Acceleration of Cosmic Rays

One of the most intriguing problems in high energy astrophysics is the mechanism by which high energy particles are accelerated to ultrarelativistic energies. The specific features of particle acceleration which we have to account for are as follows:

1. A power-law energy spectrum for particles of all types. The energy spectrum of cosmic rays and the electron energy spectrum of many non-thermal sources have the form :

$$\mathrm{dN}(E) \propto E^{-x}\mathrm{dE},$$

   where the exponent x lies in the range roughly 2.2-2.3. For the cosmic rays, $x = 2.5 - 2.7$ at energies $\sim$TeV, with slightly flatter spectra for primary nuclei such as iron. The typical spectra of radio-sources correspond to electron spectra with $x \approx 2.6$ with a scatter of about 0.4 about this mean value. The continuum spectra of quasars in the optical and X-ray wave-bands correspond to $x \approx 3$.

2. The acceleration of cosmic rays to energies $E \sim 10^{20}$eV.

3. The acceleration mechanism should result in chemical abundances for the cosmic rays which are similar to the cosmic abundances of the elements.

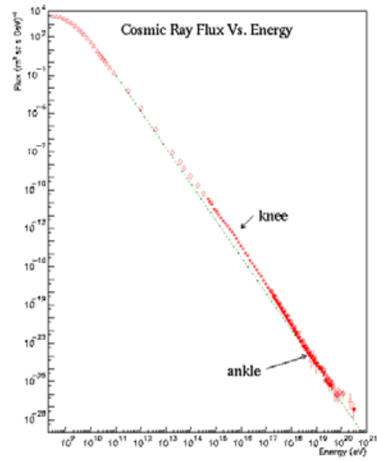

Figure 1. Spectrum of Cosmic Rays as a function of Energy.

1 The Fermi Acceleration - 2nd-Order process

The Fermi mechanism was proposed by Fermi in 1949 as a stochastic means by which particles colliding with coulds in the interstellar medium could be accelerated to high energies.

In Fermi's original picture, charged particles are reflected from magnetic mirrors associated with irregularities in the Galactic magnetic field. The mirrors are assumed to move randomly with typical velocity V, and Fermi showed that the particle gain energy statistically in these reflections. If the particles only remain within the acceleration region for some characteristic time τ_{esc}, a power-law distribution of particle energies is found.

Let's assume that the collision between a particle and a mirror, or massive cloud, takes place such that the angle between the initial direction of the particle and the normal to the surface of the mirror is θ. Let us work out the change of energy of the particle in a single collision. We suppose the cloud is infinitely massive so that its velocity is unchanged in the collision. The centre of momentum frame is therefore that of the cloud moving at velocity V ($\gamma_V = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$).

1. Express the energy of a particle in the moving cloud's frame, as a function of θ and γ_V ;
2. Write down the x-component of the relativistic 3-momentum in the CoM Frame p'_x ;
3. How is the particle's energy transformed during the collision ? What about p'_x ?
4. Transforming back in the observer's frame, find the energy after the collision E'' ;
5. Expand this expression assuming $\frac{V}{c} \ll 1$;
6. We now have to average over all possible angles θ.

Because of scattering by hydro-magnetic waves or irregularities in the magnetic field, it is likely that the particle is randomly scattered in pitch angle between encounters with the clouds. There is a slightly greater probability of head-on encounters as opposed to the following collisions. For simplicity, let us consider the case of a relativistic particle with $v \approx c$. The probability of collision at angle θ is proportional to $\gamma_V\left(1 + \frac{V}{c}\cos\theta\right)$.

The probability of the pitch angle lying in the angular range $[\theta, \theta + d\theta]$ is proportional to $\sin\theta\, d\theta$, so that averaging over all angles the expression found in 5. in the limit $v \to c$, you can express the average gain per collision in this case.

This illustrate the famous result derived by Fermi that the average increase in energy is only second order in $\frac{V}{c}$. It is also immediately apparent that this result leads to an exponential increase in the energy of the particles since the same fractional increase occurs per collision.

