An overview of Cosmology

CERN Student Summer School

7-9 August 2002

Julien Lesgourgues
(CERN Theory Division
& LAPTH, Annecy)

**This draft is based on the full lectures, and includes
additional considerations on inflation and quintessence at the end.**

**Please don't hesitate to report any mistake or misprint to
Julien.Lesgourgues@cern.ch.**

What is the difference between astrophysics and cosmology?

While astrophysicists study the surrounding celestial bodies, like planets, stars, galaxies, clusters of galaxies, gas clouds, etc., cosmologists try to describe the evolution of the Universe as a whole, on the largest possible distances and time scales. While purely philosophical in the early times, and still very speculative at the beginning of the XX-th century, cosmology has gradually entered into the realm of experimental science over the past 80 years. Today, as we will see in chapter 2, astronomers are even able to obtain very precise maps of the surrounding Universe a few billion years ago, i.e. over distances ranging up to billions of light-years.

Cosmology has raised some fascinating questions like: is the Universe static or expanding ? How old is it and what will be its future evolution ? Is it flat, open or closed ? Of what type of matter is it composed ? How did structures like galaxies form ? In this course, we will try to give an overview of these questions, and of the partial answers that can be given today.

In the first chapter, we will introduce some fundamental concepts, in particular from General Relativity. Along this chapter, we will remain in the domain of abstraction and geometry. In the second chapter, we will apply these concepts to the real Universe and deal with concrete results, observations, and testable predictions.

Contents

Chapter 1

The Expanding Universe

1.1 The Hubble Law

1.1.1 The Doppler effect

At the beginning of the XX-th century, the understanding of the global structure of the Universe beyond the scale of the solar system was still relying on pure speculation. In 1750, with a remarkable intuition, Thomas Wright noticed that the luminous stripe observed in the night sky and called the Milky Way could be a consequence of the spatial distribution of stars: they could form a thin plate, what we call now a galaxy. At that time, with the help of telescopes, many faint and diffuse objects had been already observed and listed, under the generic name of nebulae - in addition to the Andromeda nebula which is visible by eye, and has been known many centuries before the invention of telescope. Soon after the proposal of Wright, the philosopher Emmanuel Kant suggested that some of these nebulae could be some other clusters of stars, far outside the Milky Way. So, the idea of a galactic structure appeared in the mind of astronomers during the XVIII-th century, but even in the following century there was no way to check it on an experimental basis.

At the beginning of the nineteenth century, some physicists observed the first spectral lines. In 1842, Johann Christian Doppler argued that if an observer receives a wave emitted by a body in motion, the wavelength that he will measure will be shifted proportionally to the speed of the emitting body with respect to the observer (projected along the line of sight):

$$\Delta\lambda/\lambda = \vec{v}.\vec{n}/c \tag{1.1}$$

where c is the celerity of the wave (See figure 1.1). He suggested that this effect could be observable for sound waves, and maybe also for light. The later assumption was checked experimentally in 1868 by Sir William Huggins, who found that the spectral lines of some neighboring stars were slightly shifted toward the red or blue ends of the spectrum. So, it was possible to know the projection along the line of sight of star velocities, v_r, using

$$z \equiv \Delta\lambda/\lambda = v_r/c \tag{1.2}$$

where z is called the redshift (it is negative in case of blue-shift) and c is the speed of light. Note that the redshift gives no indication concerning the distance of the star. At the beginning of the XX-th century, with increasingly good instruments, people could also measure the redshift of some nebulae. The first measurements, performed on the brightest objects, indicated some arbitrary distribution of red and blue-shifts, like for stars. Then, with more observations,
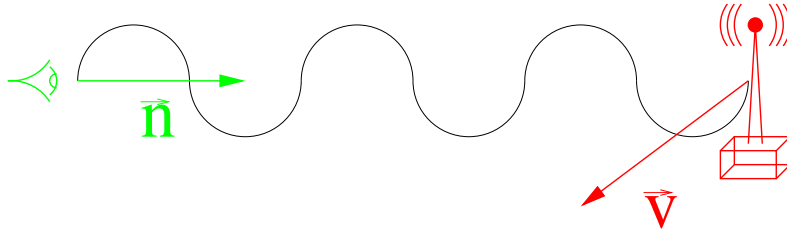
Figure 1.1: The Doppler effect

it appeared that the statistics was biased in favor of red-shifts, suggesting that a majority of nebulae were going away from us, unlike stars. This was raising new questions concerning the distance and the nature of nebulae.

1.1.2 The discovery of the galactic structure

In the 1920's, Leavitt and Shapley studied some particular stars, called the cepheids, known to have a periodic time-varying luminosity. They could show that the period of cepheids is proportional to their absolute luminosity L (the absolute luminosity is the total amount of light emitted by unit of time, i.e., the flux integrated on a closed surface around the star). They were also able to give the coefficient of proportionality. So, by measuring the apparent luminosity, i.e. the flux l per unit of surface through an instrument pointing to the star, it was easy to get the distance of the star r from

$$L = l \times \left(4\pi r^2\right) \ . \tag{1.3}$$

Using this technique, it became possible to measure the distance of various cepheids inside our galaxies, and to obtain the first estimate of the characteristic size of the stellar disk of the Milky Way (known today to be around 80.000 light-years).

But what about nebulae? In 1923, the 2.50m telescope of Mount Wilson (Los Angeles) allowed Edwin Hubble to make the first observation of individual stars inside the brightest nebula, Andromeda. Some of these were found to behave like cepheids, leading Hubble to give an estimate of the distance of Andromeda. He found approximately 900.000 light-years (but later, when cepheids were known better, this distance was established to be around 2 million light-years). That was the first confirmation of the galactic structure of the Universe: some nebulae were likely be some distant replicas of the Milky Way, and the galaxies were separated by large voids.

1.1.3 The Cosmological Principle

This observation, together with the fact that most nebulae are redshifted (excepted for some of the nearest ones like Andromeda), was an indication that on the largest observable scales, the Universe was expanding. At the beginning, this idea was not widely accepted. Indeed, in the most general case, a given dynamics of expansion takes place around a center. Seeing the Universe in expansion around us seemed to be an evidence for the existence of a center in the Universe, very close to our own galaxy.

Until the middle age, the Cosmos was thought to be organized around mankind, but the common wisdom of modern science suggests that there should
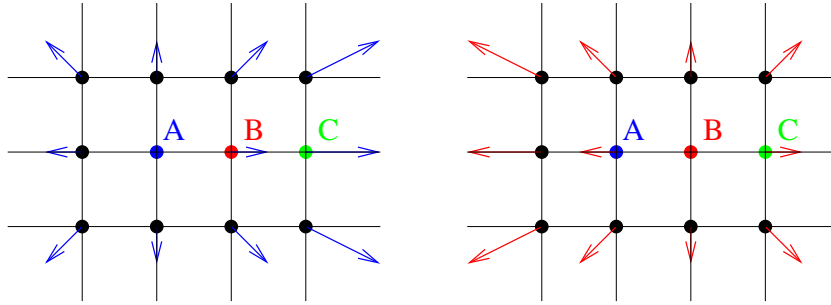
Figure 1.2: Homogeneous expansion on a two-dimensional grid. Some equally-spaced observers are located at each intersection. The grid is plotted twice. On the left, the arrays show the expansion flow measured by A; on the right, the expansion flow measured by B. If we assume that the expansion is homogeneous, we get that A sees B going away at the same velocity as B sees C going away. So, using the additivity of speeds, the velocity of C with respect to A must be twice the velocity of B with respect to A. This shows that there is a linear relation between speed and distance, valid for any observer.

be nothing special about the region or the galaxy in which we leave. This intuitive idea was formulated by the astrophysicist Edward Arthur Milne as the "Cosmological Principle": the Universe as a whole should be homogeneous, with no privileged point playing a particular role.

Was the apparently observed expansion of the Universe a proof against the Cosmological Principle? Not necessarily. The homogeneity of the Universe is compatible either with a static distribution of galaxies, or with a very special velocity field, obeying to a linear distribution:

$$\vec{v} = H \ \vec{r} \tag{1.4}$$

where $\vec{v}$ denotes the velocity of an arbitrary body with position $\vec{r}$, and H is a constant of proportionality. An expansion described by this law is still homogeneous because it is left unchanged by a change of origin. To see this, one can make an analogy with an infinitely large rubber grid, that would be stretched equally in all direction: it would expand, but with no center (see figure 1.2). This result is not true for any other velocity field. For instance, the expansion law

$$\vec{v} = H \ |\vec{r}| \ \vec{r} \tag{1.5}$$

is not invariant under a change of origin: so, it has a center.

1.1.4 Hubble's discovery

So, a condition for the Universe to respect the Cosmological Principle is that the speed of galaxies along the line of sight, or equivalently, their redshift, should be proportional to their distance. Hubble tried to check this idea, still using the cepheid technique. He published in 1929 a study based on 18 galaxies, for which he had measured both the redshift and the distance. His results were showing roughly a linear relation between redshift and distance (see figure 1.3). He concluded that the Universe was in homogeneous expansion, and gave the first estimate of the coefficient of proportionality H, called the Hubble parameter.

This conclusion has been checked several time with increasing precision and is widely accepted today. It can be considered as the starting point of experimental cosmology. It is amazing to note that the data used by Hubble was
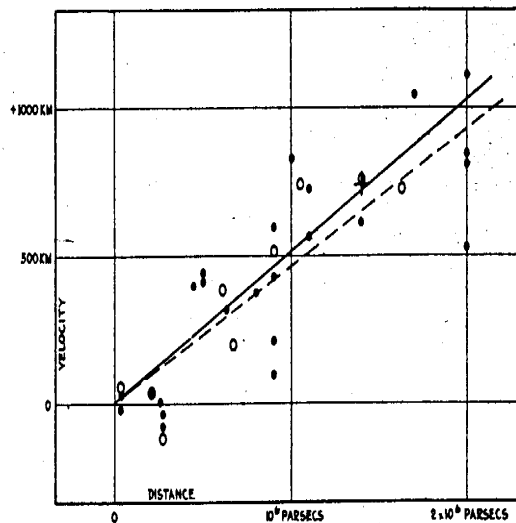
FIGURE 1

Figure 1.3: The diagram published by Hubble in 1929. The labels of the horizontal (resp. vertical) axis are 0, 1, 2 Mpc (resp. 0, 500, 1000 km.s^{-1}). Hubble estimated the expansion rate to be 500 km.s^{-1}Mpc^{-1}. Today, it is known to be around 70 km.s^{-1}Mpc^{-1}.

so imprecise that Hubble's conclusion was probably a bit biaised... Anyway, current data leaves no doubt about the proportionality, even if there is still an uncertainty concerning the exact value of H. The Hubble constant is generally parametrized as

$$H = 100 \ h \ \text{km s}^{-1}\text{Mpc}^{-1} \tag{1.6}$$

where h is the dimensionless "reduced Hubble parameter", currently known to be in the range $h = 0.67 \pm 0.15$, and Mpc denotes a Mega-parsec, the unity of distance usually employed for cosmology (1 Mpc $\simeq 3 \times 10^{22}$m $\simeq 3 \times 10^{6}$ light-years). So, for instance, a galaxy located at 10 Mpc goes away at a speed close to 700 km s^{-1}.

1.1.5 Homogeneity and inhomogeneities

Before leaving this section, we should clarify one point about the "Cosmological Principle", i.e., the assumption that the Universe is homogeneous. Of course, nobody has ever claimed that the Universe was homogeneous on small scales, since compact objects like planets or stars, or clusters of stars like galaxies are inhomogeneities in themselves. The Cosmological Principle only assumes homogeneity after smoothing over some characteristic scale. By analogy, take a grid of step l (see figure 1.4), and put one object in each intersection, with a randomly distributed mass (with all masses obeying to the same distribution of probability). Then, make a random displacement of each object (again with all
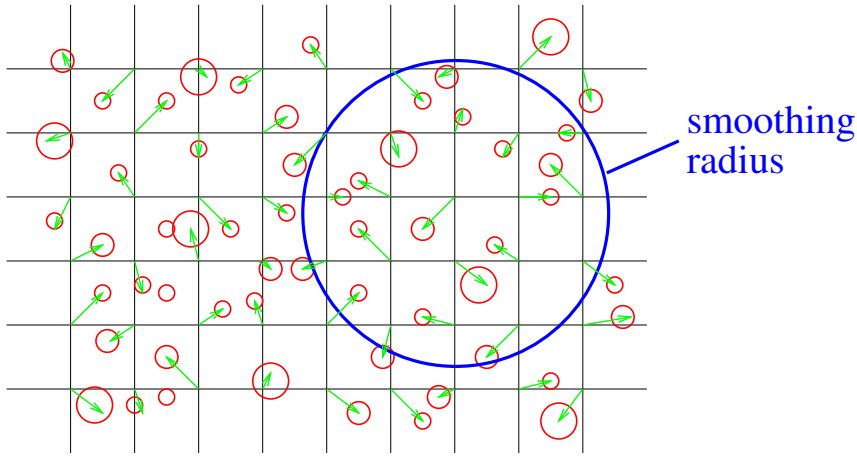
7

Figure 1.4: We build an inhomogeneous distribution of objects in the following way: starting from each intersection of the grid, we draw a random vector and put an object of random mass at the extremity of the vector. Provided that all random vectors and masses obey to the same distributions of probability, the mass density is still homogeneous when it is smoothed over a large enough smoothing radius (in our example, the typical length of the vectors is smaller than the step of the grid; but our conclusion would still apply if the vectors were larger than the grid step, provided that the smoothing radius is even larger). This illustrates the concept of homogeneity on a given scale, like in the Universe.

displacements obeying to the same distribution of probability). At small scales, the mass density is obviously inhomogeneous for three reasons: the objects are compact, they have different masses, and they are separated by different distances. However, since the distribution has been obtained by performing a random shift in mass and position, starting from an homogeneous structure, it is clear even intuitively that the mass density smoothed over some large scale will remain homogeneous again.

The Cosmological Principle should be understood in this sense. Let us suppose that the Universe is almost homogeneous at a scale corresponding, say, to the typical intergalactic distance, multiplied by thirty or so. Then, the Hubble law doesn't have to be verified exactly for an individual galaxy, because of peculiar motions resulting from the fact that galaxies have slightly different masses, and are not in a perfectly ordered phase like a grid. But the Hubble law should be verified in average, provided that the maximum scale of the data is not smaller than the scale of homogeneity. The scattering of the data at a given scale reflects the level of inhomogeneity, and when using data on larger and larger scales, the scattering must be less and less significant. This is exactly what is observed in practice. An even better proof of the homogeneity of the Universe on large scales comes from the Cosmic Microwave Background, as we shall see in section 2.2.

We will come back to these issues in section 2.2, and show how the formation of inhomogeneities on small scales are currently understood and quantified within some precise physical models.

1.2 The Universe Expansion from Newtonian Gravity

It is not enough to observe the galactic motions, one should also try to explain it with the laws of physics.

1.2.1 Newtonian Gravity versus General Relativity

On cosmic scales, the only force expected to be relevant is gravity. The first theory of gravitation, derived by Newton, was embedded later by Einstein into a more general theory: General Relativity (thereafter denoted GR). However, in simple words, GR is relevant only for describing gravitational forces between bodies which have relative motions comparable to the speed of light[1]. In most other cases, Newton's gravity gives a sufficiently accurate description.

Precisely, the speed of neighboring galaxies is always much smaller than the speed of light. So, *a priori*, Newtonian gravity should be able to explain the Hubble flow. One could even think that historically, Newton's law led to the prediction of the Universe expansion, or at least, to its first interpretation. Amazingly, and for reasons which are more mathematical than physical, it happened not to be the case: the first attempts to describe the global dynamics of the Universe came with GR, in the 1910's. In this course, for pedagogical purposes, we will not follow the historical order, and start with the Newtonian approach.

Newton himself did the first step in the argumentation. He noticed that if the Universe was of finite size, and governed by the law of gravity, then all massive bodies would unavoidably concentrate into a single point, just because of gravitational attraction. If instead it was infinite, and with an approximately homogeneous distribution at initial time, it could concentrate into several points, like planets and stars, because there would be no center to fall in. In that case, the motion of each massive body would be driven by the sum of an infinite number of gravitational forces. Since the mathematics of that time didn't allow to deal with this situation, Newton didn't proceed with his argument.

