

SUPERSYMMETRY AND STRING THEORY

EUROPEAN SUMMER UNIVERSITY

PARTICLES AND THE UNIVERSE

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ECOLE POLYTECHNIQUE

CPHT

PART 1 : SUPERSYMMETRY IN SPACE-TIME

1. Why Symmetry? outstanding tool in physics (and elsewhere) for centuries

- Lagrange and Hamilton: *insight into dynamics*



Noether : Symmetry and conservation laws

example: translation symmetry \leftrightarrow conservation of momentum

- Symmetry unifies: in relativistic electrodynamics \vec{E} and \vec{B}

are unified into

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

thanks to the Lorentz Symmetry

- Symmetry reduces the effective number of degrees of freedom
particle in a central force \rightarrow radial one-dim. motion
- Symmetry classifies: in atomic physics, energy levels \rightarrow quantum numbers $\rightarrow l, m$: angular momentum
 \rightarrow rotation symmetry
in general: $\mathcal{H} = \bigoplus$ representations of the symmetry group

Central question for 100 years: what is the symmetry of space-time?

- before 1905: Galilean symmetry
- after 1905: Poincaré symmetry
(1917: \hookrightarrow in empty space though)
- after 2008: Supersymmetry ?

2. Reminder : continuous groups of transformations

- Note : discrete symmetries play also an important role C P T
- Example of continuous group : rotations in 3-dim. Euclidean space

— angle ϑ around z-axis

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{v}^1 \\ \vec{v}^2 \\ \vec{v}^3 \end{pmatrix} = \begin{pmatrix} \vec{v}'^1 \\ \vec{v}'^2 \\ \vec{v}'^3 \end{pmatrix}$$

— in general $\vec{v}' = M \vec{v}$ $M M^T = 1$ $M \in SO(3)$

M depends on 3 parameters (Euler angles)

Infinitesimal transformations: small ϑ

$$\vec{v}' = \vec{v} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vartheta \vec{v}$$

$$= \vec{v} + i \vartheta \vec{J}_3 \vec{v}$$

- Hermitean generators of $SO(3)$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a Lie algebra $[J_i, J_j] = i \epsilon_{ijk} J_k$

- In physical systems : generators of rotations \equiv angular momentum $\vec{L} = \vec{r} \times \vec{p}$
 - in classical mechanics : Poisson brackets
 - in quantum mechanics
 - operator on scalar wave function: $L_x = i\hbar(y\partial_z - z\partial_y)$
similarly for L_y, L_z
 - Hilbert space decomposed on representations
 - labelled by ℓ $\|L\|^2 = \hbar^2 \ell(\ell+1)$
 - of dimension $2\ell+1$ $m = -\ell, -\ell+1, \dots, \ell$

$$L_z = m\hbar$$

3. The symmetry of the space-time

- Empty space-time is Minkowski flat 4-dim manifold with metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + d\vec{x}^2$ $\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- The space-time symmetries \equiv isometries of Minkowski (i.e. transformations which do not alter the ds^2)
 - Translations : in time (1)
in space (3)
 - Rotations : around each space axis (3)
 - Lorentz boosts: along each space axis (3)

Last two items : Lorentz group.

Total : Poincaré group

- Generators of the Poincaré group acting on scalar wave functions

- Translations

$$P_\mu = i\hbar \partial_\mu$$

Energy and linear momentum

- Rotations and boosts

$$J_{\mu\nu} = i\hbar(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

- generators of rotations : $L_i = J_{jk}$ angular momentum
- generators of boosts : $K_i = J_{0i}$

- Algebra of these operators ($\hbar = 1$)

- $[P_\mu, P_\nu] = 0$

abelian part

- $[J_{\mu\nu}, J_{\rho\sigma}] = i(\gamma_{\mu\rho} J_{\nu\sigma} + \gamma_{\nu\rho} J_{\mu\sigma} - \gamma_{\mu\rho} J_{\nu\sigma} - \gamma_{\nu\rho} J_{\mu\sigma})$