Before looking at this part of the calculation a little more deeply, let us complete the essence of Fermi's original argument. If the mean free path between clouds along a field line is L, the time between collisions is $\frac{L}{c\cos\varphi}$, where φ is the pitch angle of the particle with respect to the magnetic field direction. We need to average $\cos\varphi$ over the pitch angle to find the average time between collisions, which is just $\frac{2L}{c}$. Therefore, we find a typical rate of energy increase :

$$\frac{\mathrm{dE}}{\mathrm{dt}} = \frac{4}{3}\frac{V^2}{cL}E = \alpha E.$$

It is assumed that the particles remain in the accelerating region for a characteristic time τ_{esc}. The diffusion-loss equation for the particles is :

$$\frac{\partial N}{\partial t} = \text{Diffusion Term} + \frac{\partial}{\partial E}[\text{EnergyLoss}(E)\, N(E)] - \frac{N}{\tau_{\mathrm{esc}}} + \text{Source Term}.$$

We are interested in the steady-state solution, and hence $\frac{\partial N}{\partial t} = 0$. We will not consider diffusion so that the diffusion term is $= 0$, and since we assume there are no sources, the source term is equal to 0. The energy loss term is $-\frac{\mathrm{dE}}{\mathrm{dt}} = -\alpha E$. Therefore :

$$-\frac{d}{\mathrm{dE}}[\alpha E N(E)] = \frac{N(E)}{\tau_{\mathrm{esc}}}.$$

Finally, :

$$N(E) \propto E^{-x}, \ (\star)$$

where $x = 1 + \frac{1}{\alpha\,\tau_{\mathrm{esc}}}$. Thus we have succeeded in deriving a power-law energy spectrum.

2 Acceleration by 1st-Order Process

We can rewrite the essence of the Fermi acceleration mechanism in a rather simple fashion if we let $E = \beta E_0$ be the average energy of the particle after one collision and P the probability that the particle remains within the accelerating region after one collision. Then, after k collisions, there are $N = N_0 P^k$ particles with energies $E = E_0\beta^k$. If we eliminate k between these quantities,

$$\frac{\ln\left(\frac{N}{N_0}\right)}{\ln\left(\frac{E}{E_0}\right)} = \frac{\ln P}{\ln \beta} \Rightarrow \frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{\frac{\ln P}{\ln \beta}}$$

Therefore $N(E)\mathrm{dE} \propto E^{-1+\frac{\ln P}{\ln \beta}} dE$. It is clear from this formulation that we have again recovered a power law. To make the equivalence between the first and the second versions of Fermi acceleration complete, we see from equation ($\star$) and the definition of β that $\beta = 1 + \frac{\alpha}{M}$, where α/M is the increment of energy per collision and P is related to τ_{esc}.

In the Fermi mechanism α is proportional to $(V/c)^2$, because of the decelerating effect of the following collisions. The original version of Fermi's theory is known as second order Fermi acceleration and is a very slow process. We would do much better if there were only head-on collisions. In this case the energy increase is $\frac{\Delta E}{E} \propto 2\frac{V}{c}$, that is, first order in $\frac{V}{c}$. This is called first order Fermi acceleration.

A very attractive version of the first order Fermi acceleration in the presence of strong shock waves was also proposed.

To illustrate the basic physics of the acceleration process, let's consider the case of a strong shock, for example, that caused by a supernova explosion, propagating through the interstellar medium. A flux of high energy particles is assumed to be present both in front and behind the shock front. The particles are considered to be of very high energy, and so the velocity of the shock is very much less than the velocities of the high energy particles.

The key point about the acceleration mechanisms is that the high energy particles hardly notice the shock at all, since its thickness will normally be very much smaller than the gyroradius of a high energy particle. Because of turbulence behind the shock front and irregularities ahead of it, when the particles pass through the shock in either direction, they are scattered so that their velocity distribution rapidly becomes isotropic on either side of the shock front. The key point is that the distributions are isotropic with respect to the frames of reference in which the fluid is at rest on either side of the shock.

Let us consider the case of a strong shock. This is the case, for example, for the material ejected in supernova explosions, where the velocities can be up to about km/s, compared with the sound speed of the interstellar medium, which is at most about 10 km/s.

In the case of a strong shock, the shock wave travels at a highly supersonic velocity $U \gg c_s$, where c_s is the speed of the sound in the ambient medium. It is often convenient to transform into the frame of reference in which the shock front is at rest, and then the upstream gas flows into the shock front at velocity $v_1 = |U|$ (density ρ_1) and leaves the shock front with a downstream velocity v_2 (density ρ_2).