1.2.2 The rate of expansion from Gauss theorem

In fact, using Gauss theorem, this problem turns out to be quite simple. Suppose that the Universe consists in many massive bodies distributed in an isotropic and homogeneous way (i.e., for any observer, the distribution looks the same in all directions). This should be a good modelization of the Universe on sufficiently large scales. We wish to compute the motion of a particle located at a distance r away from us. Because the Universe is assumed to be isotropic, the problem is spherically symmetric, and we can employ Gauss theorem on a sphere centered on us, and of radius r (see figure 1.5). So, the acceleration of any particle on the surface of the sphere reads

$$\ddot{r} = -\frac{\mathcal{G}M(r)}{r^2} \tag{1.7}$$

where $\mathcal{G}$ is Newton's constant and $M(r)$ is the mass contained inside the sphere of radius r. In other words, the particle feels the same force as if it had a two-body interaction with the mass of the sphere concentrated at the center. Note that r varies with time, but $M(r)$ remains constant: because of spherical

---

[1]Going a little bit more into details, it is also relevant when an object is so heavy and so close that the speed of liberation from this object - defined like the speed of liberation from the Earth - is comparable to the speed of light.
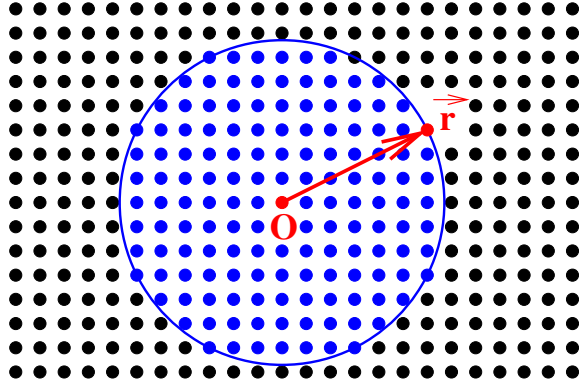
Figure 1.5: Gauss theorem applied to the local Universe.

symmetry, no particles can enter or leave the sphere, which contains always the same mass.

Since Gauss theorem allows us to make completely abstraction of the mass outside the sphere, we can make an analogy with the motion e.g. of a satellite ejected vertically from the Earth. We know that this motion depends on the initial velocity, compared with the speed of liberation from the Earth: if the initial speed is large enough, the satellites goes away indefinitely, otherwise it stops and falls down. We can see this mathematically by multiplying equation (1.7) by $\dot{r}$, and integrating it over time:

$$\frac{\dot{r}^2}{2} = \frac{\mathcal{G}M(r)}{r} - \frac{k}{2} \tag{1.8}$$

where k is a constant of integration. We can replace the mass $M(r)$ by the volume of the sphere multiplied by the homogeneous mass density ρ_{mass}, and rearrange the equation as

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi\mathcal{G}}{3}\rho_{\text{mass}} - \frac{k}{r^2} \ . \tag{1.9}$$

The quantity $\dot{r}/r$ is called the rate of expansion. Since $M(r)$ is time-independent, the mass density evolves as $\rho_{\text{mass}} \propto r^{-3}$ (i.e., matter is simply diluted when the Universe expands). The behavior of $r(t)$ depends on the sign of k. If k is positive, $r(t)$ can grow at early times but it always decreases at late times, like the altitude of the satellite falling back on Earth: this would correspond to a Universe expanding first, and then collapsing. If k is zero or negative, the expansion lasts forever.

In the case of the satellite, the critical value, which is the speed of liberation (at a given altitude), depends on the mass of the Earth. By analogy, in the case of the Universe, the important quantity that should be compared with some critical value is the homogeneous mass density. If at any time, ρ_{mass} it is bigger than the critical value

$$\rho_{\text{mass}} = \frac{3(\dot{r}/r)^2}{8\pi\mathcal{G}} \tag{1.10}$$

then k is positive and the Universe will re-collapse. Physically, what it means is that the gravitational force wins against inertial effects. In the other case, the Universe expands forever, because the density is too small with respect to the expansion velocity, and gravitation never takes over inertia. The case $k = 0$ corresponds to a kind of equilibrium between gravitation and inertia. The Universe expands forever, following a power–law: $r(t) \propto t^{2/3}$.
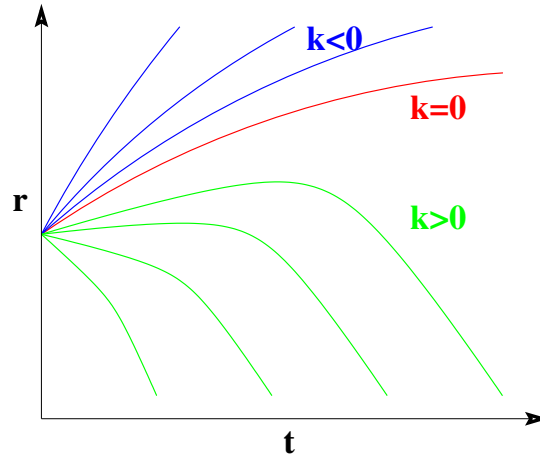
10

Figure 1.6: The motion of expansion in a Newtonian Universe is equivalent to that of a body ejected from Earth. It depends on the initial rate of expansion, compared with the equivalent of the speed of liberation. When the parameter k is zero or negative, the expansion lasts forever, otherwise the Universe re-collapses ($r \to 0$).

1.2.3 The limitations of Newtonian predictions

In the previous calculation, we cheated a little bit: we assumed that the Universe was isotropic around us, but we didn't check that it was isotropic everywhere (and therefore homogeneous). Following what we said before, homogeneous expansion requires proportionality between speed and distance. Looking at equation (1.9), we see immediately that this is true at a given time only when $k = 0$. So, it seems that the other solutions are not compatible with the Cosmological Principle.

This argument shouldn't be taken seriously, because the link that we made between homogeneity and linear expansion was based on the additivity of speed (look for instance at the caption of figure 1.2), and therefore, on Newtonian mechanics. But Newtonian mechanics cannot be applied at large distances, where v becomes large and comparable to the speed of light. This would occur for a characteristic scale called the Hubble radius R_H :

$$R_H = cH^{-1}, \tag{1.11}$$

at which the Newtonian expansion law gives $v = HR_H = c$.

So, the full problem has to be formulated in relativistic terms. In the GR results, we will see again some solutions with $k \neq 0$, but they will remain compatible with the homogeneity of the Universe.

1.3 General relativity and the Friedmann-Lemaître Universe

It is far beyond the scope of this course to introduce General Relativity, and to derive step by step the relativistic laws governing the evolution Universe. We will simply write these laws, asking the reader to admit them - and in order to give a perfume of the underlying physical concepts, we will comment on the differences with their Newtonian counterparts.
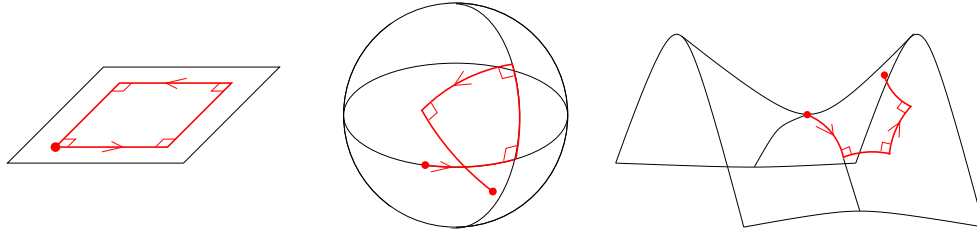
Figure 1.7: Measuring the curvature of some two-dimensional spaces. By walking four times along a distance d, turning 90 degrees left between each walk, a small man on the plane would find that he is back at his initial point. Doing the same thing, a man on the sphere would cross his trajectory, and would stop way from his departure point. Instead, a man on the hyperboloid would not close his trajectory.

1.3.1 The curvature of space-time

When Einstein tried to build a theory of gravitation compatible with the invariance of the speed of light, he found that the minimal price to pay was :

- to abandon the idea of a gravitational potential, related to the distribution of matter, and whose gradient gives the gravitational field in any point.

- to assume that our four-dimensional space-time is curved by the presence of matter.

- to impose that free-falling objects describe geodesics in this space-time.

What does that mean in simple words?

First, let's recall briefly what a curved space is, first with only two-dimensional surfaces. Consider a plane, a sphere and an hyperboloid. For us, it's obvious that the sphere and the hyperboloid are curved, because we can visualize them in our three-dimensional space: so, we have an intuitive notion of what is flat and what is curved. But if there were some two-dimensional people living on these surfaces, not being aware of the existence of a third dimension, how could they know whether they leave in a flat or a in curved space-time?

There are several ways in which they could measure it. One would be to obey the following prescription: walk in straight line on a distance d; turn 90 degrees left; repeat this sequence three times again; see whether you are back at your initial position. The people on the three surfaces would find that they are, as long as they walk along a small square, smaller than the radius of curvature. But a good test is to repeat the operation on larger and larger distances. When the size of the square will be of the same order of magnitude as the radius of curvature, the habitant of the sphere will notice that before stopping, he crosses the first branch of his trajectory (see figure 1.7). The one on the hyperboloid will stop without closing his trajectory.

It is easy to think of the curvature of a two-dimensional surface because we can visualize it embedded into three-dimensional space. Getting an intuitive representation of a three-dimensional curved space is much more difficult. A 3-sphere and a 3-hyperboloid could be defined analytically as some 3-dimensional spaces obeying to the equation $a^2 + b^2 + c^2 \pm d^2 = R^2$ inside a 4-dimensional Euclidian space with coordinates (a, b, c, d). If we wanted to define them by making use of only three dimensions, the problem would be exactly like for drawing a planisphere of the Earth. We would need to give a map of the spaces,
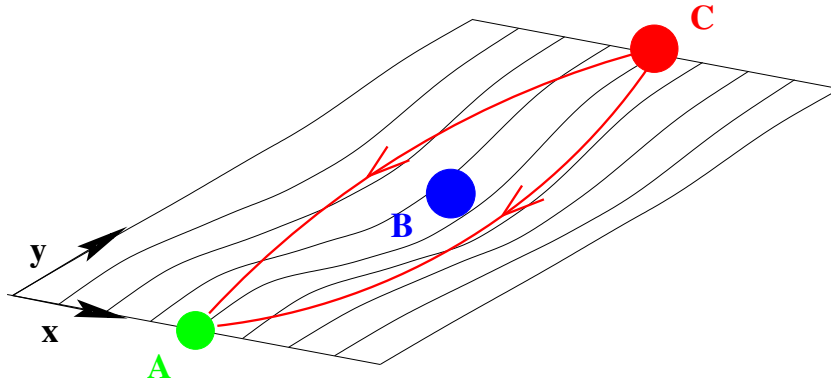
Figure 1.8: Gravitational lensing. Somewhere between an object C and a terrestrial observer A, a massive object B - for instance, a galaxy - curves its surrounding space-time. Here, for simplicity, we only draw two spatial dimensions. In absence of gravity and curvature, the only possible trajectory of light between C and A would be a straight line. But because of curvature, the straight line is not anymore the shortest trajectory. Photons prefer to follow two geodesics, symmetrical around B. So, the observer will not see one image of C, but two distinct images. In fact, if we restore the third spatial dimension, and if the three points are perfectly aligned, the image of C will appear as a ring around B. This phenomenon is observed in practice.

together with a crucial information: the scale of the map as a function of the location on the map - the scale on a planisphere is not uniform! This would bring us to a mathematical formalism, called Riemann geometry, that we don't have time to introduce here.

That was still four three dimensions. Finally, the curvature of a four-dimensional space-time is impossible to visualize intuitively, first because it has even more dimensions, and second because even in special/general relativity, there is a difference between time and space (for the readers who are familiar with special relativity, what is referred here is the negative signature of the metric).

The Einstein theory of gravitation says that four-dimensional space-time is curved, and that the curvature in each point is given entirely in terms of the matter content in this point. In simple words, this means that the curvature plays more or less the same role as the potential in Newtonian gravity. But the potential was simply a function of space and time coordinates. In GR, the full curvature is described not by a function, but by something more complicated - like a matrix of functions obeying to particular laws - called a tensor.

Finally, the definition of geodesics (the trajectories of free-falling bodies) is the following. Take an initial point and an initial direction. They define a unique line, called a geodesic, such that any segment of the line gives the shortest trajectory between the two points (so, for instance, on a sphere of radius R, the geodesics are all the great circles of radius R, and nothing else). Of course, geodesics depend on curvature. All free-falling bodies follow geodesics, including light rays. This leads for instance to the phenomenon of gravitational lensing (see figure 1.8).

So, in General Relativity, gravitation is not formulated as a force or a field, but as a curvature of space-time, sourced by matter. All isolated systems follow geodesics which are bent by the curvature. In this way, their trajectories are

affected by the distribution of matter around them: this is precisely what gravity means.

1.3.2   Building the first cosmological models

After obtaining the mathematical formulation of General Relativity, around 1916, Einstein studied various testable consequences of his theory in the solar system (e.g., corrections to the trajectory of Mercury, or to the apparent diameter of the sun during an eclipse). But remarkably, he immediately understood that GR could also be applied to the Universe as a whole, and published some first attempts in 1917. However, Hubble's results concerning the expansion were not known at that time, and most physicists had the prejudice that the Universe should be not only isotropic and homogeneous, but also static – or stationary. As a consequence, Einstein (and other people like De Sitter) found some interesting cosmological solutions, but not the ones that really describe our Universe.

A few years later, some other physicists tried to relax the assumption of stationarity. The first was the russo-american Friedmann (in 1922), followed closely by the Belgian physicist and priest Lemaître (in 1927), and then by some americans, Robertson and Walker. When the Hubble flow was discovered in 1929, it became clear for a fraction of the scientific community that the Universe could be described by the equations of Friedmann, Lemaître, Roberston and Walker. However, many people – including Hubble and Einstein himself – remained reluctant to this idea for many years. Today, the Friedmann – Lemaître model is considered as one of the major achievements of the XXth century.

Before giving these equations, we should stress that they look pretty much the same as the Newtonian results given above - although some terms seem to be identical, but have a different physical interpretation. This similarity was noticed only much later. Physically, it has to do with the fact that there exists a generalization of Gauss theorem to GR, known as Birkhoff theorem. So, as in Newtonian gravity, one can study the expansion of the homogeneous Universe by considering only matter inside an sphere. But in small regions, General Relativity admits Newtonian gravity as an asymptotic limit, and so the equations have many similarities.

1.3.3   Our Universe is curved

The Friemann-Lemaître model is defined as the most general solution of the laws of General Relativity, assuming that the Universe is isotropic and homogeneous. We have seen that in GR, matter curves space-time. So, the Universe is curved by its own matter content, along its four space and time dimensions. However, because we assume homogeneity, we can decompose the total curvature into two parts:

- the spatial curvature, i.e., the curvature of the usual 3-dimensional space (x, y, z) at fixed time. This curvature can be different at different times. It is maximally symmetric, i.e., it makes no difference between the three directions (x, y, z). There are only three maximally symmetric solutions: space can be Euclidean, a 3-sphere with finite volume, or a 3-hyperboloid. These three possibilities are referred to as a flat, a closed or an open Universe. *A priori*, nothing forbids that we leave in a closed or in an open Universe: if the radius of curvature was big enough, say, bigger than the size of our local galaxy cluster, then the curvature would show up only in long range astronomical observations. In the next chapter, we will see how recent observations are able to give a precise answer to this question.

- the two-dimensional space-time curvature, i.e., for instance, the curvature of the (t, x) space-time. Because of isotropy, the curvature of the (t, y) and the (t, z) space-time have to be the same. This curvature is the one responsible for the expansion. Together with the spatial curvature, it fully describes gravity in the homogeneous Universe.

The second part is a little bit more difficult to understand for the reader who is not familiar with GR, and causes a big cultural change in one's intuitive understanding of space and time.

1.3.4 Comoving coordinates

In the spirit of General Relativity, we will consider a set of three variables that will represent the spatial coordinates (i.e., a way of mapping space, and of giving a label to each point), but NOT directly a measure of distance!

