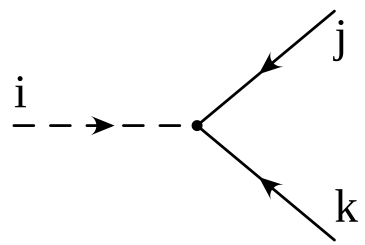
$$D^a = -g(\phi^* T^a \phi)$$

and the scalar potential becomes

$$V(\phi) = F_i^* F_i + \frac{1}{2} D^a D^a = W_i^* W_i + \frac{1}{2} g^2 (\phi^* T^a \phi)(\phi^* T^a \phi)$$

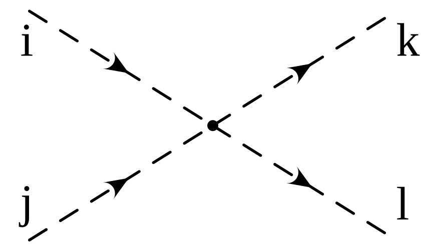
completely determined by the auxiliary fields. This fact is relevant for the discussion of spontaneous supersymmetry breaking.

Couplings in a supersymmetric gauge theory



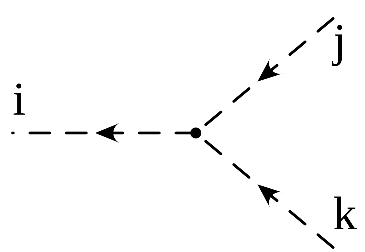
(a)

y_{ijk}



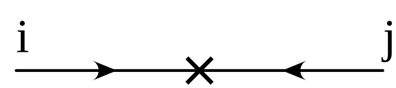
(b)

$y_{ijm}y^*_{klm}$



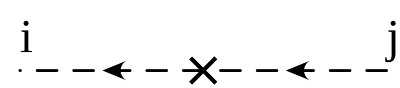
(a)

$M^*_{im}y_{jkm}$



(b)