1. Write the equation of continuity for the mass through the shock ; in the case of a strong shock $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$, where γ is the ratio of specific heats of the gas. Taking γ for a monoatomic gas (or fully ionised gas), **compute $\frac{\rho_2}{\rho_1}$, and express v_2 as a function of v_1** ;

Now let us consider the high energy particles ahead of the shock. Scattering ensures that the particle distribution is isotropic in the frame of reference in which the gas is at rest.

Let us consider the upstream particles first. The shock advances through the medium at velocity U, but the gas behind the shock travels at a velocity $\frac{3}{4}U$ relative to the upstream gas. When a high energy particle crosses the shock front, it obtains a small increase in energy of the order $\frac{\Delta E}{E} \propto \frac{U}{c}$. The particles are then scattered by the turbulence behind the shock front, so that their velocity distribution become isotropic with respect to that flow.

Now let's consider the opposite process of the particle diffusing from behind the shock to the upstream region in front of the shock. Now the velocity distribution of the particles is isotropic, behind the shock, and, when they cross the shock front, they encounter gas moving towards the shock front, again with the same velocity $\frac{3}{4}U$. In other words, the particle undergoes exactly the same process of receiving a small increase ΔE in energy on crossing the shock from downstream to upstream as it did in traveling from upstream to downstream.

This is the clever aspect of this acceleration mechanism. Every time the particle crosses the shock front it receives an increase in energy-and the increment in energy is the same going in both directions. Thus, unlike the standard Fermi mechanism, in which there are both head-on and following collisions, in the case of the shock front, the collisions are always head on and the energy is transferred to the particles. The beauty of the mechanism is the complete symmetry between the passage of the particles from upstream to downstream and from downstream to upstream through the shock wave.

3. Let us consider the process of acceleration in a somewhat more quantitative way. We can work out the expressions of β and P for this cycle by using some simple arguments. First, we evaluate the average increase in the energy of the particle on crossing from the upstream to the downstream sides of the shock. The gas on the downstream side approaches the particle at a velocity $V = \frac{3}{4}U$. **By performing a Lorentz transformation find the particle's energy when it passes into the downstream region (taking the x-coordinate perpendicular to the shock)** ;

4. We assume that the shock is non-relativistic so that $V \ll c$, $\gamma_V \approx 1$, but that the particles are relativistic, so that $E \approx \mathrm{pc}$ and $p_x = \frac{E}{c}\cos\theta$. **Rewrite the energy gain $\boldsymbol{\Delta E}$ and $\boldsymbol{\frac{\Delta E}{E}}$;**

5. We now seek the probability that the particles which cross the shock waves arrive at an angle θ per unit time. This is a standard piece of kinetic theory. The number of particles within the angle θ to $\theta + d\theta$ is proportional to $\sin\theta d\theta$, but the rate at which they approach the shock front is proportional to the x-component of their velocities, $c\cos\theta$. Therefore the probability of the particle crossing the shock is proportional to $\sin\theta d\theta c\cos\theta$. Normalizing so that the integral of the probability distribution over all the particles approaching the shock is equal to unity, that is, those with $\frac{\theta}{2}$ in the range 0 to $\frac{\pi}{2}$, we find $p(\theta) = 2\sin\theta\cos\theta d\theta$. **You can now write the average gain in energy on crossing the shock**.

The particle's velocity vector is randomized without any energy loss by scattering in the downstream region and it then re-crosses the shock, when it gains another fractional increase in energy $\frac{2V}{3c}$, so that, in making one round trip across the shock and back again, the fractional energy increase is, on average :

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3}\frac{V}{c}.$$

Consequently, $\beta = \frac{E}{E_0} = 1 + \frac{4V}{3c}$ in one round trip.

According to classical particle theory, the number of particles crossing the shock is $Nc/4$, where N is the number density of particles. This is the average number of particles crossing the shock in either direction, since, as noted above, the particles scarcely notice the shock. Downstream, however, the particles are swept away or "advected" from the shock, because the particles are isotropic in that frame. It can be seen that the particles are removed from the region of the shock at a rate $NV = N\frac{U}{4}$. Thus, the fraction of the particles lost per unit time is $\frac{NU/4}{Nc/4} = \frac{U}{c}$. Since we assume that the shock is non-relativistic, it can be seen that only a very small fraction of the particles is lost per cycle. Thus $P = 1 - \frac{U}{c}$. This solves the problem since we need $\ln\beta$ and $\ln P$ to insert into Eq. (9). Therefore, since $\ln P = \ln(1 - \frac{U}{c}) \approx -\frac{U}{c}$ and $\ln\beta = \ln(1 + \frac{4V}{3c}) \approx \frac{4V}{3c} = \frac{U}{c}$, one finds :