- $[J_{\mu\nu}, P_\rho] = i(\gamma_{\nu\rho} P_\mu - \gamma_{\mu\rho} P_\nu)$

Poincaré = Lorentz \times Translations

- Unitary representations of the Poincaré group

a free particle in relativistic quantum mechanics
is a state of a unitary representation of Poincaré

- label : eigenvalues of two Casimirs

- o $P_\mu P^\mu \rightarrow -m^2 \in \mathbb{R} \quad (c=1)$

- o $W_\mu W^\mu \rightarrow m^2 s(s+1) \quad s \in \mathbb{Z}/2$

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} \quad \text{Pauli-Lubanski}$$

- each state in a given representation $|\vec{P}, \sigma\rangle$

- o massless $\sigma = \pm s$

- o massive $\sigma = -s, -s+1, \dots, s-1, s$

s : spin or helicity (intrinsic angular momentum)

- Examples
 - scalar function $\phi(x)$ only momentum P_μ
 $\square\phi = m^2\phi$ example: π^0 , Higgs, ...
 - vector function $A^\mu(x)$ P_μ and $s=1$
 $\square A^\mu - \partial^\mu (\partial_\nu A^\nu) = 0$ example: photon $\sigma=\pm 1$
 - a massive Dirac spinor $\psi(x)$ P_μ and $s=\frac{1}{2}$
 $(\gamma^\mu \partial_\mu - m)\psi = 0$ (Dirac matrices: 4×4)
example: electron-positron
 - spin $3/2$ massive or massless
example: Ω^- , gravitino
 - spin 2 massive or massless
example: $f_2^{(1270)}^+$, graviton

4. Internal symmetries

- extra symmetries which characterize a family of particles subject to certain interaction
- not to be confused with spacetime symmetries (kinematical symmetries) with which they commute
- - Hilbert space \equiv tensor product of “spacetime” Hilbert space and “internal” Hilbert space
 - Internal symmetry multiplet \equiv collection of particles of same mass and spin but different internal quantum numbers.

- examples :
 - the charged pion π^\pm $\begin{pmatrix} \phi \\ \phi^* \end{pmatrix}$
 → singlet of space-time ; doublet in the "charge" space
 - the neutron and the proton $\begin{pmatrix} \psi_n \\ \psi_p \end{pmatrix}$
 → spinors of space-time ; isospin doublet : $I = 1/2$
 $SU(2)_{\text{Isospin}}$ is the internal symmetry (referring to strong interaction)
 - the left-handed neutrino and the left-handed electron $\begin{pmatrix} \psi_{\nu_L} \\ \psi_{e_L} \end{pmatrix}$
 → spinors of space-time ; weak-isospin doublet $I_{\text{weak}} = 1/2$
 $SU(2)_{\text{weak}}$ is the internal symmetry of weak interactions
- notice the symmetry breaking : \neq masses

- Internal symmetries are manifestations of interactions : the way a particle transforms under the action of an internal group translates into the way it is charged under the corresponding interaction

- Standard model exact internal symmetry group

$U(1)$ hypercharge \times $SU(2)$ weak \times $SU(3)$ colour local

- Standard model approximate internal symmetries (global)

$SU(2)$ isospin

more and more \downarrow $SU(3)$ plus strangeness

approximate $SU(4)$ plus charm

:

- Because of symmetry breakings (e.g. $U(1)_{\text{hyper}} \times SU(2)_{\text{weak}} \rightarrow U(1)_{\text{electr.}}$)
the mass degeneracy is lifted in multiplets

Inspecting the masses is not enough
to conclude about multiplets.