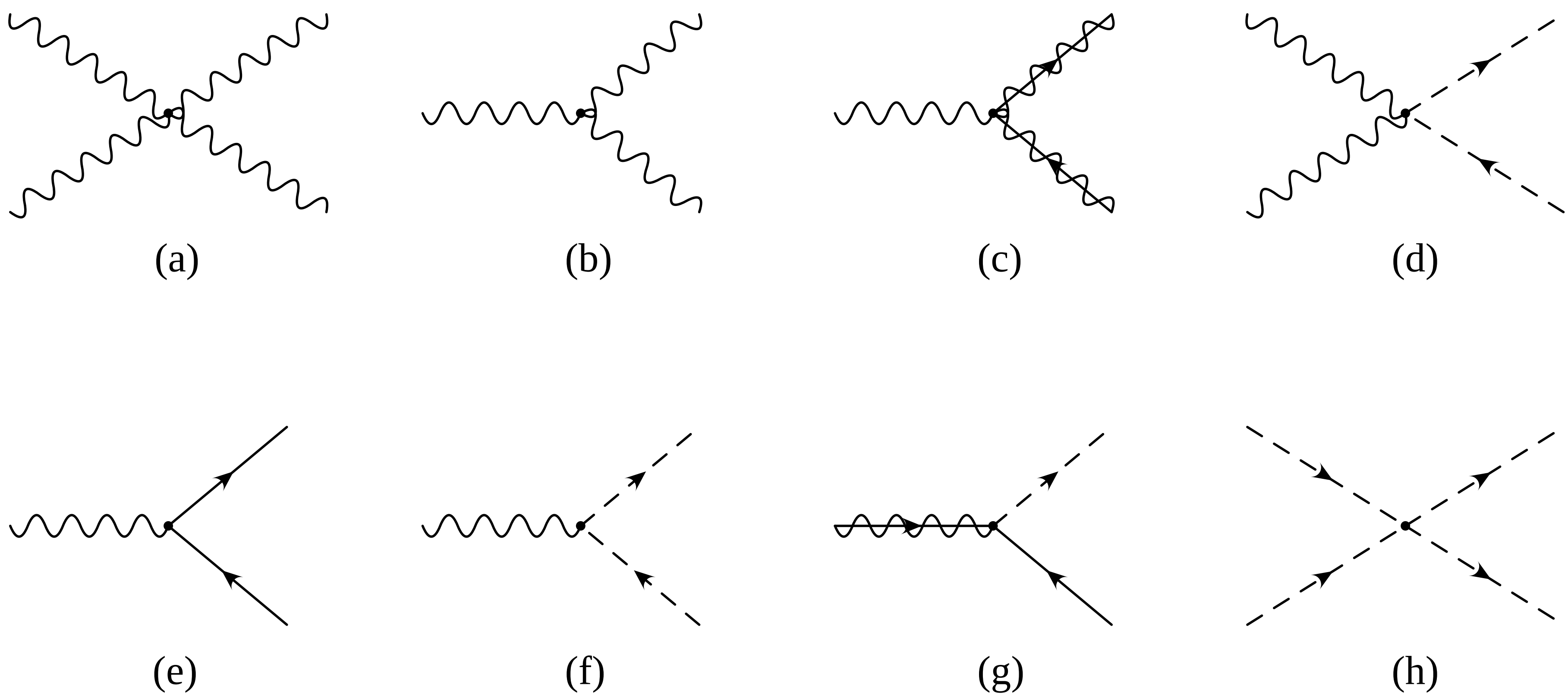
M_{ij}



(c)

$M^*_{ik}M_{kj}$

Gauge couplings



Soft supersymmetry breaking

The most general soft supersymmetry breaking lagrangian is given by

$$\mathcal{L}_{\rm soft} = -(m^2)_{ij}\phi_i\phi_j^* - \left(\frac{1}{2}\, m_\lambda\, \lambda^a\lambda^a + \frac{1}{2}\, b_{ij}\phi_i\phi_j + \frac{1}{6}\, a_{ijk}\phi_i\phi_j\phi_k + \text{h.c.}\right)$$

It can be shown that a theory with exact supersymmetry, plus $\mathcal{L}_{\rm soft}$, is free of quadratic divergences.

In principle, there could also be

$$-\frac{1}{2}\, c_{ijk}\phi_i^*\phi_j\phi_k + \text{h.c.}$$

Usually negligible in most supersymmetry breaking scenarios.

A word on gaugino masses. A mass term for a left-handed Weyl spinor

$$\xi^T \epsilon \xi + \text{h.c.}$$

is called a Majorana mass term. It is allowed by Lorentz invariance, but generally forbidden if ξ belongs to a complex representation of a gauge group. A simple example: a phase transformation $\delta\xi = i\alpha\xi$. Matter fermions in the Standard Model (with the possible exception of a right-handed neutrino) cannot have a Majorana mass term.

The case of gauginos λ^a is different: they transform according to the adjoint representation,

$$\delta_{\text{gauge}}\lambda_a = g\, f_{abc}\lambda_b\alpha_c$$

which is self-conjugate. This gives

$$\delta_{\text{gauge}}(\lambda_a^T \epsilon\lambda_a) = g\, f_{abc}\alpha_c(\lambda_b^T \epsilon\lambda_a + \lambda_a^T \epsilon\lambda_b) = 0$$

because f_{abc} is completely antisymmetric.

The minimal supersymmetric Standard Model

With some specifications, there is a unique superpotential which is allowed by the Standard Model gauge invariance and by renormalizability:

$$W_{\mathrm{MSSM}} = \tilde{u}_R^*\, y_u\, \tilde{q}_L H_u - \tilde{d}_R^*\, y_d\, \tilde{q}_L H_d - \tilde{e}_R^*\, y_l\, \tilde{\ell}_L H_d + \mu\, H_u\, H_d$$

