SUPERSYMMETRY AND STRING THEORY

ESU – Particles and the Universe

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Strasbourg July 2006

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<u>Reminder</u>: assuming a Lagrangian density $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$, the Euler–Lagrange equations read:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0,$$

for each field ϕ_i.

1. The Poincaré algebra

Using the expressions for the generators P_μ and $J_{\mu\nu}$ in terms of space-time derivative operators acting on scalar functions, check the commutation relations of the Poincaré algebra.

2. Graded algebras

Show that the structure constants of a graded algebra satisfy

$$C_{JI}{}^{K} = -(-1)^{\eta_I \eta_J} C_{IJ}{}^{K}.$$

Determine explicitly the quadratic relation among these structure constants using the super-Jacobi identity:

$$\begin{aligned}(-1)^{\eta_K \eta_I} \{\{T_I, T_J\}, T_K\} + (-1)^{\eta_I \eta_J} \{\{T_J, T_K\}, T_I\} \\ +(-1)^{\eta_J \eta_K} \{\{T_K, T_I\}, T_J\} = 0.\end{aligned}$$

3. The Wess–Zumino model

The Wess–Zumino Lagrangian density is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}\partial_\mu B \partial^\mu B - \frac{1}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi \\ &\quad + \frac{1}{2}\left(F^2 + G^2\right) + m\left[FA + GB - \frac{1}{2}\bar{\psi}\psi\right] \\ &\quad + g\left[F\left(A^2 + B^2\right) + 2GAB - \bar{\psi}(A + i\gamma_5 B)\psi\right].\end{aligned}$$

The dynamical fields are two scalars A and B and a Majorana spinor (i.e. a self-charge-conjugate Dirac spinor) ψ; F and G are auxiliary fields.

– Show that the above density is invariant under the following supersymmetry transformations:

$$\delta A = \bar{\alpha}\psi, \quad \delta B = -i\bar{\alpha}\gamma_5\psi,$$
$$\delta\psi = \partial_\mu(A + i\gamma_5 B)\gamma^\mu\alpha + (F - i\gamma_5 G)\alpha,$$
$$\delta F = \bar{\alpha}\gamma^\mu\partial_\mu\psi, \quad \delta G = -i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi,$$

where α,$\bar{\alpha}$ are Grassmanian (anti-commuting) infinitesimal parameters in a Majorana spinor representation, γ_μ are the ordinary Dirac matrices and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

– Eliminate the non-propagating auxiliary fields from the Lagrangian by using their equations of motion.

4. Free particle

Show the classical equivalence of the following free-particle actions:

$$S_1 = -m\int_{\mathrm{i}}^{\mathrm{f}} \mathrm{d}s$$

and

$$S_2 = m\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} \mathrm{d}\tau\, \eta_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu,$$

where m is the mass, ds the world-line length element and τ a time-like parameter for the curve ($\dot{} \equiv \mathrm{d}/\mathrm{d}\tau$).

5. The Nambu–Goto's action

Show that the Nambu–Goto's action for a free string of tension T reads explicitly:

$$S_1 = -T\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} \mathrm{d}\tau \int_0^{\sigma_\mathrm{end}} \mathrm{d}\sigma \sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2\, X'^2},$$

where $\dot{X}^\mu \equiv \partial X^\mu/\partial\tau$ and $X^{\mu\prime} \equiv \partial X^\mu/\partial\sigma$.

– Write down the equations of motion and show that, when satisfied, the variation of the action under $X^\mu \to X^\mu + \delta X^\mu(\tau,\sigma)$ is

$$\delta S_1 = \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} \mathrm{d}\tau \left[\delta X^\mu \mathcal{P}^\sigma_\mu\right]_0^{\sigma_\mathrm{end}}$$

where $\mathcal{P}^\sigma_\mu \equiv \partial\mathcal{L}/\partial X^{\mu\prime}$.

– Discuss the boundary conditions of open versus closed strings.

– Choose a parameterization where

$$\dot{X} \cdot X^{'} = 0 \quad \text{and} \quad \dot{X}^2 + X^{'2} = 0$$

and show that the equations of motion then read:

$$\ddot{X}^\mu - X^{''\mu} = 0.$$

– Solve these equations for closed string and open string with free endpoints (i.e. with boundary conditions $\mathcal{P}^\sigma_\mu(\tau,0) = \mathcal{P}^\sigma_\mu(\tau,\sigma_{\text{end}}) = 0$).

6. The Polyakov's action

Consider the action

$$S_2 = -\frac{T}{2} \int_{\rm i}^{\rm f} \mathrm{d}^2\zeta \, \sqrt{h} \, h^{\alpha\beta} \, \partial_\alpha X^\mu \, \partial_\beta X^\nu \, \eta_{\mu\nu},$$

where $h_{\alpha\beta}$ is the two-dimensional intrinsic world-sheet metric and $h = -\det\{h_{\alpha\beta}\}$.

– Show that this action is invariant under
  – reparameterizations $\zeta^\alpha \to \tilde{\zeta}^\alpha\left(\zeta^\beta\right)$,
  – Weyl rescalings $h_{\alpha\beta} \to \Omega\left(\zeta^\beta\right) h_{\alpha\beta}$.

– Write down the equations of motion for the fields $X^\mu\left(\zeta^\beta\right)$ and $h_{\alpha\beta}\left(\zeta^\beta\right)$.

Further reading

- *The Quantum Theory of Fields*, Volume III Supersymmetry, Steven Weinberg, Cambridge, 2000
- *A First Course in String Theory*, Barton Zwiebach, Cambridge, 2004