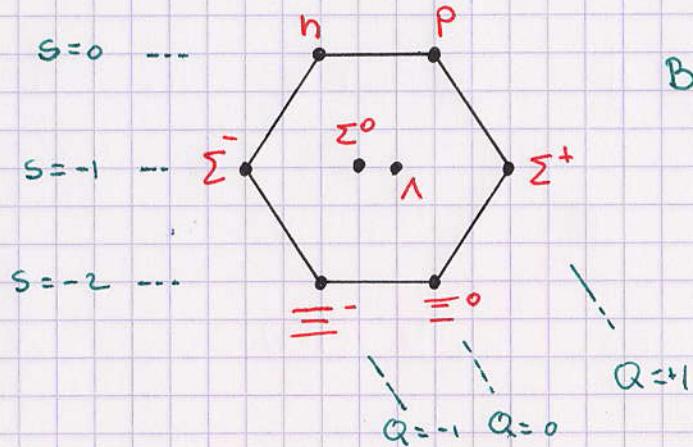
- Key question: is there any symmetry we have missed?

5. Space-time versus internal symmetries

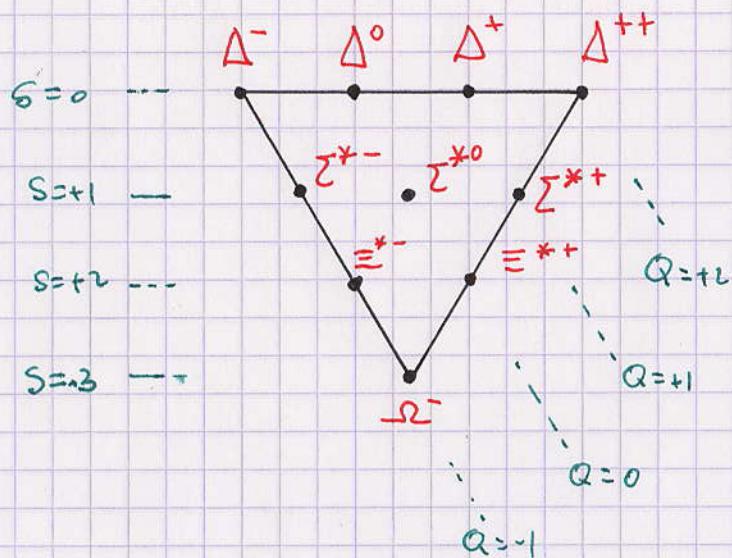
- in the '60's baby-boom of hadronic resonances
 - success of the $SU(3)$ _{flavour} - eightfold way of Gell-Mann and Ne'eman as an internal symmetry
 - classification of particles with respect to their behaviour regarding strong interactions
 - many $SU(3)$ _{flavour} multiplets containing particles of approximately identical mass, same spin but very different charges (electric, strangeness).

→ Total symmetry of nature \equiv Poincaré \otimes internal

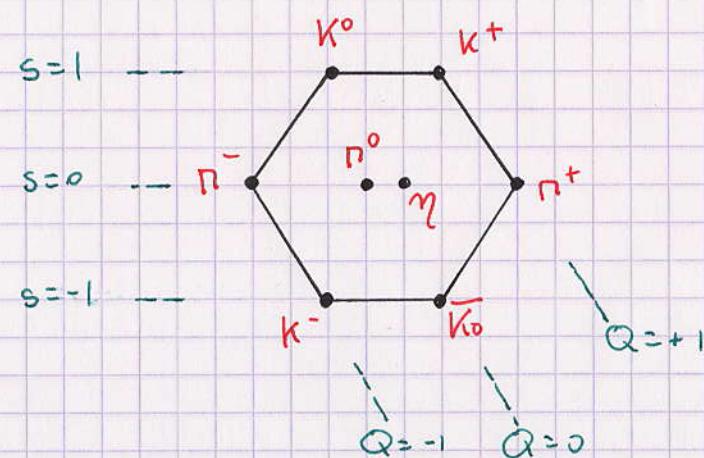
• Exemplos



Baryon octet spin $1/2$



Baryon decuplet
spin $3/2$



meson octet
spin zero

- natural question : Is the symmetry of nature necessarily Poincaré \otimes Internal ?

Could we have a symmetry group
that cannot be put in the
form of a tensor product
although having Poincaré as a subgroup?