The laws of GR allow us to work with any system of coordinates that we prefer. For simplicity, in the Friedmann model, people generally employ a particular system of spherical coordinates (r, θ, ϕ) called "comoving coordinates", with the following striking property: the physical distance dl between two infinitesimally close objects with coordinates (r, θ, ϕ) and $(r + dr, \theta + d\theta, z + dz)$ is not given by

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \ d\phi^2) \tag{1.12}$$

as in usual Euclidean space, but

$$dl^2 = a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \ d\phi^2) \right] \tag{1.13}$$

where $a(t)$ is a function of time, called the scale factor, whose time-variations account for the curvature of two-dimensional space-time; and k is a constant number, related to the spatial curvature: if $k = 0$, the Universe is flat, if $k > 0$, it is closed, and if $k < 0$, it is open. In the last two cases, the radius of curvature R_c is given by

$$R_c(t) = \frac{a(t)}{\sqrt{|k|}}. \tag{1.14}$$

When the Universe is closed, it has a finite volume, so the coordinate r is defined only up to a finite value: $0 \le r < 1/\sqrt{k}$.

If k was equal to zero and a was constant in time, we could redefine the coordinate system with $(r', \theta', \phi') = (ar, \theta, \phi)$, find the usual expression (1.12) for distances, and go back to Newtonian mechanics. So, we stress again that the curvature really manifests itself as $k \ne 0$ (for spatial curvature) and $\dot{a} \ne 0$ (for the remaining space-time curvature).

Note that when $k = 0$, we can rewrite dl^2 in Cartesian coordinates:

$$dl^2 = a^2(t) \left(dx^2 + dy^2 + dz^2 \right) \ . \tag{1.15}$$

In that case, the expressions for dl contains only the differentials dx, dy, dz, and is obviously left unchanged by a change of origin - as it should be, because we assume a homogeneous Universe. But when $k \ne 0$, there is the additional factor $1/(1 - kr^2)$, where r is the distance to the origin. So, naively, one could think that we are breaking the homogeneity of the Universe, and privileging a particular point. This would be true in Euclidean geometry, but in curved geometry, it is only an artifact. Indeed, choosing a system of coordinates is equivalent to mapping a curved surface on a plane. By analogy, if one draws a planisphere of half-of-the-Earth by projecting onto a plane, one has to chose a particular point defining the axis of the projection. Then, the scale of the map
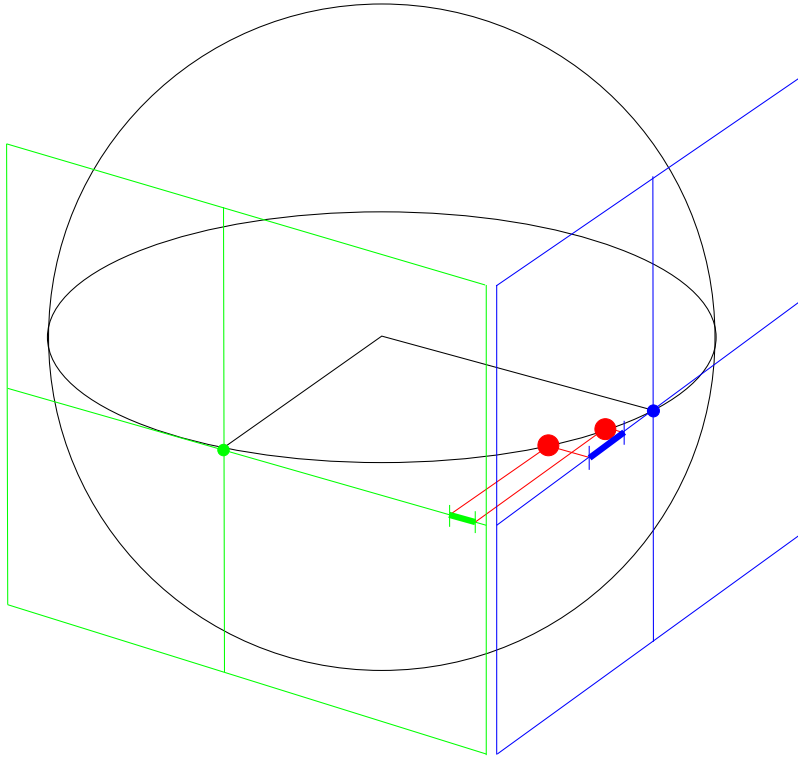
Figure 1.9: Analogy between the Friedman coordinates for an open/closed Universe, and the mapping of a sphere by parallel projection onto a plane. Two maps are shown, with different axes of projection. On each map, the scale depends on the distance to the center of the map: the scale is smaller at the center than on the borders. However, the physical distance between too point on the sphere can be computed equally well using the two maps and their respective scaling laws.

is not the same all over the map. However, if one reads too maps with different projection axes (see figure 1.9), and computes the distance between Geneva and Rome from the two maps, using the appropriate scaling laws, he will of course find the same physical distance. This is exactly what happens in our case: in a particular coordinate system, the expression for physical lengths depends on the coordinate *with respect to the origin*, but physical lengths are left unchanged by a change of coordinates. In figure 1.10 we push the analogy and show how the factor $1/(1 - kr^2)$ appears explicitly for the parallel projection of a 2-sphere onto a plane. The rigorous proof that equation (1.13) is invariant by a change of origin would be more complicated, but essentially similar.

Free-falling bodies follow geodesics in this curved space-time. Among bodies, we can distinguish between photons, which are relativistic because they travel at the speed of light, and ordinary matter like galaxies, which are non-relativistic (i.e., an observer sitting on them can ignore relativity at short distance, as we do on Earth).

The geodesics followed by non-relativistic bodies in space-time are given by $dl = 0$, i.e., fixed spatial coordinates. So, galaxies are at rest in comoving coordinates. But it doesn't mean that the Universe is static, because all distances grow proportionally to $a(t)$: so, the scale factor accounts for the homogeneous
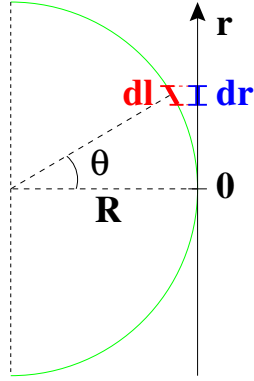
Figure 1.10: We push the analogy between the Friedmann coordinates in a closed Universe, and the parallel projection of a half–sphere onto a plane. The physical distance dl on the surface of the sphere corresponds to the coordinate interval dr, such that $dl = dr/\cos\theta = dr/\sqrt{1 - \sin^2\theta} = dr/\sqrt{1 - (r/R)^2}$, where R is the radius of the sphere. This is exactly the same law as in a closed Friedman Universe.

expansion. A simple analogy helps in understanding this subtle concept. Let us take a rubber balloon and draw some points on the surface. Then, we inflate the balloon. The distances between all the points grow proportionally to the radius of the balloon. This is not because the points have a proper motion on the surface, but because all the lengths on the surface of the balloon increase with time.

The geodesics followed by photons are straight lines in 3-space, but not in space-time. Locally, they obey to the same relation as in Newtonian mechanics: $c\,dt = dl$, i.e.,

$$c^2 dt^2 = a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\;d\phi^2)\right]\;. \tag{1.16}$$

So, on large scales, the distance traveled by light in a given time interval is not $\Delta l = c\Delta t$ like in Newtonian mechanics, but is given by integrating the infinitesimal equation of motion. We can write this integral, taking for simplicity a photon with constant (θ, ϕ) coordinates (i.e., the origin is chosen on the trajectory of the photon). Between t_1 and t_2, the change in *comoving coordinates* for such a photon is given by

$$\int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - kr^2}} = \int_{t_1}^{t_2} \frac{c}{a(t)}dt. \tag{1.17}$$

This propagation equation of light is extremely important - probably, one of the two most important of cosmology, together with the Friedmann equation, that we will give soon. It is with this equation that we are able today to measure the curvature of the Universe, its age, its acceleration, and other fundamental quantities.

1.3.5 Bending of light in the expanding Universe

Lets give a few examples of the implications of equation (1.16), which give the bending of the trajectories followed by photons in our curved space-time, as illustrated in figure 1.11.
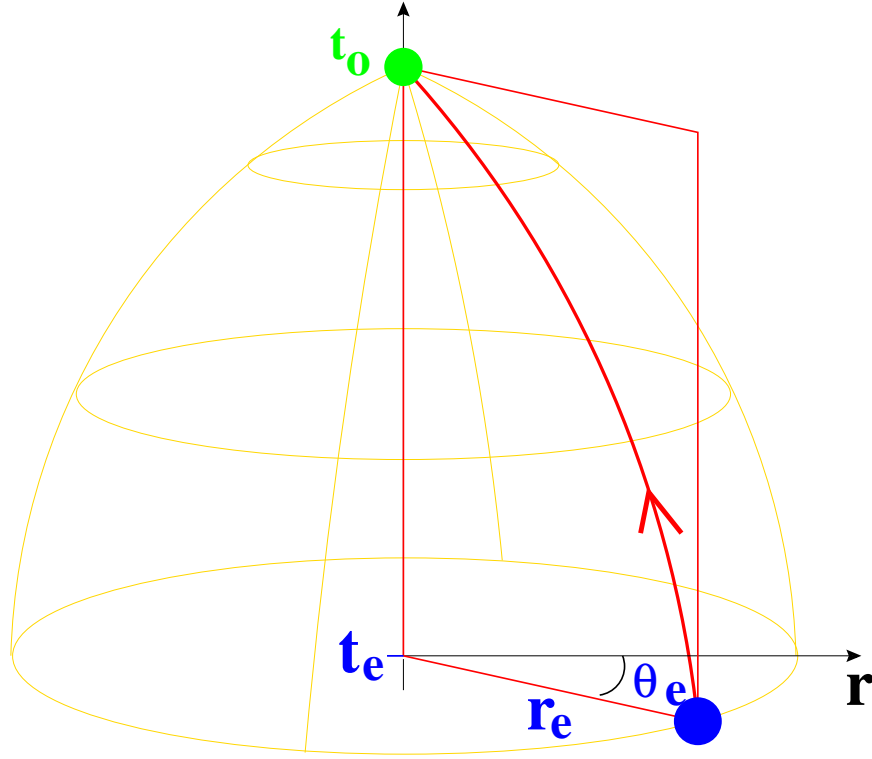
Figure 1.11: An illustration of the propagation of photons in our Universe. The dimensions shown here are (t, r, θ): we skip ϕ for the purpose of representation. We are siting at the origin, and at a time t_0, we can see a the light of a galaxy emitted at (t_e, r_e, θ_e). Before reaching us, the light from this galaxy has traveled over a curved trajectory. In any point, the slope dr/dt is given by equation (1.16). So, the relation between r_e and $(t_0 - t_e)$ depends on the spatial curvature and on the scale factor evolution. The trajectory would be a straight line in space-time only if $k = 0$ and $a = constant$, i.e., in the limit of Newtonian mechanics in Euclidean space. The ensemble of all possible photon trajectories crossing $r = 0$ at $t = t_0$ is called our "past light cone", visible here in orange. Asymptotically, near the origin, it can be approximated by a linear cone with $dl = cdt$, showing that at small distance, the physics is approximately Newtonian.

The redshift.

First, a simple calculation based on equation (1.16) - we don't include it here - gives the redshift associated with a given source of light. Take two observers sitting on two galaxies (with fixed comoving coordinates). A light signal is sent from one observer to the other. At the time of emission t_1, the first observer measures the wavelength λ_1. At the time of reception t_2, the second observer will measure the wavelength λ_2 such that

$$z = \frac{\delta\lambda}{\lambda} = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{a(t_2)}{a(t_1)} - 1 \ . \tag{1.18}$$

So, the redshift depends on the variation of the scale-factor between the time of emission and reception, but not on the curvature parameter k. This can be understood as if the scale-factor represented the "stretching" of light in out

18

curved space-time. We can check that when the Universe expands ($a(t_2) > a(t_1)$), the wavelengths are enhanced ($\lambda_2 > \lambda_1$), and the spectral lines are red-shifted (contraction would lead to blue-shift, i.e., to a negative redshift parameter $z < 0$).

In real life, what we see when we observe the sky is never directly the distance, size, or velocity of an object, but the light traveling from it. So, by studying spectral lines, we can easily measure the redshifts. This is why astronomers, when they refer to an object or to an event in the Universe, generally mention the redshift rather than the distance or the time. It's only when the function $a(t)$ is known - and as we will see later, we almost know it in our Universe - that a given redshift can be related to a given time and distance.

The importance of the redshift as a measure of time and distance comes from the fact that we don't see our full space-time, but only our past line-cone, i.e., a three-dimensional subspace of the full four-dimensional space-time, corresponding to all points which could emit a light signal that we would receive today on Earth. If we remove the coordinate ϕ and draw the past line-cone of a given observer, it does look like a cone, of course curved like all photon trajectories (see figure 1.11).

Also note that in Newtonian mechanics, the redshift was defined as $z = v/c$, and seemed to be limited to $|z| < 1$. The true GR expression doesn't have such limitations, since the ratio of the scale factors can be arbitrarily large without violating any fundamental principle. And indeed, observations do show many objects - like quasars - at redshifts of 2 or even bigger. We'll see later that we also observe the Cosmic Microwave Background at a redshift of approximately $z = 1000$!

The Hubble parameter in General Relativity.

In the limit of small redshift, we expect to recover the Newtonian results, and to find a relation similar to $z = v/c = Hr/c$. To show this, let's assume that t_0 is the present time, and that our galaxy is at $r = 0$. We want to compute the redshift of a nearby galaxy, which emitted the light that we receive today at a time $t_0 - dt$. In the limit of small dt, the equation of propagation of light shows the physical distance between the galaxy and us is simply

$$dl = c \, dt \tag{1.19}$$

while the redshift of the galaxy is

$$z = \frac{a(t_0)}{a(t_0 - dt)} - 1 = \frac{1}{1 - \frac{\dot{a}(t_0)}{a(t_0)}dt} - 1 = \frac{\dot{a}(t_0)}{a(t_0)}dt \ . \tag{1.20}$$

By combining these two relations we obtain

$$z = \frac{\dot{a}(t_0)}{a(t_0)} \frac{dl}{c} \ . \tag{1.21}$$

So, at small redshift, we recover the Hubble law, and the role of the Hubble parameter is played by $\dot{a}(t_0)/a(t_0)$. In the Friedmann Universe, we will directly define the Hubble parameter as the expansion rate of the scale factor:

$$H(t) = \frac{\dot{a}(t)}{a(t)} \ . \tag{1.22}$$

The present value of H is generally noted H_0:

$$H_0 = 100h \text{ km s}^{-1}\text{Mpc}^{-1}, \qquad 0.5 < h < 0.8. \tag{1.23}$$

Angular diameter – redshift relation.

When looking at the sky, we don't see directly the size of the objects, but only their angular diameter. In Euclidean space, i.e. in absence of gravity, the angular diameter $d\theta$ of an object would be related to its size dl and distance r through

$$d\theta = \frac{dl}{r} \ . \tag{1.24}$$

Recalling that $z = v/c$ and $v = Hr$, we easily find an angular diameter – redshift relation valid in Euclidean space:

$$d\theta = \frac{H \ dl}{c \ z} \ . \tag{1.25}$$

In General Relativity, because of the bending of light by gravity, the steps of the calculation are different. Using the definition of infinitesimal distances (1.13), we see that the physical size dl of an object is related to its angular diameter $d\theta$ through

$$dl = a(t_e) \ r_e \ d\theta \tag{1.26}$$

where t_e is the time at which the galaxy emitted the light ray that we observe today on Earth, and r_e is the comoving coordinate of the object. The equation of motion of light gives a relation between r_e and t_e:

$$\int_{r_e}^{0} \frac{dr}{\sqrt{1 - kr^2}} = \int_{t_e}^{t_0} \frac{c}{a(t)} dt \tag{1.27}$$

where t_0 is the time today. So, the relation between r_e and t_e depends on $a(t)$ and k.