(call it W from now on). The first term in detail:

$$\tilde{u}_R^*\, y_u\, \tilde{q}_L H_u = (\tilde{u}_R^*)_j{}^f\, (y_u)_{fg}\, (\tilde{q}_L)_\alpha^{gj}\, (H_u)_\beta\, \epsilon^{\alpha\beta}$$

j **is an** $SU(3)_{\mathrm{color}}$ **index,** $j = 1, 2, 3$

α, β **are** $SU(2)_{\mathrm{L}}$ **indices,** $\alpha = 1, 2$, $\beta = 1, 2$

f, g **are generation indices,** $f = 1, 2, 3$, $g = 1, 2, 3$

If ϕ_1 and ϕ_2 are $SU(2)$ doublets,

$$\delta\phi_1 = i\alpha^a \sigma_a \, \phi_1 \qquad \delta\phi_2 = i\alpha^a \sigma_a \, \phi_2$$

then

$$\phi_1^T \, \epsilon \, \phi_2, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv i\sigma_2$$

is an $SU(2)$ scalar:

$$\delta(\phi_1^T \, \epsilon \, \phi_2) = i\alpha^a \phi_1^T \, (\sigma_a^T \epsilon + \epsilon \sigma_a) \, \phi_2 = 0$$

because σ_1, σ_3 are symmetric, σ_2 antisymmetric, and $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$. The quantities

$$\phi_1^\dagger \phi_1 \quad \phi_1^\dagger \phi_2 \quad \phi_2^\dagger \phi_2$$

are also, obviously, $SU(2)$ invariants, by they cannot enter the superpotential, which is an analytic function of all scalar fields.

Some comments:

- **It is now clear why we need two Higgs doublets: in the Standard Model, H gives mass to up-type quarks, while $H^c = \epsilon H^*$ gives mass to down quarks and lepons. Here, H and H^* cannot appear simultaneously in the superpotential.**
- y_u, y_d, y_l **are 3×3 matrices in family space of dimensionless Yukawa couplings. They contain fermion masses and generation mixing. A good approximation:** $y_u^{33} = y_t, y_b^{33} = y_b, y_l^{33} = y_\tau$, **all other entries equal to 0.**
- **No bilinear term other than $\mu H_u H_d$ is allowed by gauge symmetry. μ is the only dimensionful parameter in the superpotential; in particular, mass terms for fermions are forbidden.**

The superpotential originates a large variety of interaction vertices, with coupling constants related by supersymmetry.

Recall our recipe,

$$\mathcal{L}_{\rm ng} = -W_i\, W_i^* - \frac{1}{2}\, W_{ij}\, \psi_i \psi_j - \frac{1}{2}\, W_{ij}^*\, \psi_i^\dagger \psi_j^\dagger$$

where

$$W_i = \frac{\partial W}{\partial \phi_i}, W_{ij} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}$$

We find e.g.

- **Yukawa couplings, such as**

$$\frac{\partial W}{\partial \tilde{u}_R^* \partial \tilde{u}_L} \bar{u} u = \bar{u}\, y_u\, u\, H_u^0$$

  as in the Standard Model.

- **quark-squark-higgsino couplings,**

$$\frac{\partial W}{\partial \tilde{u}_R^* \partial H_u^0} \bar{u} \tilde{h}_u^0 = \bar{u}\, y_u\, \tilde{q}_L\, h_u^0$$

- **Quartic scalar couplings:**

$$\left| \frac{\partial W}{\partial \tilde{u}_R^*} \right|^2 = |y_u\, \tilde{q}_L\, H_u|^2$$

The term $\mu H_u H_d$ provides a higgsino mass term

$$-\mu\,(\tilde{h}_u^+\tilde{h}_d^- - \tilde{h}_u^0\tilde{h}_d^0 + \text{h.c.})$$

and a scalar Higgs mass term

$$-\left|\mu\right|^2\left(\left|H_u^+\right|^2 + \left|H_u^0\right|^2 + \left|H_d^0\right|^2 + \left|H_d^-\right|^2\right)$$

This term cannot induce spontaneous breaking of the electroweak gauge symmetry. We must rely upon soft breaking terms.

The value of μ is a problem: we expect it to be of the same order (~ 100 GeV) as the soft breaking terms, in order to provide the correct value of the W and Z masses, but it has a completely different origin.

Accidental symmetries and R-parity

Baryon and lepton number conservation in the Standard Model follow from accidental symmetries: a consequence of renormalizability.