$$\frac{\ln P}{\ln\beta} = -1,$$

and hence the differential energy spectrum of the high energy particles is :

$$N(E)\,\mathrm{dE} \propto E^{-2}\mathrm{dE}.$$

This is the result we have been seeking. It may be objected that we have obtained a value of 2 rather than 2.5 for the exponent of the differential energy spectrum, and that problem cannot be neglected.

3 Maximum attainable Energy - Hillas Plot

Derive, by using Maxwell's equations, the maximum amount of energy a particle of charge Ze can attain in a magnetic field B and accelerating scale L.

What is the Larmor Radius of a particle of charge Ze in a magnetic field B as a function of its energy E ? What constraint does that impose on the accelerating scale L ? How those 2 relationships compare ?

All estimates lead to $E_{\max} \sim \Gamma Z e B R$ (where Γ is the Lorentz factor in the shocks, in *e.g.* GRBs) - Hillas results (see the plot).

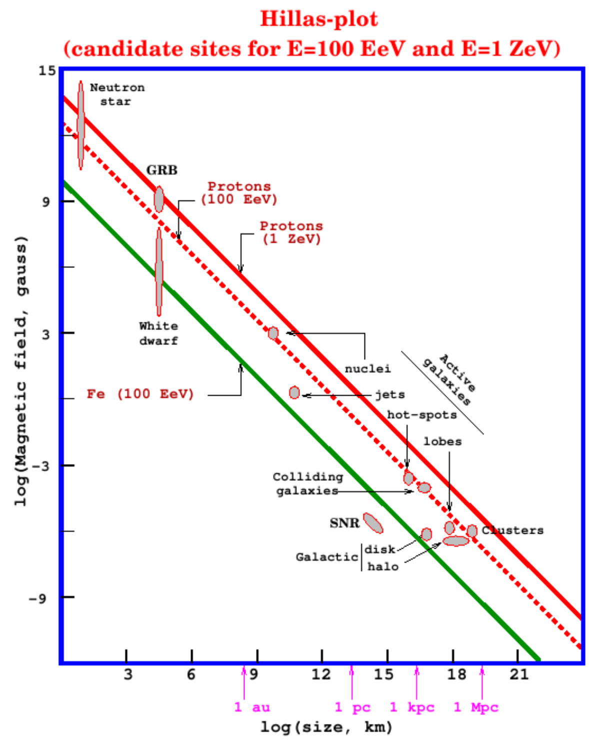

Figure 2. The So-called Hillas-Plot.

4 GZK Cut-off

The cosmic medium is filled by background radiation of relic photons, left over from the big bang, of typical energy 10^{-3}eV. We consider the propagation of a high energy proton through this medium.

Let $p_\mu = (\varepsilon, \vec{p})$ be the photon 4-momentum. Let $P_\mu = (E, \vec{P})$ be the proton 4-momentum. Evaluate the center-of-mass energy ε_{CM}.

Calculate the threshold for the reaction $\gamma + p \rightarrow \Delta \rightarrow \pi + N$ to occur.

This results in a mean free path for protons of about 100 Mpc at 10^{20}eV. This is called the Gresein-Zatsepin-Kuzmin cutoff, which is seen in the following figure.

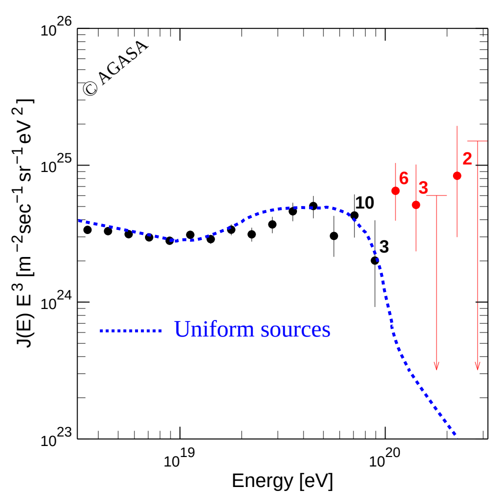

Figure 3. UHECR Spectrum as measured by AGASA.