If we knew the function $a(t)$ and the value of k, we could calculate the integral (1.27) and re-express equation (1.26) in the form

$$d\theta = \frac{H \ dl}{c \ z} f(z) \tag{1.28}$$

where $f(z)$ would be a function of redshift, accounting for general relativistic corrections, and depending on the curvature. We conclude that in the Friedmann Universe, for an object of fixed size and redshift, the angular diameter depends on the curvature - as illustrated graphically in figure 1.12. Therefore, if we know in advance the physical size of an object, we can simply measure its redshift, its angular diameter, and immediately obtain some informations on the geometry of the Universe.

It seems very difficult to know in advance the physical size of a remote object. However, we will see in the next chapter that some beautiful developments of modern cosmology enabled physicists to know the physical size of some objects visible at a redshift of $z \simeq 1000$ (namely, the anisotropies of the Cosmological Microwave Background). So, the angular diameter – redshift relation has been used in the past decade in order to measure the spatial curvature of the Universe. We will show the results in the last section of chapter two.

Apparent Luminosity – redshift relation.

We have seen that in Newtonian mechanics, the absolute luminosity of an object and the apparent luminosity l that we measure on Earth per unit of surface are related by

$$l = \frac{L}{4\pi r^2} = \frac{LH^2}{4\pi c^2 z^2} \ . \tag{1.29}$$

So, if for some reason we know independently the absolute luminosity of a celestial body (like for cepheids), and we measure its redshift, we can obtain the value of H, as Hubble did in 1929.
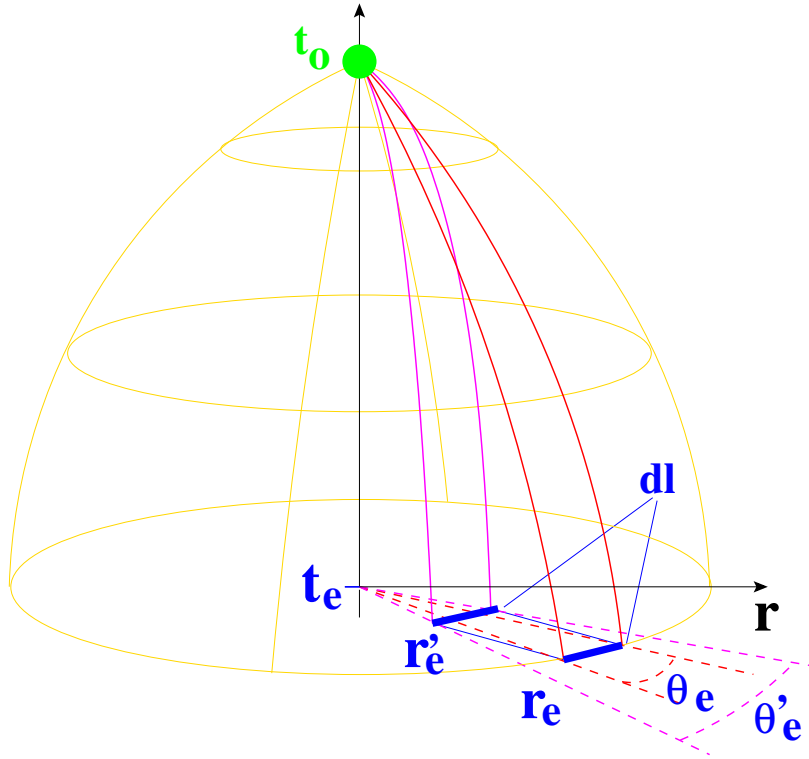
Figure 1.12: Angular diameter – redshift relation. We consider an object of fixed size dl and fixed redshift, sending a light signal at time t_e that we receive at present time t_0. All photons travel by definition with θ =constant. However, the bending of their trajectories in the (t,r) plane depends on the spatial curvature and on the scale factor evolution. So, for fixed t_e, the comoving coordinate of the object, r_e, depends on curvature. The red lines are supposed to illustrate the trajectory of light in a flat Universe with $k = 0$. If we keep dl, $a(t)$ and t_e fixed, but choose a positive value $k > 0$, we know from equation (1.27) that the new coordinate $r_e{}'$ has to be smaller. But dl is fixed, so the new angle $d\theta'$ has to be bigger, as easily seen on the figure for the purple lines. So, in a closed Universe, objects are seen under a larger angle. Conversely, in an open Universe, they are seen under a smaller angle.

But we would like to extend this technique to very distant objects (in particular, supernovae, that are observed up to a redshift of two, and are thought to have a calibrated absolute luminosity, like cepheids). So, one needs to recompute this relation in the framework of general relativity. Because of the bending of light rays, the trajectory of the photons traveling from an object located at a given redshift depends both on the curvature and the scale factor. So, today, the light will be dispersed on a sphere of variable size.

If we know several objects with the same absolute magnitude, and we plot them on an apparent magnitude - redshift diagram, we will get a curve whose shape at small redshift is given by (1.29), and at large redshift by General Relativity corrections (depending on the spatial curvature and the scale factor.) We will see in the next chapter that such an experiment has been performed for supernovae, leading to one of the most intriguing discovery of the past years.

In summary of this section, according to General Relativity, the homogeneous Universe is curved by its own matter content, and the curvature can be specified with just one number and one function: the curvature k, and the scale factor $a(t)$. We should be able to relate these two quantities with what causes the curvature: the matter density.

1.3.6 The Friedmann law

The Friedmann law relates the scale factor $a(t)$, the spatial curvature parameter k and the homogeneous energy density of the Universe $\rho(t)$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\mathcal{G}}{3}\frac{\rho}{c^2} - \frac{kc^2}{a^2} \ . \tag{1.30}$$

Together with the propagation of light equation, this law is the key ingredient of the Friedmann-Lemaître model.

In special/general relativity, the total energy of a particle is the sum of its rest energy $E_0 = mc^2$ plus its momentum energy. So, if we consider only non-relativistic particles like those forming galaxies, we get $\rho = \rho_{\mathrm{mass}}c^2$. Then, the Friedmann equation looks exactly like the Newtonian expansion law (1.9), excepted that the function $r(t)$ (representing previously the position of material points) is replaced by the scale factor $a(t)$. Of course, the equations look the same, but are not equivalent . First, we have already seen that only at very small distance, the distinction between the scale factor $a(t)$ and the classical position $r(t)$ is irrelevant; on large distances – like the Hubble radius – the difference of interpretation between the two is crucial. Second, we have seen that the term proportional to k seems to break the homogeneity of the Universe in the Newtonian formalism, while in General Relativity, when it is correctly interpreted as the spatial curvature term, it is perfectly consistent with the Cosmological Principle.

Actually, there is a third crucial difference between the Friedmann law and the Newtonian expansion law. In the previous paragraph, we only considered non-relativistic matter like galaxies. But the Universe also contains relativistic particles traveling at the speed of light, like photons (and also neutrinos if their mass is very small). *A priori*, their gravitational effect on the Universe expansion could be important. How can we include them?

1.3.7 Relativistic matter and Cosmological constant

The Friedmann equation is true for any types of matter, relativistic or non-relativistic; if there are different species, the total energy density ρ is the sum over the density of all species.

What depends on the type of matter is the relation between ρ and a, i.e., the way in which energy gets diluted by the expansion.

For non-relativistic matter, the answer is obvious. Take a distribution of particles with fixed comoving coordinates. Their energy density equals their mass density times c^2. Look only at the matter contained into a comoving sphere centered around the origin, of comoving radius r. If the sphere is small with respect to the curvature, the physical volume inside the sphere is just $V = \frac{4\pi}{3}(a(t)r)^3$. Since both the sphere and the matter particles have fixed comoving coordinates, no matter can enter or leave from inside the sphere during the expansion. Therefore, the mass (or the energy) inside the sphere is conserved. We conclude that ρV is constant and that $\rho \propto a^{-3}$.

For ultra–relativistic matter like photons, the energy is not given by the rest mass but by the frequency ν or the wavelength λ:

$$E = h\nu = hc/\lambda. \tag{1.31}$$

But we knows that physical wavelengths are stretched proportionally to the scale factor $a(t)$. So, if we repeat the argument of the sphere, assuming now that it contains a homogeneous bath of photons with equal wavelength, we see that the total energy inside the sphere evolves proportionally to a^{-1}. So, the energy density of relativistic matter is proportional to $\rho \propto a^{-4}$. If the scale factor increases, the photon energy density decreases faster than that of ordinary matter.

This result could be obtained differently. For any gas of particles with a distribution of velocities, one can define a pressure (corresponding physically to the shock of particles against the borders if the gas was enclosed into a box). This can be extended to a gas of relativistic particles, for which the speed equals the speed of light. A calculation based on statistical mechanics gives the famous result that the pressure of a relativistic gas is related to its density by $p = \rho/3$.

In the case of the Friedmann universe, General Relativity provides several equations: the Friedmann law, and also, for each specie, the equation of conservation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \ . \tag{1.32}$$

This is consistent with what we already said. For non-relativistic matter, like galaxies, the pressure is negligible (like in a gas of still particles), and we get

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \qquad \Rightarrow \qquad \rho \propto a^{-3}. \tag{1.33}$$

For relativistic matter like photons, we get

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1 + \frac{1}{3})\rho = -4\frac{\dot{a}}{a}\rho \qquad \Rightarrow \qquad \rho \propto a^{-4}. \tag{1.34}$$

Finally, in quantum field theory, it is well–known that the vacuum can have an energy density different from zero. The Universe could also contain this type of energy (which can be related to particle physics, phase transitions and spontaneous symmetry breaking). For vacuum, one can show that $p = -\rho$. This means that the vacuum energy is never diluted and remains constant. This constant energy density was called by Einstein – who introduced it in a completely different way and with other motivations – the Cosmological Constant. We will see that this term is probably important in our Universe.

Chapter 2

The Standard Cosmological Model

The real Universe is not homogeneous: it contains stars, galaxies, clusters of galaxies...

In cosmology, all quantities – like the density and pressure of each specie – are decomposed into a spatial average, called the *background*, plus some inhomogeneities. The later are assumed to be small with respect to the background in the early Universe: so, they can be treated like linear perturbations. As a consequence, the evolution of each Fourier mode is independent of the others. During the evolution, if the perturbations of a given quantity become large, the linear approximation breaks down, and one has to employ a full non–linear description, which is very complicated in practice. However, for many purposes in cosmology, the linear theory is sufficient in order to make testable predictions.

In section 2.1, we will describe the evolution of the homogeneous background. Section 2.2 will give some hints about the evolution of linear perturbations – and also, very briefly, about the final non-linear evolution of matter perturbations. Altogether, these two sections provide a brief summary of what is called the standard cosmological model, which depends on a few free parameters. In section 2.3, we will show that the main cosmological parameters have already been measured with quite good precision. Finally, in section 2.4, we will introduce the theory of inflation, which provides some initial conditions both for the background and for the perturbations in the very early Universe. We will conclude with a few words on the so-called quintessence models.

2.1 The Hot Big Bang scenario

A priori, we don't know what type of fluid or particles gives the dominant contributions to the energy density of the Universe. According to the Friedmann equation, this question is related to many fundamental issues, like the behavior of the scale factor, the spatial curvature, or the past and future evolution of the Universe...

2.1.1 Various possible scenarios for the history of the Universe

We will classify the various types of matter that could fill the Universe according to their pressure-to-density ratio. The three most likely possibilities are:

1. ultra–relativistic particles, with $v = c$, $p = \rho/3$, $\rho \propto a^{-4}$. This includes photons, massless neutrinos, and eventually other particles that would have a very small mass and would be traveling at the speed of light. The generic name for this kind of matter, which propagates like electromagnetic radiation, is precisely "radiation".

2. non-relativistic pressureless matter – in general, simply called "matter" by opposition to radiation – with $v \ll c$, $p \simeq 0$, $\rho \propto a^{-3}$. This applies essentially to all structures in the Universe: planets, stars, clouds of gas, or galaxies seen as a whole.

3. a possible cosmological constant, with time–invariant energy density and $p = -\rho$, that might be related to the vacuum of the theory describing elementary particles, or to something more mysterious. Whatever it is, we leave such a constant term as an open possibility. Following the definition given by Einstein, what is actually called the "cosmological constant" Λ is not the energy density ρ_Λ, but the quantity

$$\Lambda = 8\pi\mathcal{G}\rho_\Lambda/c^2 \tag{2.1}$$

which has the dimension of the inverse square of a time.

We write the Friedmann equation including these three terms:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\mathcal{G}}{3c^2}\rho_{\mathrm{R}} \ + \ \frac{8\pi\mathcal{G}}{3c^2}\rho_{\mathrm{M}} \ - \ \frac{kc^2}{a^2} \ + \ \frac{\Lambda}{3} \tag{2.2}$$

where ρ_{R} is the radiation density and ρ_{M} the matter density. The order in which we wrote the four terms on the right–hand side – radiation, matter, spatial curvature, cosmological constant – is not arbitrary. Indeed, they evolve with respect to the scale factor as a^{-4}, a^{-3}, a^{-2} and a^0. So, if the scale factors keeps growing, and if these four terms are present in the Universe, there is a chance that they all dominate the expansion of the Universe one after each other (see figure 2.1). Of course, it is also possible that some of these terms do not exist at all, or are simply negligible. For instance, some possible scenarios would be:

- only matter domination, from the initial singularity until today (we'll come back to the notion of Big Bang later).

- radiation domination $\rightarrow$ matter domination today.

- radiation dom. $\rightarrow$ matter dom. $\rightarrow$ curvature dom. today

- radiation dom. $\rightarrow$ matter dom. $\rightarrow$ cosmological constant dom. today

But all the cases that do not respect the order (like for instance: curvature domination $\rightarrow$ matter domination) are impossible.

During each stage, one component strongly dominates the others, and the behavior of the scale factor, of the Hubble parameter and of the Hubble radius are given by:

1. Radiation domination:

$$\frac{\dot{a}^2}{a^2} \propto a^{-4}, \qquad a(t) \propto t^{1/2}, \qquad H(t) = \frac{1}{2t}, \qquad R_H(t) = 2ct. \tag{2.3}$$

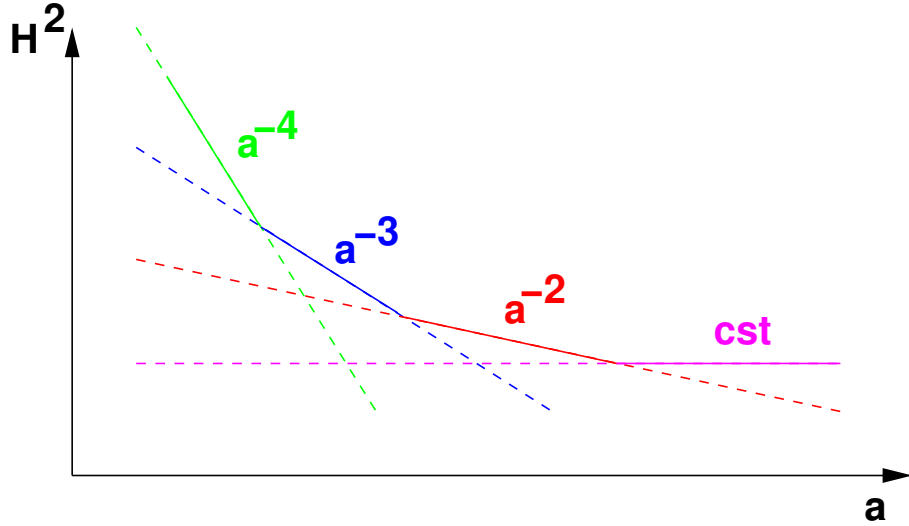So, the Universe is in decelerated power–law expansion.

Figure 2.1: Evolution of the square of the Hubble parameter, in a scenario in which all typical contributions to the Universe expansion (radiation, matter, curvature, cosmological constant) dominate one after each other.

2. Matter domination:

$$\frac{\dot{a}^2}{a^2} \propto a^{-3}, \qquad a(t) \propto t^{2/3}, \qquad H(t) = \frac{2}{3t}, \qquad R_H(t) = \frac{3}{2}ct. \qquad (2.4)$$

Again, the Universe is in power–law expansion, but it decelerates more slowly than during radiation domination.

3. Negative curvature domination ($k < 0$):

$$\frac{\dot{a}^2}{a^2} \propto a^{-2}, \qquad a(t) \propto t, \qquad H(t) = \frac{1}{t}, \qquad R_H(t) = ct. \qquad (2.5)$$

An open Universe dominated by its curvature is in linear expansion.