This is no longer true in supersymmetry:

$$W_{\Delta L=1} = \frac{1}{2}\lambda_{ijk}\,\tilde{\ell}^i_L\tilde{\ell}^j_L(\tilde{e}^k_R)^* + \frac{1}{2}\lambda'_{ijk}\,\tilde{\ell}^i_L\tilde{q}^j_L(\tilde{d}^k_R)^* + \mu'_i\,\tilde{\ell}^i_L H_u$$

$$W_{\Delta B=1} = \frac{1}{2}\lambda''_{ijk}\,(\tilde{u}^i_R)^*(\tilde{d}^j_R)^*(\tilde{d}^k_R)^*$$

are allowed by renormalizability and gauge symmetry.

In order to implement B and L conservation we must impose some extra symmetry (not B and L themselves: they are violated by nonperturbative effects!).

Consider the so-called matter parity:

$$P_M = (-1)^{3(B-L)}$$

where $B = 1/3$ for quarks and squarks and 0 otherwise, $L = 1$ for leptons and sleptons, 0 otherwise.

P_M commutes with supersymmetry: all particles in the same supermultiplet have the same P_M (-1 for quarks and leptons, $+1$ for Higgs and gauge).

Requiring $P_M = +1$ for all terms in the lagrangian (or in the superpotential) eliminates B and L violating terms.

A different, but equivalent language: define R−parity as

$$P_R = (-1)^{3(B-L)+2s}$$

Conserved if P_M is conserved, but does not commute with supersymmetry:

$$P_R = +1 \quad \text{for standard particles}$$
$$P_R = -1 \quad \text{for superpartners}$$

R-parity conserved $\Leftrightarrow$ matter parity conserved.

Useful for phenomenology: each vertex must contain an even number of fields with $P_R = -1$.

Consequences of R-parity conservation

- **The lightest particle with $P_R = -1$ is stable. It is called the LSP. A candidate for dark matter.**
- **Supersymmetric particles decay into states with an odd number of $P_R = -1$ particles.**
- **Supersymmetric particles are always produced in even numbers (usually 2).**

Soft supersymmetry breaking in the MSSM

The most general soft supersymmetry breaking term for the MSSM is

$$
\begin{aligned}
\mathcal{L}_{\mathrm{soft}} &= -\frac{1}{2}\Big(M_3\tilde{g}\tilde{g} + M_2\tilde{w}\tilde{w} + M_1\tilde{b}\tilde{b} + \mathrm{h.c.}\Big) \\
&\quad -\Big(\tilde{u}_R^*\,A_u\,\tilde{q}_L\,H_u - \tilde{d}_R^*\,A_d\,\tilde{q}_L\,H_d - \tilde{e}_R^*\,A_e\,\tilde{\ell}_L\,H_d\Big) + \mathrm{h.c.} \\
&\quad -\tilde{q}_L^\dagger\,m_Q^2\,\tilde{q}_L - \tilde{\ell}_L^\dagger\,m_L^2\,\tilde{\ell}_L - \tilde{u}_R^\dagger\,m_U^2\,\tilde{u}_R - \tilde{d}_R^\dagger\,m_D^2\,\tilde{d}_R - \tilde{e}_R^\dagger\,m_E^2\,\tilde{e}_R \\
&\quad -m_{H_u}^2\,H_u^\dagger\,H_u - m_{H_d}^2\,H_d^\dagger\,H_d - (B\,H_u\,H_d + \mathrm{h.c.})
\end{aligned}
$$

An enormous number of new independent parameters.

All of them are expected to be of order m_{soft} (to the appropriate power), something between 100 GeV and 1 TeV.

Soft breaking terms arise after spontaneous breaking of supersymmetry has taken place; we postpone an analysis of this problem.

An anticipation: two main scenarios have been proposed:

- **gravity-mediated supersymmetry breaking**
- **gauge-mediated supersymmetry breaking**

The resulting spectrum and couplings (and experimental strategies) are quite different in the two cases.

Do we have any guiding principle that can help us reducing the number of arbitrary parameters?

$m_Q^2, m_U^2, m_D^2, m_E^2$ **are arbitrary** 3×3 **matrices in flavour space; they are subject from strong contraints from**

- **flavour changing neutral current phenomena (e.g.** $K_0 \bar{K}_0$ **mixing)**
- **individual leptonic number conservations (e.g.** $\mu \to e\gamma$**)**
- **CP violation**

The (unknown) mechanism that generates soft breaking terms must respect these constraints.