- In the first case the multiplets of the internal group are degenerate in mass and spin since they are tensor products of spacetime reprs. with internal reprs.
- In the second case this is no longer true
 \rightarrow multiplets may contain several spins

- A two-stage answer

— 1967

Coleman and Mandula

• assume a collection of generators $\{A_I\} = \{P_S, J_{\mu\nu}, B_A\}$

$$[A_I, A_J] = i f_{IJ}^K A_K$$

with $[P_S, B_A] = 0 \rightarrow$ equal-mass multiplets

$$\text{then } \Rightarrow \{A_I\} = \{P_S, J_{\mu\nu}\} \otimes \{B_A\}$$

(Key point: the Lorentz group is non compact)

therefore $\{B_A\}$ is an ordinary internal symmetry and all members of a multiplet have equal spin and mass

— 1974

Wess and Zumino

Way out and possible extension of space-time symmetry: SUPERSYMMETRY

- How?

Remember that **bosons** are quantized with **commutators**
 whereas **fermions** are quantized with **anti-commutators**
 as a consequence of the Pauli's exclusion principle

- $[a, a^\dagger] = 1$ $|a|_0\rangle = 0$ **bosonic oscillator**

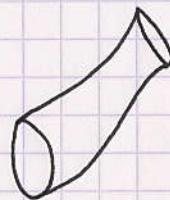
$$[a, a] = [a^\dagger, a^\dagger] = 0 \quad (a^\dagger)^n |0\rangle = |n\rangle \quad n \in \mathbb{N}$$

- $\{b, b^\dagger\} = 1$ $|b|_0\rangle = 0$ **fermionic oscillator**

$$\{b, b\} = \{b^\dagger, b^\dagger\} = 0 \quad (b^\dagger)^n |0\rangle = |n\rangle \quad n=0, 1$$

Mixing **bosons** and **fermions** (integer and half-integer spins)
 in a multiplet requires the algebra to have **commutors**
 and **anticommutators** among its generators

- Historically supersymmetry appeared in two space-time dimensions : necessary symmetry of the dynamics of the fermionic string world-sheet, (Gervais, Sakita)



- Independently discovered by Gol'fand Likhman 1971
Volkov and Akulov 1973

6. Spacetime supersymmetry

- General framework : **graded algebras**
- instead of $[T_I, T_J] = i f_{IJ}^{K} T_K$
 one uses **graded generators** $T_I \rightarrow \text{grade } \eta_I = 0 \text{ or } 1$
 $\rightarrow \eta(T_I T_J \dots) = \eta_I + \eta_J + \dots \pmod{2}$
bosonic or fermionic
- instead of $[,]$ one uses $\{T_I, T_J\} = T_I T_J - (-)^{\eta_I \eta_J} T_J T_I$
 $\rightarrow \{ \text{bosonic, any} \} = \text{commutator}$
 $\{ \text{fermionic, fermionic} \} = \text{anticommutator}$
- the general structure is $\{T_I, T_J\} = i C_{IJ}^{K} T_K$

- Supersymmetry algebras

- Poincaré appears as a subalgebra
- The full algebra is not factorized:

$\exists Q_i \ i=1, \dots, N$ fermionic generators

$[Q_i, P_\mu] = 0 \rightarrow$ multiplets will have the same mass

$[Q_i, J_{\mu\nu}] \neq 0 \rightarrow \gg \gg \gg$ different spins

$\{Q_i, Q_j\} = T_{ij} \rightarrow$ combination of Poincaré and some "internal"

- for $N=1$: minimal supersymmetry } number of supercharges
 $N > 1$: extended supersymmetry } $4N$
in 4 dimensions

- Representations : space-time supermultiplets which contain several multiplets of the Poincaré with equal mass and both $1/2$ integer and integer spins with equal fermionic and bosonic degrees of freedom

- Some examples for massless multiplets in four dimensions

- $N=1$

chiral

2 scalars

1 helicity $\frac{1}{2}$ spinor

vector

1 vector

gauge boson

1 helicity $\frac{1}{2}$ spinor

gaugino

gravitational

1 helicity 2 tensor

graviton

1 helicity $\frac{3}{2}$ vector-spinor

gravitino

- $N=2$

hypermultiplet

2 complex scalars

2 helicity $\frac{1}{2}$ spinors

vector

2 scalars

2 helicity $\frac{1}{2}$ spinors

1 vector

gravitational

1 helicity 2 tensor

2 helicity $\frac{3}{2}$ vector spinor

:

1 vector

graviphoton

- Similarly for massive multiplets

7. Supersymmetric quantum field theories

- Quantum field theory describes
 - a collection of **multiplets** of the full symmetry group (space-time and internal)
 - a set of **interaction terms** (**vertices**) allowed by the symmetry

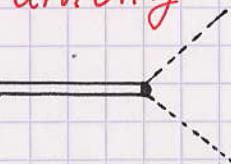
example: a scalar field and Poincaré $\times \mathbb{Z}_2$

$$\mathcal{L} = -\frac{1}{2} m^2 \phi^2 - \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

$$V(\phi) = \lambda_4 \phi^4 + \lambda_6 \cancel{\phi^6} + \dots \cancel{\phi^8} \quad \text{even powers}$$

- If we require **supersymmetry** the lagrangian density is constrained and determines **couplings among bosons and fermions**: Yukawa's

$\Phi \bar{\Psi} \Psi$



\Rightarrow more predictive theory

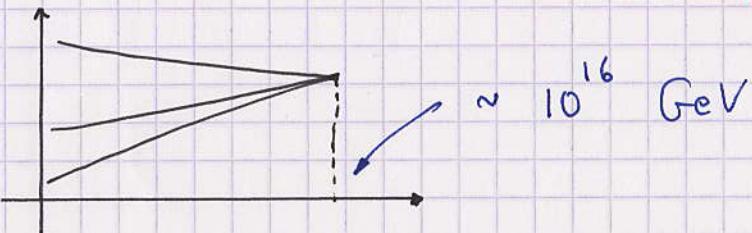
- The ultraviolet behavior of the theory is **improved** (less severe ultraviolet divergences).
 - Many phenomenological consequences have triggered interest for experimental search
- Some popular marketing items :

- Supersymmetry improves the hierarchy problem

$$m_e \sim 1/2 \text{ MeV} \quad m_W \sim 100 \text{ GeV} \quad m_{\text{Higgs}} \sim 150 \text{ GeV (?)}$$

$$M_{\text{Planck}} \sim 10^{19} \text{ GeV}$$

- supersymmetry suggests the unification of interactions



- More symmetry \Rightarrow more constraints \Rightarrow more predictive

- Many practical open problems
 - many more parameters than the standard model
 - none of the observed particles can be superpartner of another → they all define supermultiplets with missing partners at higher energies
→ Supersymmetry must be broken
 - How? Where? at LHC?
- many formal challenges
 - demanding local - supersymmetry invariance brings new degrees of freedom graviton, gravitinos,...
 - are supergravities better-behaved than general relativity in the ultraviolet?

Yes, supergravities have less
severe ultraviolet divergences,
but are still non-renormalizable !

They however arise as
low-energy effective theories
of superstring theories !

PART 2 : STRING THEORY

1. Again back to the 60's

- Despite the hadronic baby-boom and the success of the "eight fold way" for classifying the hadrons in $SO(3)$ _{flavour} multiplets very little was known about the strong interactions !
- Questions :
 - Compute the S-matrix elements ?
Questionable whether a field theory could do the job !
 - Were the hadrons elementary ?

- The eight fold way was suggesting the existence of an underlying structure à la Mendeleev periodic classification

Missing particles in some multiplets were even found (e.g. Ω^-)

- This led to the Quark model (u, d, s) of Gell-Mann and Zweig (1964).
- But not a single experimental evidence came before the deep-inelastic scattering experiments (SLAC very late '60's)
- Alternative models with elementary hadrons were flourishing with the power of computing S-matrix:

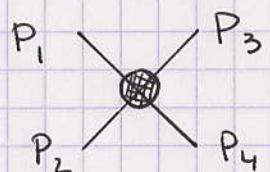
The DUAL models

2. The essence of dual models

- Aim: compute scattering amplitudes
- Axioms:
 - duality in kinematic channels $A(s,t) = A(t,s)$
 - soft (exponentially decreasing) behaviour with energy
 - Existence of an infinite number of elementary hadrons
→ infinite number of poles in the amplitudes
 - Relation among mass and spin: Regge trajectories

$$\alpha = \alpha(m^2) = \alpha_0 + \alpha' m^2 \quad \alpha' \approx 1 \text{ GeV}^{-2}$$

- Output: Veneziano amplitude (1968) and many variations of



$$A(s,t) = \Gamma(-\alpha(s)) \Gamma(-\alpha(t)) / \Gamma(-\alpha(s) - \alpha(t))$$

$$s = -(P_1 + P_2)^2 \quad t = -(P_2 + P_3)^2 \quad u = -(P_1 + P_3)^2$$

- Intriguing features:
 - no local field theory could reproduce such a behaviour!
 - a string theory was proven to do so! (Nambu, Nielsen, Svartholm 1970)
- "fortunately"
 - deep-inelastic-scattering experiments demonstrated the existence of partons: hadron constituents, "free" inside them
 - gauge theories were proven to be renormalizable
 - QCD was developed around $SU(3)$ colour and proven to be asymptotically free.
- string theory further developed as a unification theory

3. String dynamics versus particle dynamics

- Relativistic particle dynamics ($c = \hbar = 1$)

 A hand-drawn sketch of a curve representing a world-line, starting at point i and ending at point f .

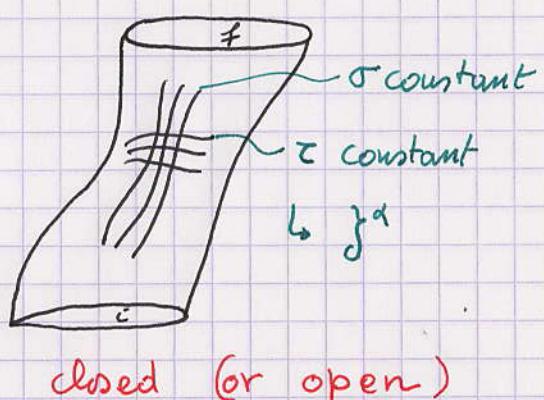
$$S = -m \int_i^f ds = -m \int_i^f d\tau \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad \dot{x}^\mu = dx^\mu / d\tau$$

$\delta S = 0 \Leftrightarrow$ minimal world-line length

m : mass

small m : quantum regime

- Relativistic string dynamics : Nambu-Goto action



$$S = -T \int_i^f dA = -T \int_i^f d\tau d\sigma \sqrt{-\det \hat{g}} \quad \hat{g}_{\alpha\beta} = \frac{\partial X^\mu}{\partial \tau^\alpha} \frac{\partial X^\nu}{\partial \tau^\beta} g_{\mu\nu}$$

$\delta S = 0 \Leftrightarrow$ minimal world-sheet area

T : string tension $\equiv 1/(2\pi\alpha')$

4. The string classical motion

- Remarks about symmetries

- in a covariant formulation, the effective number of degrees of freedom is less than the naïve counting

- free particle $3 x^i, 3 p^i$

- although x^μ and p^μ are 8

reason: reparametrization invariance $\zeta \rightarrow \tilde{\zeta} = f(\zeta)$

↳ set $\dot{x}^\mu \dot{x}^\nu \gamma_{\mu\nu} = -1 \Leftrightarrow p^2 = m^2$

↳ use residual symmetry to eliminate x^0 .

- free string D-2 independent directions

- due to $s^\alpha \rightarrow \tilde{s}^\alpha (s^\beta)$

- other example $A_\mu(x)$ has 2 polarizations: $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$

- The action for the free string is invariant under space-time Poincaré symmetry:

$$X^\mu \rightarrow X^\mu + a^\mu \quad X^\mu \rightarrow \lambda^\mu \nu X^\nu$$

Upon quantization, the Hilbert space will be a collection of states, representations of the Poincaré group with different spins and masses.