4. Positive curvature domination: if $k > 0$, and if there is no cosmological constant, the right–hand side finally goes to zero: expansion stops. After, the scale factor starts to decrease. H is negative, but the right–hand side of the Friedmann equation remains positive. The Universe recollapses. We know that we are not in such a phase, because we observe the Universe expansion. But *a priori*, we might be living in a closed Universe, slightly before the expansion stops.

5. Cosmological constant domination:

$$\frac{\dot{a}^2}{a^2} \rightarrow \text{constant}, \qquad a(t) \propto \exp(\Lambda t/3), \qquad H = c/R_H = \sqrt{\Lambda/3}. \qquad (2.6)$$

The Universe ends up in exponentially accelerated expansion.

So, in all cases, there seems to be a time in the past at which the scale factor goes to zero, called the initial singularity or the "Big Bang". The Friedmann description of the Universe is not supposed to hold until $a(t) = 0$. At some time, when the density reaches a critical value called the Planck density, we believe

that gravity has to be described by a quantum theory, where the classical notion of time and space disappears. Proposals for such theories exist, called "string theories". Sometimes, they try to address the initial singularity problem, and to build various scenarios for the origin of the Universe. Anyway, this field is still very speculative, and of course, our understanding of the origin of the Universe will always break down at some point. A reasonable goal is just to go back as far as possible, on the basis of testable theories.

The future evolution of the Universe heavily depends on the existence of a cosmological constant. If the later is exactly zero, then the future evolution is dictated by the curvature (if $k > 0$, the Universe will end up with a "Big Crunch", where quantum gravity will show up again, and if $k \leq 0$ there will be eternal decelerated expansion). If instead there is a positive cosmological term, then the Universe necessarily ends up in eternal accelerated expansion.

2.1.2 The matter budget today

In order to know the past and future evolution of the Universe, it would be enough to measure the present density of radiation, matter and Λ, and also to measure H_0. Then, thanks to the Friedmann equation, it would be possible to extrapolate $a(t)$ at any time. Let us express this idea mathematically. We take the Friedmann equation, evaluated today, and divide it by H_0^2:

$$1 = \frac{8\pi\mathcal{G}}{3H_0^2 c^2}\left(\rho_{R0} + \rho_{M0}\right) - \frac{kc^2}{a_0^2 H_0^2} + \frac{\Lambda}{3H_0^2}. \tag{2.7}$$

where the subscript 0 means "evaluated today". Since by construction, the sum of these four terms is one, they represent the relative contributions to the present Universe expansion. These terms are usually noted

$$\Omega_R = \frac{8\pi\mathcal{G}}{3H_0^2 c^2}\rho_{R0}, \tag{2.8}$$

$$\Omega_M = \frac{8\pi\mathcal{G}}{3H_0^2 c^2}\rho_{M0}, \tag{2.9}$$

$$\Omega_k = \frac{kc^2}{a_0^2 H_0^2}, \tag{2.10}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \tag{2.11}$$

$$\tag{2.12}$$

and the "matter budget" equation is

$$\Omega_R + \Omega_M - \Omega_k + \Omega_\Lambda = 1. \tag{2.13}$$

The Universe is flat provided that

$$\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda \tag{2.14}$$

is equal to one. In that case, as we already know, the total density of matter, radiation and Λ is equal at any time to the critical density

$$\rho_c(t) = \frac{3c^2 H^2(t)}{8\pi\mathcal{G}}. \tag{2.15}$$

Note that the parameters Ω_x, where $x \in \{R, M, \Lambda\}$, could have been defined as the present density of each specie divided by the present critical density:

$$\Omega_x = \frac{\rho_{x0}}{\rho_{c0}}. \tag{2.16}$$

So far, we conclude that the evolution of the Friedmann Universe can be described entirely in terms of four parameters, called the "cosmological parameters":

$$\Omega_{\mathrm{R}}, \Omega_{\mathrm{M}}, \Omega_{\Lambda}, H_0. \tag{2.17}$$

One of the main purposes of cosmology is to measure the value of the cosmological parameters.

2.1.3   The Cold and Hot Big Bang alternatives

Curiously, after the discovery of the Hubble expansion and of the Friedmann law, there were no rapid progresses in cosmology for a few decades. The most likely explanation is that most physicists were not considering seriously the possibility of studying the Universe in the far past, near the initial singularity, because they thought that it would always be impossible to test experimentally any model of this type.

Nevertheless, a few pioneers tried to think about the origin of the Universe. At the beginning, for simplicity, they assumed that the expansion of the Universe is always dominated by a single component, the one forming galaxies, i.e., pressureless matter. Since going back in time, the density of matter increases as a^{-3}, matter had to be very dense at early times. This was formulated as the "Cold Big Bang" scenario.

According to Cold Big Bang, in the early Universe, the density is so high that matter has to consist in a gas of nucleons and electrons. Then, when the density falls below a critical value, some nuclear reactions form the first nuclei - this era is called nucleosynthesis. But later, due to the expansion, the dilution of matter is such that nuclear reactions are suppressed (in general, the expansion freezes out all processes when their characteristic time–scale becomes smaller than the so–called Hubble time–scale H^{-1}). So, only the lightest nuclei have time to form in a significant amount. After nucleosynthesis, matter consists in a gas of nuclei and electrons, with electromagnetic interactions. When the density becomes even smaller, they finally combine into atoms – this second transition is called the recombination. At late time, any small density inhomogeneity in the gas of atoms is enhanced by gravitational interactions. The atoms get accumulated into clumps like stars and planets - but this is a different story.

In the middle of the XX-th century, a few particle physicists tried to build the first models of nucleosynthesis – the era of nuclei formation. In particular, four groups – each of them not being aware of the work of the others – reached approximately the same negative conclusion: in the Cold Big Bang scenario, nucleosynthesis does not work properly, because the formation of hydrogen is strongly suppressed with respect to that of heavier elements. But this conclusion is at odds with observations: using spectrometry, astronomers know that there is a lot of hydrogen in stars and clouds of gas. The groups of the Russo-American Gamow in the 1940's, of the Russian Zel'dovitch (1964), of the British Hoyle and Tayler (1964), and of Peebles in Princeton (1965) all reached this conclusion. They also proposed a possible way to reconcile nucleosynthesis with observations. If one assumes that during nucleosynthesis, the dominant energy density is that of photons, the expansion is driven by $\rho_{\mathrm{R}} \propto a^{-4}$, and the rate of expansion is different. This affects the kinematics of the nuclear reactions is such way that enough hydrogen can be created.

In that case, the Universe would be described by a Hot Big Bang scenario, in which the radiation density dominates at early time. Before nucleosynthesis and recombination, the mean free path of the photons is very small, because they are continuously interacting – first, with electrons and nucleons, and then, with electrons and nuclei. So, their motion can be compared with the Brownian
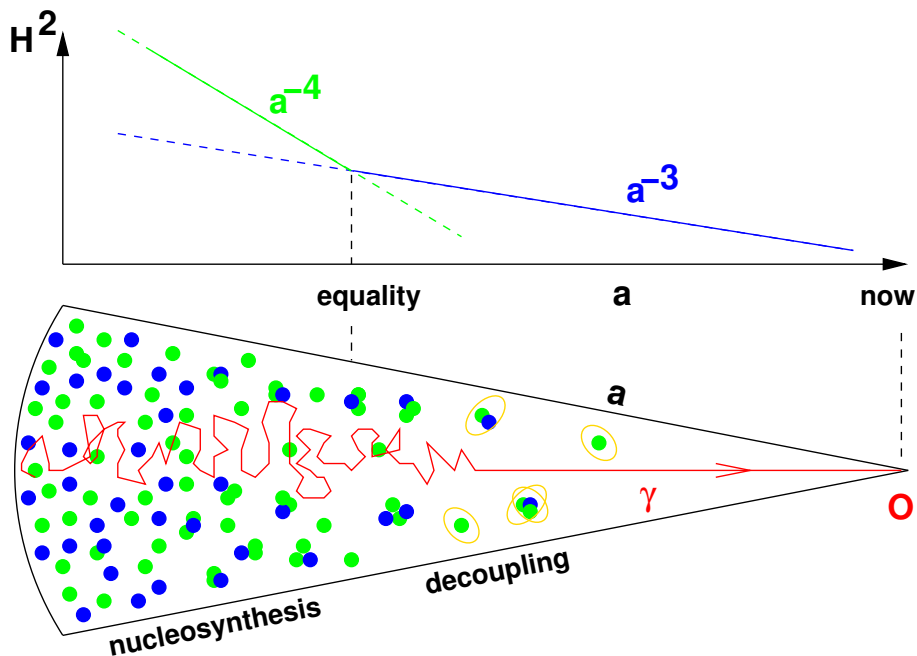
Figure 2.2: On the top, evolution of the square of the Hubble parameter as a function of the scale factor in the Hot Big Bang scenario. We see the two stages of radiation and matter domination. On the bottom, an idealization of a typical photon trajectory. Before decoupling, the mean free path is very small due to the many interactions with baryons and electrons. After decoupling, the Universe becomes transparent, and the photon travels in straight line, indifferent to the surrounding distribution of electrically neutral matter.

motion in a gas of particles: they form what is called a "black–body". In any black–body, the many interactions maintain the photons in thermal equilibrium, and their spectrum (i.e., the number density of photons as a function of wavelength) obeys to a law found by Planck in the 1890's. Any "Planck spectrum" is associated with a given temperature.

Following the Hot Big Bang scenario, after recombination, the photons do not see any more charged electrons and nuclei, but only neutral atoms. So, they stop interacting significantly with matter. Their mean free path becomes infinite, and they simply travel along geodesics – excepted a very small fraction of them which interacts accidentally with atoms, but since matter gets diluted, this phenomenon rapidly becomes negligible. So, essentially, the photons travel freely from recombination until now, keeping the same energy spectrum as they had before, i.e., a Planck spectrum, but with a temperature that decreases with the expansion. This is an effect of General Relativity: the wavelength of an individual photon is proportional to the scale factor; so the shape of the Planck spectrum is conserved, but the whole spectrum is shifted in wavelength. The temperature of a black–body is related to the energy of an average photon with average wavelength: $T \sim <E> = hc/<\lambda>$. So, the temperature decreases like $1/<\lambda>$, i.e., like $a^{-1}(t)$.

The physicists that we mentioned above noticed that these photons could still be observable today, in the form of a homogeneous background radiation with a Planck spectrum. Following their calculations – based on nucleosynthesis

– the present temperature of this cosmological black–body had to be around a few Kelvin degrees. This would correspond to typical wavelengths of the order of one millimeter, like microwaves.

2.1.4   The discovery of the Cosmic Microwave Background

These ideas concerning the Hot Big Bang scenario remained completely unknown, excepted from a small number of theorists.

In 1964, two American radio–astronomers, A. Penzias and R. Wilson, decided to use a radio antenna of unprecedented sensitivity – built initially for telecommunications – in order to make some radio observations of the Milky Way. They discovered a background signal, of equal intensity in all directions, that they attributed to instrumental noise. However, all their attempts to eliminate this noise failed.

By chance, it happened that Penzias phoned to a friend at MIT, Bernard Burke, for some unrelated reason. Luckily, Burke asked about the progresses of the experiment. But Burke had recently spoken with one of his colleagues, Ken Turner, who was just back from a visit Princeton, during which he had followed a seminar by Peebles about nucleosynthesis and possible relic radiation. Through this series of coincidences, Burke could put Penzias in contact with the Princeton group. After various checks, it became clear that Penzias and Wilson had made the first measurement of a homogeneous radiation with a Planck spectrum and a temperature close to 3 Kelvins: the Cosmic Microwave Background (CMB). Today, the CMB temperature has been measured with great precision: $T_0 = 2.728$ K.

This fantastic observation was a very strong evidence in favor of Hot Big Bang scenario. It was also the first time that a cosmological model was checked experimentally. So, after this discovery, more and more physicists realized that reconstructing the detailed history of the Universe was not purely science fiction, and started to work in the field.

The CMB is present in our everyday life: fortunately, it is not as powerful as a microwave oven, but when we look at the background noise on the screen of a TV set, one fourth of the power comes from the CMB!

2.1.5   The Thermal history of the Universe

Between the 1960's and today, a lot of efforts have been made for studying with increasing precision the various stages of the Hot Big Bang scenario. Today, some epochs in the history of Universe are believed to be well understood, and are confirmed by observations; some other remain very speculative.

For the earliest stages, there are still many competing scenarios – depending, for instance, on what string theory looks like. Following the most conventional picture, gravity becomes a classical theory (with well–defined time and space dimensions) at a time called the Planck time [1]: $t \sim 10^{-36}$s, $\rho \sim (10^{18} \mathrm{GeV})^4$. Then, there would be a stage of inflation (see section 2.4), possibly related with GUT (Grand Unified Theory) symmetry breaking at $t \sim 10^{-32}$s, $\rho \sim (10^{16} \mathrm{GeV})^4$. The EW (electroweak) symmetry breaking would occur at $t \sim 10^{-6}$s, $\rho \sim (100 \text{ GeV})^4$. The generation of the baryons out of some more fundamental particles would be related to one of these two symmetry breakings (there are various possible scenarios). Then, at $t \sim 10^{-4}$s, $\rho \sim (100 \text{ MeV})^4$, the quarks would combine into hadrons.

---

[1] By convention, the origin of time is chosen by extrapolating the scale-factor to $a(0) = 0$. Of course, this is only a convention, it has no physical meaning.

After these stages, we enter into a series of phase transitions that are much better understood, and well constrained by the observations. These are:

1. at $t = 1 - 100$ s, $T = 10^9 - 10^{10}$ K, $\rho \sim (0.1 - 1 \text{ MeV})^4$, nucleosynthesis: formation of light nuclei, in particular hydrogen, helium and lithium. By comparing the theoretical predictions with the observed abundance of light elements in the present Universe, it is possible to give a very precise estimate of the total density of baryons in the Universe: $\Omega_B h^2 = 0.020 \pm 0.002$.

2. at $t \sim 10^4$ yr, $T \sim 10^4$ K, $\rho \sim (1 \text{ eV})^4$, the radiation density equals the matter density: the Universe goes from radiation domination to matter domination.

3. at $t \sim 10^5$ yr, $T \sim 2500$ K, $\rho \sim (0.1 \text{ eV})^4$, recombination of atoms and decoupling of photons. After that time, the Universe is transparent: the photons free–stream along geodesics. So, by looking at the CMB, we obtain a picture of the Universe at decoupling. Nothing has changed in the distribution of photons between 10^5 yr and today, excepted for an overall redshift of all wavelengths (implying $\rho \propto a^{-4}$, and $T \propto a^{-1}$).

4. after recombination, the small inhomogeneities of the smooth matter distribution are amplified. This leads to the formation of stars and galaxies, as we shall see later.

The success of the Hot big bang Scenario relies on the existence of a radiation–dominated stage followed by a matter–dominated stage. However, an additional stage of curvature or cosmological constant domination is not excluded.

2.1.6   A recent stage of curvature or cosmological constant domination?

If today, there was a very large contribution of the curvature and/or cosmological constant to the expansion ($\Omega_M \ll |\Omega_k|$ or $\Omega_M \ll |\Omega_\Lambda|$), the deviations from the Hot Big Bang scenario would be very strong and incompatible with many observations. However, nothing forbids that Ω_k and/or Ω_Λ are of order one. In that case, they would play a significant part in the Universe expansion only in a recent epoch (typically, starting at a redshift of one or two). Then, the main predictions of the conventional Hot Big Bang scenario would be only slightly modified. For instance, we could leave in a close or open Universe with $|\Omega_k| \sim 0.5$: in that case, there would be a huge radius of curvature, with observable consequences only at very large distances.

In both cases $\Omega_k = \mathcal{O}(1)$ and $\Omega_\Lambda = \mathcal{O}(1)$, the recent evolution of $a(t)$ would be affected (as clearly seen from the Friedmann equation). So, there would be a modification of the apparent magnitude–redshift relation, and of the angular diameter–redshift relation. We will see in section 2.3 how this has been used in recent experiments.