All these constraints are satisfied with the simple assumption that flavour mixing and CP violation are only originated by Yukawa couplings. This is achieved by assuming that

- $(m_Q^2)_{fg} = m_Q^2 \delta_{fg} \quad (m_U^2)_{fg} = m_U^2 \delta_{fg} \quad (m_D^2)_{fg} = m_D^2 \delta_{fg}$
  $(m_E^2)_{fg} = m_E^2 \delta_{fg}$
- $(A_u)_{fg} = A_u (y_u)^{fg} \quad (A_d)_{fg} = A_d (y_d)^{fg} \quad (A_e)_{fg} = A_e (y_e)^{fg}$
- **all soft breaking parameters are real.**

This (or any other) structure of soft breaking terms results from some underlying mechanism that breaks supersymmetry at a scale Q_0, presumably very large (e.g. the grand-unification scale).

Predictions at energy scales relevant for our experiments, $m_{\rm exp}$, typically of order 100 to 1000 GeV, will therefore contain large logarithms

$$\log \frac{Q_0}{m_{\rm exp}}$$

induced by radiative corrections. These must be resummed by standard renormalization group techniques.

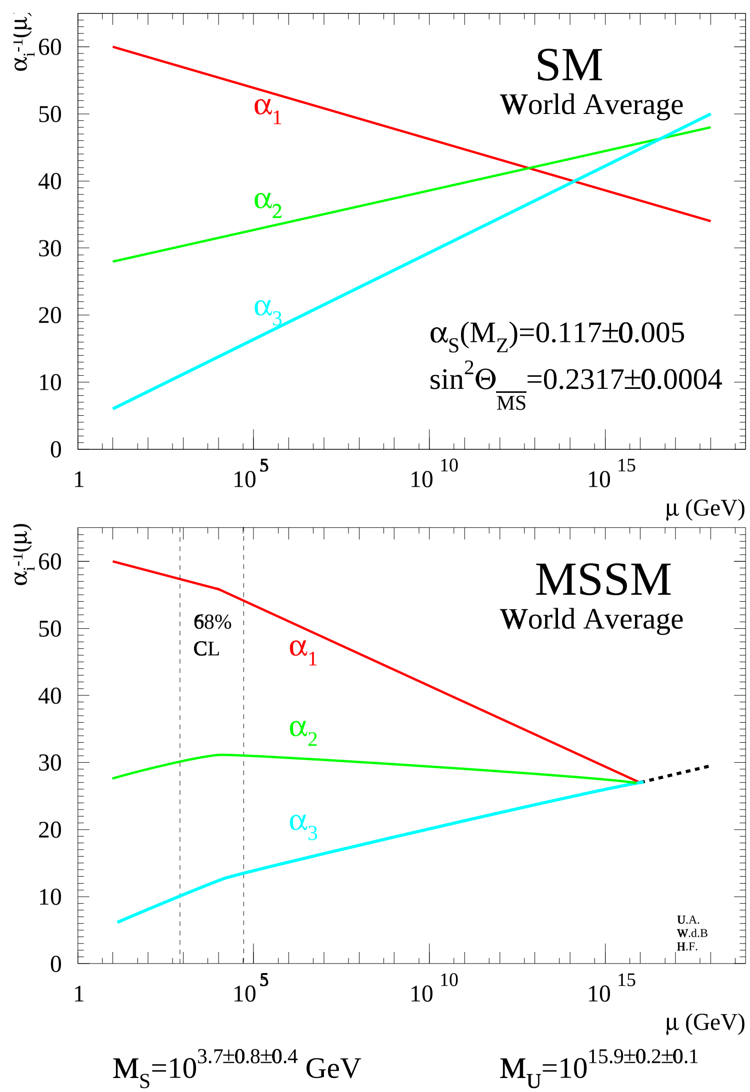
An encouraging fact: coupling constant unification works much better in the MSSM than in the Standar Model. Define

$$g_3 = g_s$$
$$g_2 = g = \frac{e}{\sin\theta_W}$$
$$g_1 = \sqrt{\frac{5}{3}}g' = \sqrt{\frac{5}{3}}\frac{e}{\cos\theta_W}$$