- Classical solutions for the closed string

- $X^\pm = X^0 \pm X^1$ are eliminated (expressed in terms of the others)

$$X^i(\sigma, z) = x_0^i + 2\alpha' p^i z + i \sum_{n \neq 0} \sqrt{\frac{\alpha'}{2}} \frac{1}{n} d_n^i e^{-2in(z-\sigma)}$$

$$(d_n^{i*} = \alpha_{-n}^i, \tilde{\alpha}_n^{i*} = \tilde{\alpha}_{-n}^i)$$

$$+ i \sum_{n \neq 0} \sqrt{\frac{\alpha'}{2}} \frac{1}{n} y_n^i e^{-2in(z+\sigma)}$$

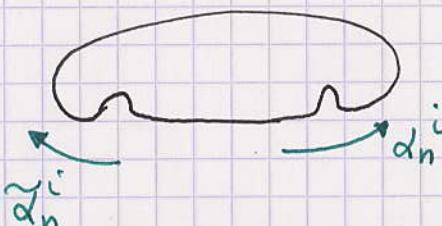
a_n^i , \tilde{a}_n^i , x_0^M , p^M are independent constants modulo

$$-P_\mu P^\mu = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^j S_{ij} = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^j S_{ij}$$

↑
 mass-shell ↑
 level-matching

• Comments

- $x_0^k + 2\alpha' p^k c$: motion of the center of mass
 - α_n^i : complex amplitudes for the right-moving modes
 - $\tilde{\alpha}_n^i$: " " " " Left - " "



- The string oscillations carry energy → for given values of the set $\{d_n^i\}$ and $\{\tilde{a}_n^i\}$, the string is in a Poincaré representation with mass

 Regge trajectories



$$M^2 = \frac{4}{\alpha'} \sum_{n=1}^{\infty} d_n^i d_n^j \tilde{a}_n^i \tilde{a}_n^j \delta_{ij}$$

- The string will also carry < spin > as a difference between left and right oscillations

● Hadronic string versus modern string

- In dual models for hadronic physics : $d' \sim 1/\text{GeV}^2$
 - Modern string theory aims at understanding gravity. When quantized a massless spin 2 particle appears. Further analysis shows
-
- that it provides the relevant interactions for being the graviton

$$d' \sim 1/M_{\text{Planck}}^2$$

$$M_{\text{Planck}} \sim 10^{19} \text{ GeV}$$

5. Features, achievements and open questions of modern quantum string

- Upon quantization

$$D_{\text{cr}} = 26$$

[Bosonic string]

- adding extra world-sheet degrees of freedom: γ^μ (Neveu-Schwarz, Ramond)

$$D_{\text{cr}} = 10$$

& space-time bosons and fermions

- Bonus nr 1 : Space-time Supersymmetry [Gliozzi, Scherk, Olive]

5 theories
at $D_{\text{cr}} = 10$

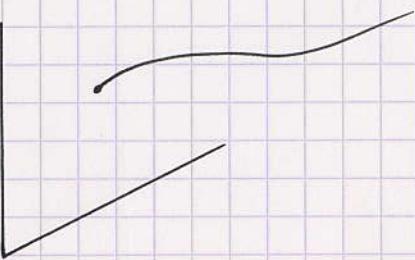
5 theories at $D_{\text{cr}} = 10$	type I	$N=1$
	Heterotic $E_8 \times E_8$ or $SO(32)$	$N=1$
	type IIA non-chiral	$N=2$
	type IIB chiral	$N=2$

- Bonus nr 2 : Spectrum and interactions are given

- Studying the low-energy behaviour and interactions:

ordinary field theory : SUPERGRAVITY THEORY

- Adding D-branes as dynamical objects



allows to establish unexpected duality properties among the theories

→ M-theory

- Goal n° 1 : phenomenology

- compactify $10 \rightarrow 4$ and dynamically justify the pattern
- fit the known spectrum of elementary particles
- break the space-time supersymmetry, keeping $N=1$ around the TeV

- Goal n° 2 : M_{Planck} physics

- black holes, singularities, entropy, ...
- cosmology : Λ , inflation, ...
- holography ...