Another consequence of a recent k or Λ domination would be to change the age of the Universe. Our knowledge of the age of the Universe comes from the measurement of the Hubble parameter today. Within a given cosmological scenario, it is possible to calculate the function $H(t)$ (we recall that by convention, the origin of time is such that $a \to 0$ when $t \to 0$). Then, by measuring the present value H_0, we obtain immediately the age of the Universe t_0. The result does not depend very much on what happens in the early Universe. Indeed, equality takes place very early, at 10^4yr. So, when we try to evaluate roughly the age of the Universe in billions of years, we are essentially sensitive to what happens *after* equality.

In a matter–dominated Universe, $H = 2/(3t)$. Then, using the current estimate of H_0, we get $t_0 = 2/(3H_0) \sim 9$ Gyr. If we introduce a negative curvature or a positive cosmological constant, it is easy to show (using the Friedmann equation) that H decreases more slowly in the recent epoch. In that case, for the same value of H_0, the age of the Universe is obviously bigger. For instance, with $\Omega_\Lambda \simeq 0.9$, we would get $t_0 \sim 18$ Gyr.

We leave the answer to these intriguing issues for section 2.3.

2.1.7   Dark Matter

We have many reasons to believe that the non-relativistic matter is of two kinds: ordinary matter, and dark matter. One of the well-known evidences for dark matter arises from galaxy rotation curves.

Inside galaxies, the stars orbit around the center. If we can measure the redshift in different points inside a given galaxy, we can reconstruct the distribution of velocity $v(r)$ as a function of the distance r to the center. It is also possible to measure the distribution of luminosity $I(r)$ in the same galaxy. What is not directly observable is the mass distribution $\rho(r)$. However, it is reasonable to assume that the mass distribution of the *observed luminous matter* is proportional to the luminosity distribution: $\rho_{\text{lum}}(r) = b\, I(r)$, where b is an unknown coefficient of proportionality called the bias. From this, we can compute the gravitational potential Φ_{lum} generated by the luminous matter, and the corresponding orbital velocity, given by ordinary Newtonian mechanics:

$$\rho_{\text{lum}}(r) = b\, I(r), \tag{2.18}$$

$$\Delta\Phi_{\text{lum}}(r) = 4\pi\mathcal{G}\, \rho_{\text{lum}}(r), \tag{2.19}$$

$$v_{\text{lum}}^2(r) = r\frac{\partial}{\partial r}\Phi_{\text{lum}}(r). \tag{2.20}$$

So, $v_{\text{lum}}(r)$ is known up to an arbitrary normalization factor $\sqrt{b}$. However, for many galaxies, even by varying b, it is impossible to obtain a rough agreement between $v(r)$ and $v_{\text{lum}}(r)$ (see figure 2.3). The stars rotate faster than expected at large radius. We conclude that there is some non–luminous matter, which deepens the potential well of the galaxy.

Apart from galactic rotation curves, there are many arguments – of more cosmological nature – which imply the presence of a large amount of non–luminous matter in the Universe, called dark matter. For various reasons, it cannot consist in ordinary matter that would remain invisible just because it is not lighten up. Dark matter has to be composed of particle that are intrinsically uncoupled with photons – unlike ordinary matter, made up of baryons. Within the standard model of particle physics, a good candidate for non-baryonic dark matter would be a neutrino with a small mass. Then, dark matter would be relativistic (this hypothesis is called Hot Dark Matter or HDM). However, HDM is excluded by some types of observations: dark matter particles have to be non-relativistic, otherwise galaxy cannot form during matter domination. Non–relativistic dark matter is generally called Cold Dark Matter (CDM).

There are a few candidates for CDM in various extensions of the standard model of particle physics: for instance, some supersymmetric partners of gauge bosons (like the neutralino or the gravitino), or the axion of the Peccei-Quinn symmetry. Despite many efforts, these particles have never been observed directly in the laboratory. This is not completely surprising, given that they are – by definition – very weakly coupled to ordinary particles.

In the following, we will decompose Ω_M in $\Omega_B + \Omega_{\text{CDM}}$. This introduces one more cosmological parameter. With this last ingredient, we have described the
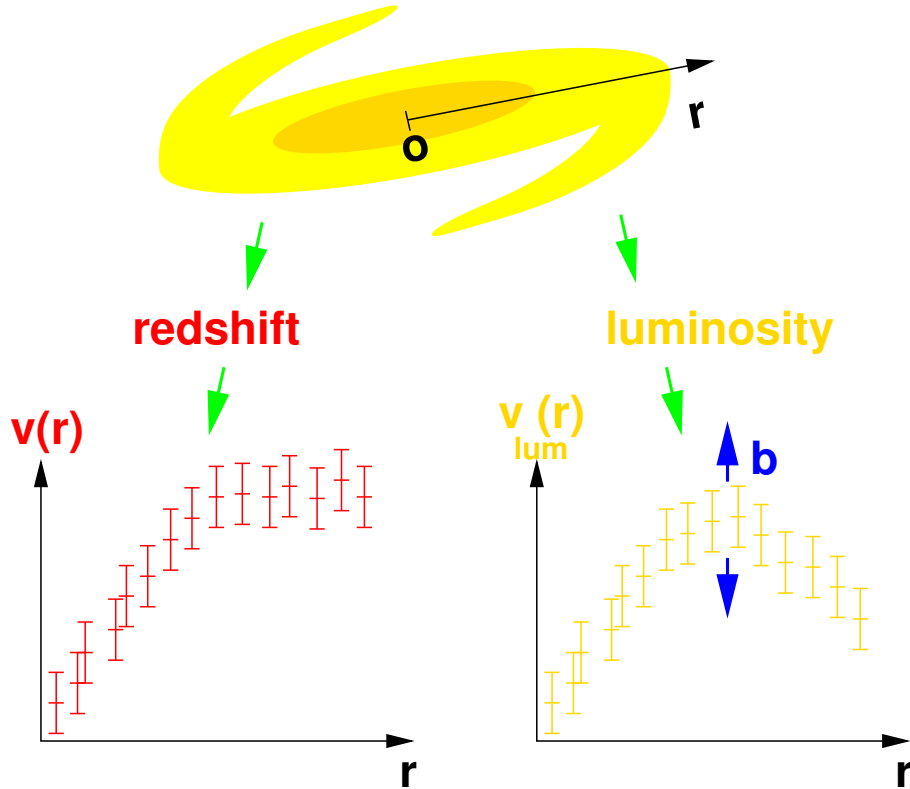
Figure 2.3: A sketchy view of the galaxy rotation curve issue. The genuine orbital velocity of the stars is measured directly from the redshift. From the luminosity distribution, we can reconstruct the orbital velocity under the assumption that all the mass in the galaxy arises form of the observed luminous matter. Even by varying the unknown normalization parameter b, it is impossible to obtain an agreement between the two curves: their shapes are different, with the reconstructed velocity decreasing faster with r than the genuine velocity. So, there has to be some non–luminous matter around, deepening the potential well of the galaxy.

main features of the Standard Cosmological Model, at the level of homogeneous quantities. We will now focus on the perturbations of this background.

2.2 Cosmological perturbations

2.2.1 Linear perturbation theory

All quantities, such as densities, pressures, velocities, curvature, etc., can be decomposed into a spatial average plus some inhomogeneities:

$$\rho_x(t, \vec{r}) = \bar{\rho}_x(t) + \delta \rho_x(t, \vec{r}). \tag{2.21}$$

We know that the CMB temperature is approximately the same in all directions in the sky. This proves that in the early Universe, at least until a redshift $z \sim 1000$, the distribution of matter was very homogeneous. So, we can treat the inhomogeneities as small linear perturbations with respect to the homogeneous

background. For a given quantity, the evolution becomes eventually non–linear when the relative density inhomogeneity

$$\delta_x(t, \vec{r}) = \frac{\delta\rho_x(t, \vec{r})}{\bar{\rho}_x(t)} \tag{2.22}$$

becomes of order one.

In linear perturbation theory, it is very useful to make a Fourier transformation, because the evolution of each Fourier mode is independent of the others:

$$\delta_x(t, \vec{k}) = \int d^3\vec{r}\, e^{-i\vec{k}.\vec{r}} \delta_x(t, \vec{r}). \tag{2.23}$$

Note that the Fourier transformation is defined with respect to the comoving coordinates $\vec{r}$. So, k is the *comoving* wavenumber, and $2\pi/k$ the *comoving* wavelength. The *physical* wavelength is

$$\lambda(t) = \frac{2\pi a(t)}{k}. \tag{2.24}$$

This way of performing the Fourier transformation is the most convenient one, leading to independent evolution for each mode. It accounts automatically for the expansion of the Universe. If no specific process occurs, $\delta_x(t, \vec{k})$ will remain constant, but each wavelength $\lambda(t)$ is nevertheless stretched proportionally to the scale factor.

Of course, to a certain extent, the perturbations in the Universe are randomly distributed. The purpose of the cosmological perturbation theory is not to predict the *individual* value of each $\delta_x(t, \vec{k})$ in each direction, but just the *spectrum* of the perturbations at a given time, i.e., the root–mean–square of all $\delta_x(t, \vec{k})$ for a given time and wavenumber, averaged over all directions. This spectrum will be noted simply as $\delta_x(t, k)$.

The full linear theory of cosmological perturbation is quite sophisticated. Here, we will simplify considerably. The most important density perturbations to follow during the evolution of the Universe are those of photons, baryons, CDM, and finally, the curvature perturbations. As far as curvature is concerned, the homogeneous background is fully described by the Friedmann expression for infinitesimal distances, i.e., by the scale factor $a(t)$ and the spatial curvature parameter k. We need to define some perturbations around this average homogeneous curvature. The rigorous way to do it is rather technical, but at the level of this course, the reader can admit that the most relevant curvature perturbation is a simple function of time and space, which is almost equivalent to the usual Newtonian gravitational potential $\Phi(t, \vec{r})$ (in particular, at small distance, the two are completely equivalent).

2.2.2 The horizon

Now that we have seen which types of perturbations need to be studied, let us try to understand their evolution. In practice, this amounts in integrating a complicated system of coupled differential equations, for each comoving Fourier mode k. But if we just want to understand *qualitatively* the evolution of a given mode, the most important is to compare its wavelength $\lambda(t)$ with a characteristic scale called the *horizon*. Let us define this new quantity.

Suppose that at a time t_1, two photons are emitted in opposite directions. The horizon d_H at time t_2 relative to time t_1 is defined as the physical distance between the two photons at time t_2 (see figure 2.4). In order to compute it, we
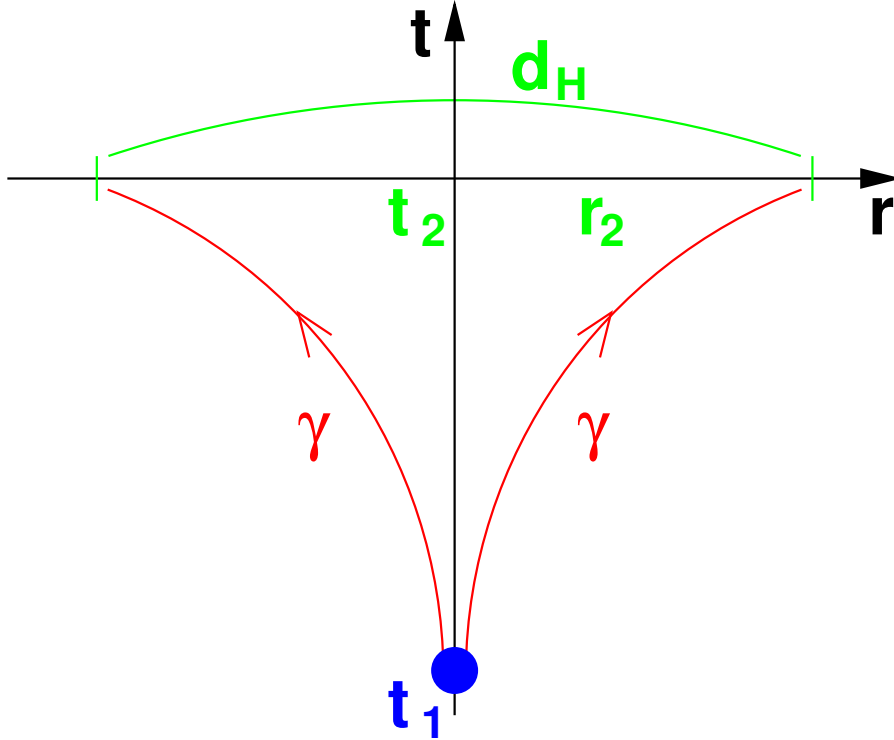
Figure 2.4: The horizon $d_H(t_1, t_2)$ is the physical distance at time t_2 between two photons emitted in opposite directions at time t_1.

note that if the origin $r = 0$ coincides with the point of emission, the horizon can be expressed as

$$d_H(t_1, t_2) = 2 \int_0^{r_2} dl = 2 \int_0^{r_2} a(t_2) \frac{dr}{\sqrt{1 - kr^2}}. \tag{2.25}$$

But the equation of propagation of light, applied to one of the photons, also gives

$$\int_{t_1}^{t_2} \frac{c\, dt}{a(t)} = \int_0^{r_2} \frac{dr}{\sqrt{1 - kr^2}}. \tag{2.26}$$

Combining the two equations, we get

$$d_H(t_1, t_2) = 2a(t_2) \int_{t_1}^{t_2} \frac{c\, dt}{a(t)}. \tag{2.27}$$

Physically, since no information can travel faster than light, the horizon represents the maximal scale on which a process starting at time t_1 can have an impact at time t_2. In the limit in which t_1 is very close to the initial singularity, d_H represents the *causal horizon*, i.e. the maximal distance at which too points in the Universe can be in causal contact. In particular, perturbations with wavelengths $\lambda(t_2) > d_H(t_1, t_2)$ are *acausal*, and cannot be affected by any physical process – if they were, it would mean that some information has traveled faster than light.

During **radiation domination**, when $a(t) \propto t^{1/2}$, it is straightforward to

35

compute

$$d_H(t_1, t_2) = 4 \ c \ t_2^{1/2}[t_2^{1/2} - t_1^{1/2}]. \qquad (2.28)$$

In the limit in which $t_1 \ll t_2$, we find

$$d_H(t_1, t_2) \to 4 \ c \ t_2 = 2 \ R_H(t_2). \qquad (2.29)$$

So, during radiation domination, the causal horizon coincides with the Hubble radius.

During **matter domination**, when $a(t) \propto t^{2/3}$, we get

$$d_H(t_1, t_2) = 6 \ c \ t_2^{2/3}[t_2^{1/3} - t_1^{1/3}]. \qquad (2.30)$$

When $t_1 \ll t_2$ this becomes

$$d_H(t_1, t_2) \to 6 \ c \ t_2 = 4 \ R_H(t_2). \qquad (2.31)$$

So, even during a matter–dominated stage following a radiation–dominated stage, the causal horizon remains of the same order of magnitude as the Hubble radius.

Since each wavelength $\lambda(t) = 2\pi a(t)/k$ evolves with time at a different rate than the Hubble radius $R_H(t)$, a perturbation of fixed wavenumber k can be super–horizon ($\lambda \gg R_H$) at some initial time, and sub–horizon ($\lambda \ll R_H$) at a later time. This is illustrated on figure 2.5. While the scale factor is always decelerating, the Hubble radius grows linearly: so, the Fourier modes enter one after each other inside the causal horizon. The modes which enter earlier are those with the smallest wavelength.

What about the perturbations that we are able to measure today? Can they be super–horizon? In order to answer this question, we first note that the largest distance that we can see today in our past–light–cone is the one between two photons emitted at decoupling, and reaching us today from opposite direction in the sky. Because the Universe was not transparent to light before decoupling, with current technology, we cannot think of any observation probing a larger distance.