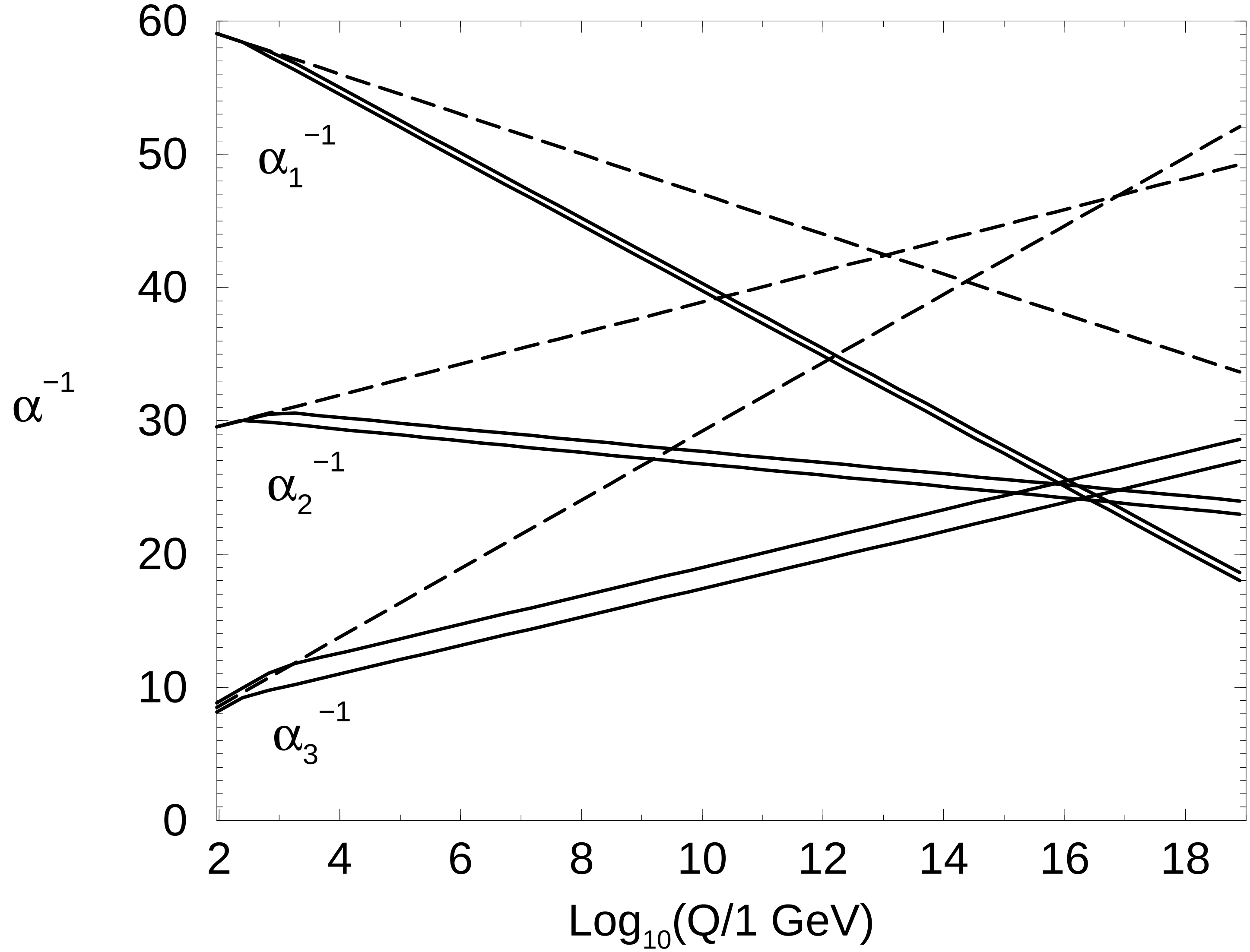
(in order to have the same normalization for symmetry generators). The corresponding RGEs at one loop are

$$\mu^2\frac{dg_i}{d\mu^2} = \frac{b_i}{8\pi}g_i^3 \quad \Rightarrow \quad \mu^2\frac{d}{d\mu^2}\frac{1}{\alpha_i} = -b_i$$

with $\alpha_i = g_i^2/(4\pi)$**.**



SM
World Average
α1
α2
α3
αS(MZ)=0.117±0.005
sin2Θ MS=0.2317±0.0004
αi-1(μ)
μ (GeV)
MSSM
World Average
68%
CL
U.A.
W.d.B
H.F.
MS=10^3.7±0.8±0.4 GeV
MU=10^15.9±0.2±0.1



α^{-1}
60
50
40
30
20
10
0
α_1^{-1}
α_2^{-1}
α_3^{-1}
2
4
6
8
10
12
14
16
18
$Log_{10}(Q/1\ GeV)$

The mass spectrum

1. The Higgs sector

The Higgs potential of the MSSM is

$$
\begin{aligned}
V_{\text{Higgs}} &= (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2) + (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) \\
&\quad + B\left(H_u^+ H_d^- - H_u^0 H_d^0\right) + \text{h.c.} \\
&\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2 \\
&\quad + \frac{1}{2} g^2 |H_u^+ H_d^{0*} + H_u^0 H_d^{-*}|^2
\end{aligned}
$$

No arbitrary quartic couplings: they are fixed by the D terms.

Minimization: **We can take $H_u^+ = H_d^- = 0$ (good news: EM unbroken) and H_u^0, H_d^0 real and positive at the minimum without loss of generality. Then we restrict to the neutral sector:**

$$\begin{aligned} V_{\text{Higgs}} &= (|\mu|^2 + m_{H_d}^2)|H_d^0|^2 + (|\mu|^2 + m_{H_u}^2)|H_u^0|^2 - (BH_u^0 H_d^0 + \text{h.c.}) \\ &\quad + \frac{1}{8}(g^2 + g'^2)(|H_u^0|^2 - |H_d^0|^2)^2 \end{aligned}$$

B **may also be taken to be real and positive: no CP violating phases in the Higgs sector.**

Perform an $SU(2)_L$ gauge rotation that makes $\langle H_u^+ \rangle = 0$; then, the condition $\partial V / \partial H_u^+ = 0$ is only fulfilled if also $\langle H_d^- \rangle = 0$.

The only place where the phases of H_u^0, H_d^0 appear is the term

$$-B H_u^0 H_d^0 + \text{h.c.}$$

We can make B real and positive by a suitable phase choice for H_u^0, H_d^0.

Clearly, the potential has a minimum if also $H_u^0 H_d^0$ is real and positive, so $\langle H_u^0 \rangle$ and $\langle H_d^0 \rangle$ must have opposite phases, which can be rotated away by a $U(1)_Y$ gauge transformation.

Conditions for electroweak spontaneous symmetry breaking:

1. **The origin $H_u^0 = H_d^0 = 0$ is not a minimum of the potential**
2. **The potential is bounded from below**

Condition 1. leads to the constraint

$$(|\mu|^2 + m_{H_d}^2)(|\mu|^2 + m_{H_u}^2) < B^2$$

In some cases, guaranteed by large negative radiative corrections to $m_{H_u}^2$.

Condition 2. is almost automatic: the quartic couplings are positive definite. There are however D-flat directions $|H_u^0| = |H_d^0|$. We must require that the potential in these directions is stabilized by quadratic terms:

$$2B < 2|\mu|^2 + m_{H_d}^2 + m_{H_u}^2$$

Define $v_u = \langle H_u^0 \rangle$ and $v_d = \langle H_d^0 \rangle$. We find

$$m_Z^2 = \frac{1}{2}(g^2 + {g'}^2)(v_u^2 + v_d^2) \;\Rightarrow\; v^2 \equiv v_u^2 + v_d^2 \simeq (174\,\mathrm{GeV})^2$$

Following tradition, we also define an angle β through

$$\tan\beta = \frac{v_u}{v_d}$$

Clearly $0 \le \beta \le \pi/2$. The minimization conditions are

$$|\mu|^2 + m_{H_d}^2 = B\tan\beta - \frac{m_Z^2}{2}\cos 2\beta$$

$$|\mu|^2 + m_{H_u}^2 = B\cot\beta + \frac{m_Z^2}{2}\cos 2\beta$$

The μ problem is now evident.