So, in order to compute the maximal wavelength accessible to observation, we can make a short calculation, quite similar to that of the horizon, but with reversed photon trajectories: the two photons are emitted at (t_1, r_1), and they reach us today at $(t_0, 0)$ from opposite directions. The propagation–of–light equation gives

$$\int_{t_1}^{t_0} \frac{c \ dt}{a(t)} = \int_{r_1}^{0} \frac{-dr}{\sqrt{1 - kr^2}}. \qquad (2.32)$$

But today, two points located at coordinate r_1 in opposite directions are separated by the physical distance

$$2 \int_0^{r_1} dl = 2 \int_0^{r_1} a(t_0) \frac{dr}{\sqrt{1 - kr^2}} = 2a(t_0) \int_{t_1}^{t_0} \frac{c \ dt}{a(t)} = d_H(t_1, t_2). \qquad (2.33)$$

We conclude that the causal horizon evaluated today gives an upper bound on the distances accessible to any observation (whatever the time of decoupling is). Our knowledge of the Universe is limited to a finite sphere of physical radius $R_H(t_0)$. So, all observable cosmological perturbations are sub–horizon today, while in the past, they were super–horizon (see figure 2.5). The largest
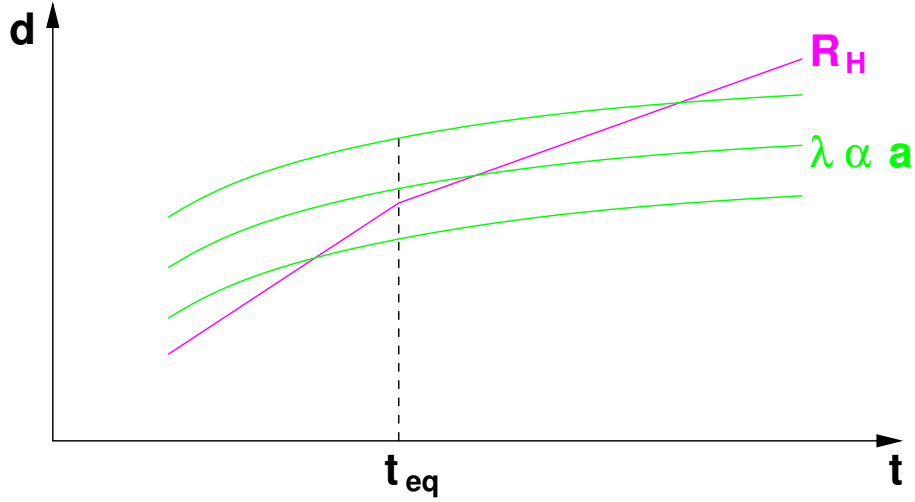
Figure 2.5: The wavelengths of some cosmological perturbations compared with the Hubble radius, during radiation domination and matter domination. Because all wavelengths $\lambda(t) = 2\pi a(t)/k$ grow with negative acceleration, and $R_H(t)$ grows linearly, the modes enter one after each other inside the horizon. The smallest perturbations enter during radiation domination, the others during matter domination. The largest cosmological perturbation observable today (in CMB observations) has $\lambda(t_0) = R_H(t_0)$.

observable scale has entered inside the horizon very recently, with a wavelength approximately equal to

$$R_H(t_0) = \frac{c}{H_0} \simeq \frac{3 \times 10^8 \text{ m s}^{-1}}{70 \text{ km s}^{-1}\text{Mpc}^{-1}} \simeq 5 \times 10^3 \text{ Mpc}. \tag{2.34}$$

All these general considerations will allow us to understand qualitatively the evolution of the different perturbations, and especially those of photons and CDM, which are the most interesting. We will try to visualize the evolution on a three-dimensional diagram, like the one of figure 1.6. This representation gives the perturbation amplitude $\delta_x(t, k)$ for a given specie x. In other words, it gives the evolution of each Fourier mode with respect to time. The purple line shows the limit between causal (sub–horizon) perturbations on the right, and acausal (super–horizon) perturbations on the left.

2.2.3 Photon perturbations

In good approximation, the photons keep a Planck spectrum during all the relevant evolution. As a consequence, the photon density perturbations $\delta_\gamma(t, k)$ can be reformulated as some temperature fluctuations around the mean black–body temperature: $\delta T/\bar{T}(t, k)$ (where $\bar{T}$ is the spatial average of the temperature at time t). The evolution is very different before and after the time of photon decoupling.

Indeed, before decoupling, photons are strongly coupled to baryons through electromagnetic interactions, and have a vanishing mean–free–path. As a consequence, the baryons and the photons are almost indistinguishable: they share
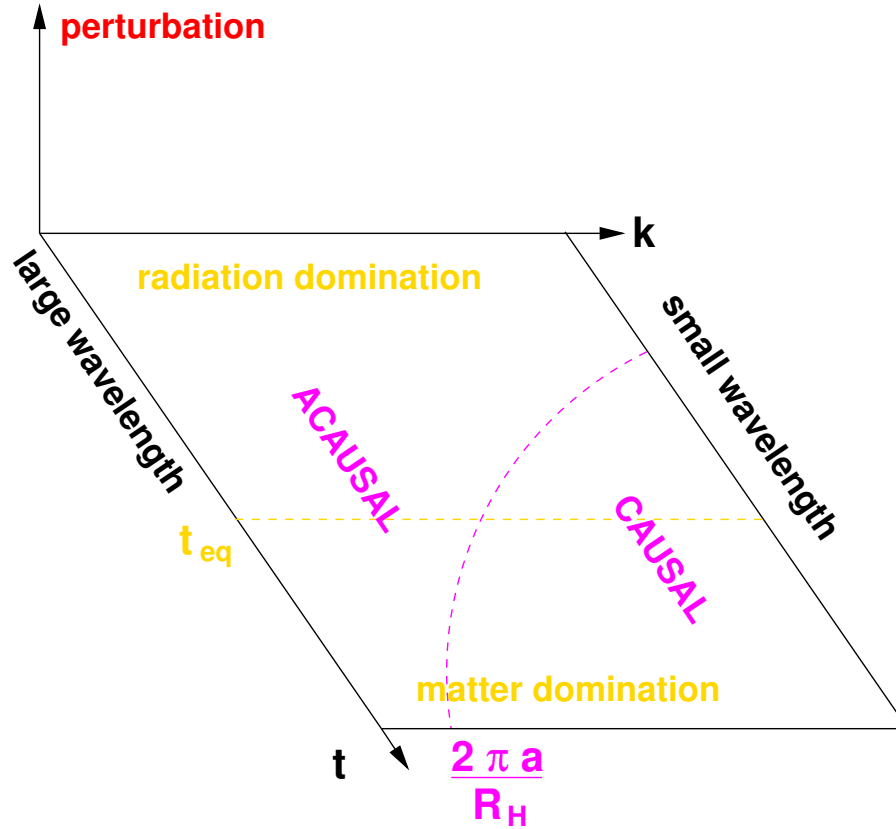
Figure 2.6: For any specie, this type of diagram can be used for understanding qualitatively the evolution of the Fourier spectrum as a function of time. Each Fourier mode of fixed comoving wavenumber k enters into the causal horizon when it crosses the purple line (which corresponds to $\lambda(t) = R_H(t)$, i.e., to $k = 2\pi a(t)/R_H(t)$).

the same density perturbations, and can be viewed as single fluid. In addition, since baryons have a mass, they feel gravitational forces: so, the entire baryon–photon fluid is coupled with the gravitational potential.

So, in each well of gravitational potential, the baryon–photon fluid tends to be compressed by gravity, but the relativistic photon pressure $p = \frac{1}{3}\rho$ tends to resist to this compression. The dynamics of the fluid is governed by two antagonist forces, as for any oscillator. So, for each Fourier mode, any departure from equilibrium – such as a non–zero initial perturbation – will lead to oscillations. These oscillations propagate through the fluid exactly like sound waves in the air: due to this analogy, the perturbations in the baryon–photon fluid before decoupling are called *acoustic oscillations*.

Of course, the oscillations can occur only for causal wavelengths: because of causality, wavelengths bigger than the Hubble radius do not feel any force. So, a given mode with a given non–zero initial amplitude starts oscillating only when it enters into the horizon (see figure 2.7). The initial amplitude for each mode at initial time is given by the *primordial spectrum* of perturbations. Of course, if the Universe was perfectly homogeneous from the very beginning (with no initial perturbations), there would be no acoustic oscillations – and no formation of structure either, so we *must* assume some primordial perturbations. At this
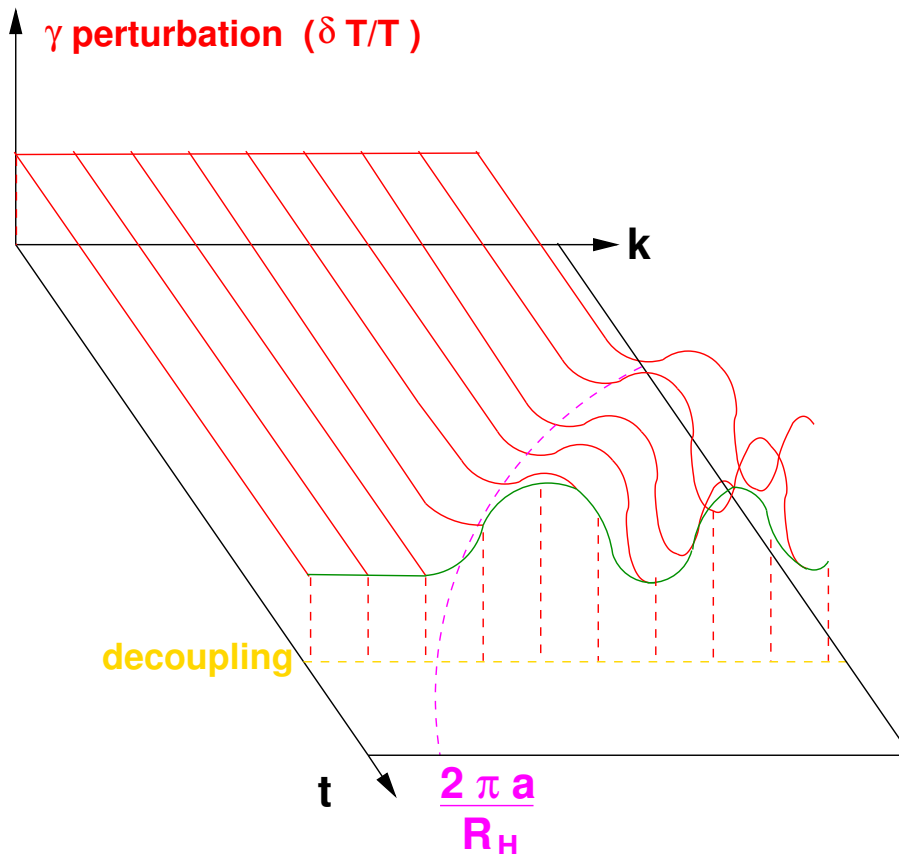
Figure 2.7: Evolution of the Fourier spectrum of photon temperature fluctuations, as a function of time. We start from a flat primordial spectrum, as predicted by inflation and as confirmed by observations. When a given Fourier mode enters inside the horizon (i.e., crosses the dashed purple line), it starts to oscillate. At decoupling, the temperature distribution is frozen (later, all temperatures are redshifted by a factor $1/a(t)$, so the relative fluctuation $\delta T/\bar{T}$ remains constant). So, the Fourier spectrum measured today is the same as at the time of decoupling. It exhibits a flat plateau for wavelengths bigger than the Hubble radius at decoupling, and then a series of peaks called *acoustic peaks*.

stage, the primordial spectrum has to be assumed arbitrarily. Only in the last section, we will see a precise mechanism called "inflation" *leading* to a given primordial spectrum. On figure 2.7, we start from a flat spectrum, independent of k – in good approximation, this corresponds to what is *observed* today, and also to what is *predicted* by the theory of inflation.

When the photons decouple from the matter, their mean–free–path grows rapidly from zero ton infinity, and they travel in straight line with essentially no interaction. During this *free-streaming*, all their characteristics – momentum, energy, wavelength, spectrum, temperature – do not evolve, excepted under the effect of homogeneous expansion. So, δT and $\bar{T}$ are both shifted proportionally to $1/a(t)$, but the distribution of $\delta T/\bar{T}(t, k)$ observed today is exactly the same as at the time of decoupling. Observing the temperature anisotropies in the sky amounts exactly in taking a picture of the Universe at the time of decoupling.

So, in order to know the present Fourier spectrum of temperature fluctua-

tions, it is enough to follow the acoustic oscillations *until decoupling.* As easily understood from figure 2.7, the Fourier modes which are still outside the Hubble radius at decoupling have no time to oscillate. Smaller wavelengths have enough time for one oscillation, even smaller ones for two oscillations, etc. The Fourier spectrum of temperature fluctuations consists in a flat plateau for $k < 2\pi a(t_{\mathrm{dec}})/R_H(t_{\mathrm{dec}})$, and then a first peak, a second peak, a third peak, etc.

Note that according to this model, the perturbations oscillate, but they keep the same order of magnitude (there is no large amplification). Therefore, if we start from small primordial perturbations, we are able to describe all the evolution of temperature fluctuations with the *linear* theory of perturbations.

2.2.4   Observing the CMB anisotropies

The measurement of the CMB by Penzias and Wilson showed no evidence for perturbations – in fact, we should say anisotropies, because we can only map the temperature along different *directions.* However, some CMB anisotropies must exist, because our Universe is not homogeneous today. The current inhomogeneities have to grow from fluctuations in the past, and at the time of decoupling there must be already at least some small fluctuations in the densities, and therefore, in the temperature. The order of magnitude of CMB anisotropies was predicted many years before being measured. By extrapolating from the present inhomogeneous structure back to the time of decoupling, many cosmologists in the 80's expected $\delta T/\bar{T}$ to be at least of order 10^{-6} – otherwise, clusters of galaxies could not have formed today.

Many experiments were devoted to the detection of these anisotropies. The first successful one was COBE-DMR, an American satellite carrying an interferometer of exquisite sensitivity. In 1992, COBE mapped the anisotropies all over the sky, and found an average amplitude $\delta T/\bar{T} \sim 10^{-5}$ (see figure 2.8). This was in perfect agreement with the theoretical predictions – another big success for cosmology. The COBE experiment had an angular resolution of a few degrees: so, anisotropies seen under one degree or less were smoothed by the detector. In a Fourier decomposition, it means that COBE could only measure the spectrum of wavelengths larger than the Hubble radius at decoupling. So, it was not probing the acoustic oscillations, but only the flat plateau.

COBE was the first experimental evidence in favor of the linear theory of cosmological perturbations. The next step was to increase the angular resolution, in order to probe smaller wavelengths, check the existence of the acoustic peaks, and compare them with the theory. In the 90's, there were some considerable efforts in this direction.

First, on the theoretical side, the spectrum of CMB anisotropies was investigated with greater and greater precision. Some people developed numerical codes computing the exact spectrum for any choice of cosmological parameters. Indeed, the details of the peaks depend on many quantities. For instance, the amplitude of the acoustic oscillations is governed by the density of baryons and photons. The whole final shape of the spectrum depends on the primordial spectrum of perturbations. Last but not least, the position of the peaks depends on the Hubble radius at decoupling. Fortunately, the theory is able to predict the *physical size* of $R_H(t_{\mathrm{dec}})$, and the redshift of decoupling – so, by measuring the angular size associated to the wavelength of the acoustic peaks, it should be possible to use the angular diameter–redshift relation (see section 1.3.5), and to measure the *curvature* of the Universe! We conclude that if the acoustic peaks could be measured, the comparison with the theoretical spectrum would reveal the values of the most crucial cosmological parameters.