Mass eigenstates:

1. Two CP-odd neutral scalars

$$\begin{pmatrix} G^0 \\ A^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} \mathrm{Im}\, H_u^0 \\ \mathrm{Im}\, H_d^0 \end{pmatrix}$$

2. Two charged scalars

$$\begin{pmatrix} G^+ \\ H^+ \end{pmatrix} = \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} H_u^+ \\ H_d^+ \end{pmatrix}$$

$G^0, G^\pm$ **are pseudo-Goldstone bosons: they can be removed from the spectrum by an appropriate gauge transformation.**

$A^0, H^\pm$ **are physical degrees of freedom.**

3. Two CP-even neutral scalars

$$\begin{pmatrix} h^0 \\ H^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, H_u^0 - v_u \\ \mathrm{Re}\, H_d^0 - v_d \end{pmatrix}$$

Finding the mass eigenvalues is a matter of algebra:

$$m_A^2 = \frac{2B}{\sin 2\beta}$$

$$m_{H^\pm}^2 = m_A^2 + m_W^2$$

$$m_{h,H}^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 \mp \sqrt{(m_A^2 + m_Z^2)^2 - 4m_Z^2 m_A^2 \cos^2 2\beta}\right)$$

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\frac{m_A^2 + m_Z^2}{m_H^2 - m_h^2} \qquad \frac{\cos 2\alpha}{\cos 2\beta} = -\frac{m_A^2 - m_Z^2}{m_H^2 - m_h^2}$$

It is easy to show that

$$m_h < |\cos 2\beta|\, m_Z$$

Radiative corrections weaken this bound. At one loop, and neglectic squark mixing, one finds

$$\Delta m_h^2 \simeq \frac{3g^2}{8\pi^2}\frac{m_t^4}{m_W^2}\log\frac{m_{\tilde{t}_1} m_{\tilde{t}_2}}{m_t^2}$$

An extra term appears if mixing is taken into account:

$$\left(\Delta m_h^2\right)_{\rm mix} \simeq \frac{3g^2}{8\pi^2}\frac{m_t^4}{m_W^2}\frac{(A_t-\mu\cot\beta)^2}{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2}\left[\log\frac{m_{\tilde{t}_1}^2}{m_{\tilde{t}_2}^2} + (A_t-\mu\cot\beta)^2 f(m_{\tilde{t}_1}^2, m_{\tilde{t}_2}^2)\right]$$

$$f(a,b) = \frac{1}{a-b}\left[1 - \frac{1}{2}\frac{a+b}{a-b}\log\frac{a}{b}\right]$$

Including all corrections, one obtains approximately (Fabiola will give details)

$$m_h \lesssim 130 \text{ GeV}$$

still very interesting in view of LHC.

Yukawa couplings are relates to fermion masses in the third generation by

$$y_t = \frac{m_t}{\sqrt{2} m_W \sin\beta} \qquad y_b = \frac{m_b}{\sqrt{2} m_W \cos\beta} \qquad y_\tau = \frac{m_\tau}{\sqrt{2} m_W \cos\beta}$$

A difference with respect to the Standard Model: y_b and y_τ may be as large as y_t, if $\tan\beta$ is large enough ($\tan\beta < 1$ strongly disfavoured).

The couplings of h, H, A to standard particles are the same as in the Standard Model, rescaled by $\alpha-$ and $\beta-$dependent factors:

	$d\bar{d}, s\bar{s}, b\bar{b}$ $e^+e^-, \mu^+\mu^-, \tau^+\tau^-$	$u\bar{u}, c\bar{c}, t\bar{t}$	W^+W^-, ZZ
h	$-\sin\alpha/\cos\beta$	$\cos\alpha/\sin\beta$	$\sin(\beta-\alpha)$
H	$\cos\alpha/\cos\beta$	$\sin\alpha/\sin\beta$	$\cos(\beta-\alpha)$
A	$-i\gamma_5\tan\beta$	$-i\gamma_5\cot\beta$	0

At tree level, the Higgs sector is described by two parameters (a common choice: m_A^2 and $\tan\beta$).