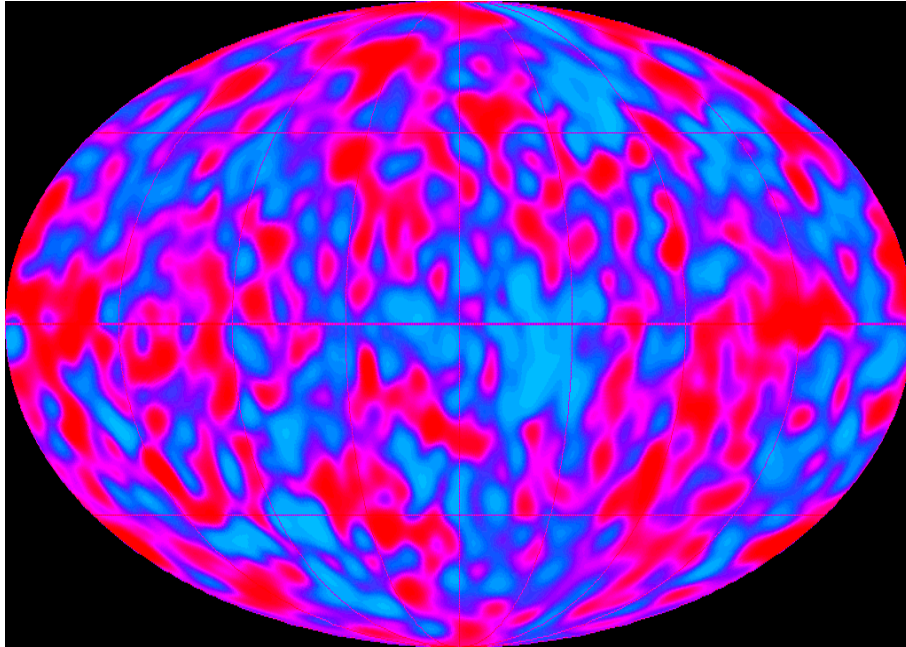On the experimental side, a decisive progress was made with Boomerang,

Figure 2.8: The first genuine "picture of the Universe" at the time of decoupling, 100 000 years after the initial singularity, and 15 billion years before the present epoch. Each blue (resp. red) spot corresponds to a slightly colder (resp. warmer) region of the Universe at that time. This map, obtained by the American satellite COBE in 1992, covers the entire sky: so, it pictures a huge sphere centered on us. The fluctuations are only of order 10^{-5} with respect to the average value $T_0 = 2.728$ K. They are the "seeds" for the present structure of the Universe: each red spot corresponds to a small overdensity of photons and baryons at the time of decoupling, that has been enhanced later, leading to galaxies and clusters of galaxies today.

a US–Italian–Canadian balloon, carrying some detectors called bolometers. In 2001, Boomerang published the map of figure 2.9. It focuses on a small patch of the sky, but with much better resolution than COBE (a few arc–minutes). The Fourier decomposition of the Boomerang map clearly shows the first three acoustic peaks (see figure 2.10). The data points are perfectly fitted with a theoretical curve obtained numerically, for a simple choice of cosmological parameters (figure 2.11). This beautiful agreement with some predictions made several decades before the experiment is one of the major successes of cosmology. By fitting theoretical curves to this data, it is possible to measure various cosmological parameters with already a good precision. For instance, the position of the first peak is strongly in favor of $k = 0$, i.e., a spatially flat Universe. We will come back to these measurements in section 2.3.

There are still many CMB experiments going on. The American satellite MAP is currently taking data in order to map the full sky with a better resolution than Boomerang. In 2007, the European satellite "Planck" will be launched. Planck is expected to perform the best possible measurement of the CMB temperature anisotropies, with such great precision that most cosmological parameters should be measured with only one per cent uncertainty.
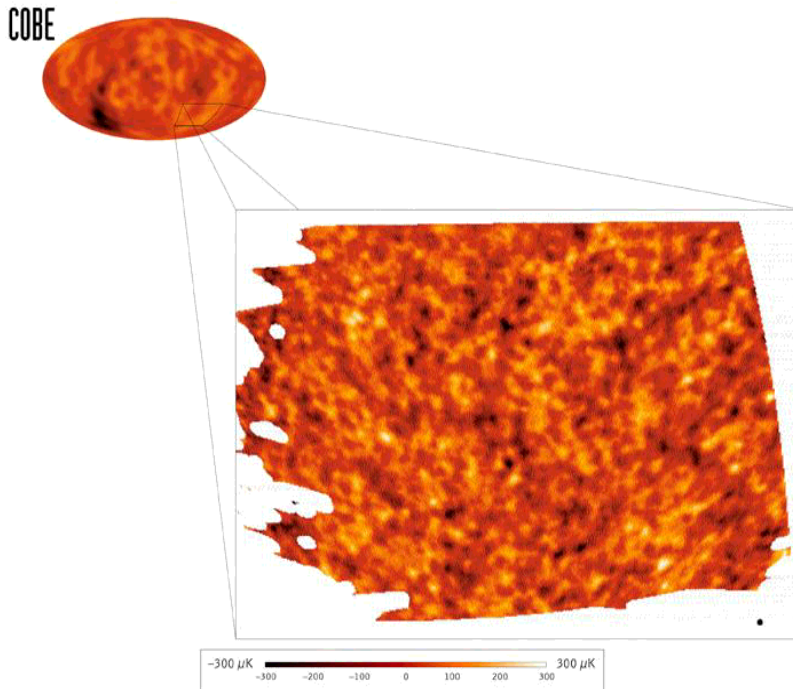
Figure 2.9: The map of CMB anisotropies obtained by the balloon experiment Boomerang in 2001. Unlike COBE, Boomerang only analyzed a small patch of the sky, but with a much better angular resolution of a few arc–minutes. The black (resp. yellow) spots correspond to colder (resp. warmer) regions.

2.2.5 Matter perturbations

We now focus on the evolution of the matter perturbations. Because we believe that the Universe contains much more CDM than baryons, in first approximation, we can forget the baryons, and do as if all the matter was in the form of CDM, with absolutely no electromagnetic interactions. This is crucial, because in absence of any coupling with the photons, the CDM fluid has a vanishing pressure: so, it tends to concentrate in each well of gravitational potential – instead of oscillating like the baryons before decoupling. This is called gravitational collapse.

Of course, as all causal mechanisms, gravitational collapse can only affect modes inside the Hubble radius. However, there is one more restriction. During matter domination, when a potential well appears, CDM concentrates in it, and deepens the gravitational potential even more. So, both CDM and gravitational potential perturbations are quickly amplified. On the other hand, during radiation domination, the gravitational potential follows the perturbations of the dominant energy density: that of radiation. So, there is no significant coupling between the CDM inhomogeneities and the gravitational potential. As a consequence, the amplification of CDM perturbations is efficient only *after* equality and *inside* R_H. If we look at the diagram of figure 2.6, this corresponds to the region on the right of the purple curve, and below the yellow line. So, even today, the Fourier spectrum of CDM perturbations has a characteristic scale corresponding to the Hubble radius at equality. The same type of numerical code which compute the spectrum of CMB anisotropies can also compute
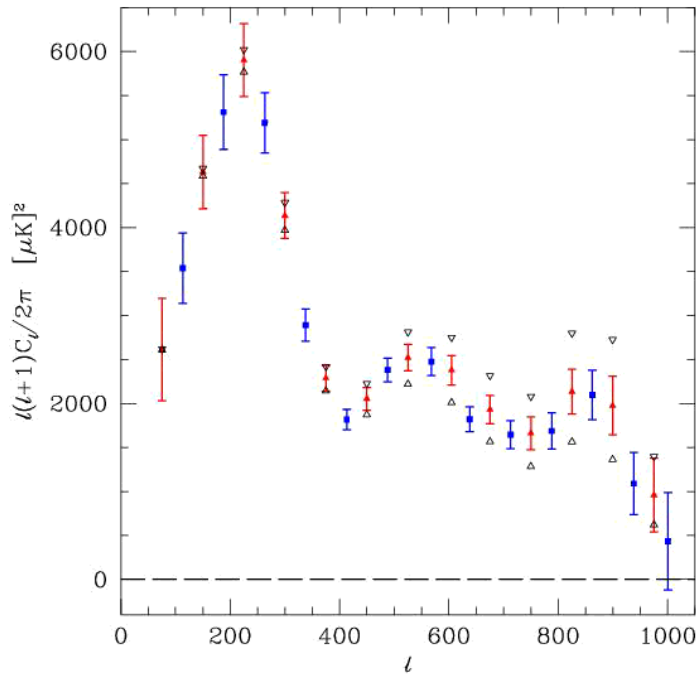
Figure 2.10: The Fourier decomposition of the Boomerang map reveals the structure of the first three acoustic oscillations. These data points account for the spectrum of $(\delta T)^2$ in $(\mu K)^2$ – we know from COBE that δT is roughly of order $10^{-5}T_0 \simeq 30\,\mu K$. The spectrum is not shown as a function of wavenumber k, but of multipole number l. This is because the anisotropies are not observed in 3-dimensional space, but on a two-dimensional sphere: the sphere of redshift $z \sim 1000$ centered on us. For purely geometric reasons, a spherical map is not expandable in Fourier modes, but in Legendre multipoles. However, the interpretation is the same. The first, second and third peak correspond to the modes that could oscillate one, two or three times between their entry into the Hubble radius and the time of decoupling.

the linear matter spectrum with great precision, for any choice of cosmological parameters.

But linear perturbation theory is valid only as long as the perturbations are small, $\delta_{CDM}(t,k) \ll 1$. After, it is no longer interesting to use a Fourier decomposition, because the evolution of each mode is not independent of the others: so, everything becomes much more complicated, and we can make some predictions only from numerical simulations of the gravitational collapse in real space. These are called an N-body simulations.

It is crucial to know at what time a given mode enters into the non-linear regime, $\delta_{CDM}(t,k) \sim 1$. If there was absolutely no growth of CDM perturbations inside the Hubble radius during radiation domination, this time would be the same for all small wavelengths, because they would grow at the same rate, starting from the same value. But because there is a very slow amplification of perturbations during this epoch, the most amplified wavelength are the smallest ones – since they entered earlier into the Hubble radius. So, the smallest wavelengths are the first ones to become non-linear. As a consequence, the smallest structures in the Universe are the older ones: this is called hierarchical structure
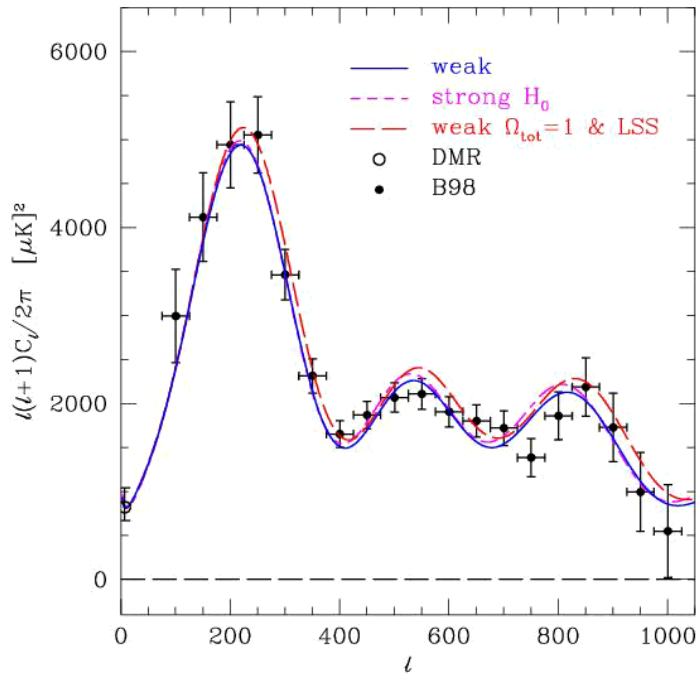
Figure 2.11: The spectrum of the Boomerang map is perfectly fitted by the theoretical predictions. For a given choice of cosmological parameters – such as Ω_B, Ω_{CDM}, Ω_Λ, Ω_R, H_0, the parameters describing the primordial spectrum, etc.– the theoretical calculation is performed by a numerical code of typically 2000 lines. By comparing with the experimental data, one can extract the parameter values, as we will see in section 2.3.

formation.

2.2.6 Hierarchical structure formation

Let's go back to equality, when no efficient amplification of the matter pertur-bations has started: the matter density in the Universe is still a continuous function, with small inhomogeneities of order 10^{-5}. Then, when the smallest wavelengths become non-linear ($\delta_{CDM}(t, k) \sim 1$), some small compact clumps start to form. But at that time, if we smooth the density distribution with a smoothing radius corresponding to the largest non-linear scale, we find again a continuous distribution (remember the analogy in figure 1.4), with a Fourier spectrum obeying to the predictions of the linear theory of perturbations.

As time goes on, some larger and larger scales become non-linear and start to collapse. So, the planets, the stars, and the clouds of gas are expected to form first; then, galaxies, and later, clusters of galaxies. Finally, in the present Universe, the scale of non-linearity is expected to be around 20 or 40 Mpc. This is one order of magnitude bigger than clusters of galaxies: so, even the clusters are not smoothly distributed. In fact, they tend to organize themselves along huge filaments, separated by voids of size 20 to 40 Mpc. The structure of the Universe on these huge scales looks very much like a spounge... But by smoothing the present distribution of matter over scales larger than 40 Mpc, one finds again a continuous distribution, with small inhomogeneities described
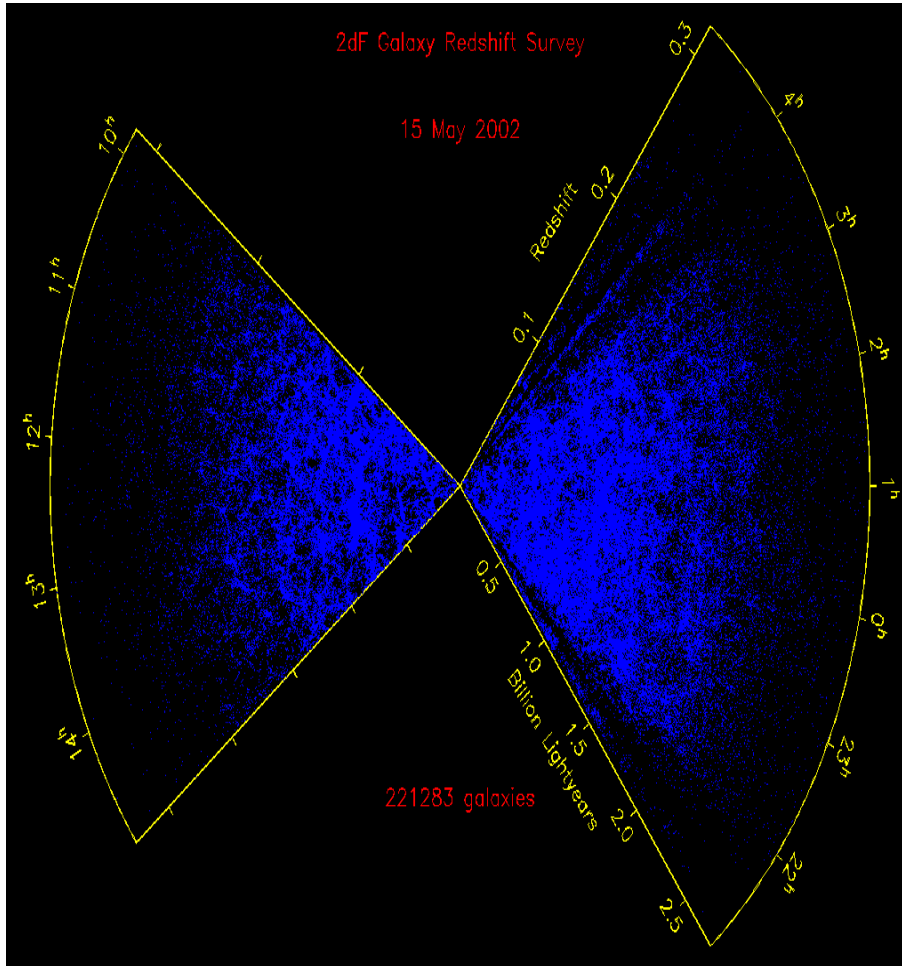
Figure 2.12: The distribution of galaxies in two thin slices of the neighboring Universe, obtained by the 2dF Galaxy Redshift Survey. The radial coordinate is the redshift, or the distance between the object and us.

by the linear Fourier spectrum of cosmological perturbation theory.

2.2.7   Observing the matter spectrum

So, if we have a large three–dimensional map of the galaxy distribution in the Universe, we can smooth it on very large scales, compare with the linear spectrum predicted by the theory, and measure the cosmological parameters on which this spectrum depends. Of course, what we observe in the sky is only luminous baryonic matter, not CDM. But astrophysicists have very good reasons to believe that on large scales, the distribution of luminous matter "traces" the distribution of mass. This means that the inhomogeneities in luminous matter would be proportional to the total density inhomogeneities – i.e., in good approximation, to the CDM inhomogeneities. The unknown coefficient of proportionality is called the bias b:

$$\delta_{\text{luminous}}(t, k) \simeq b \ \delta_{\text{CDM}}(t, k). \tag{2.35}$$
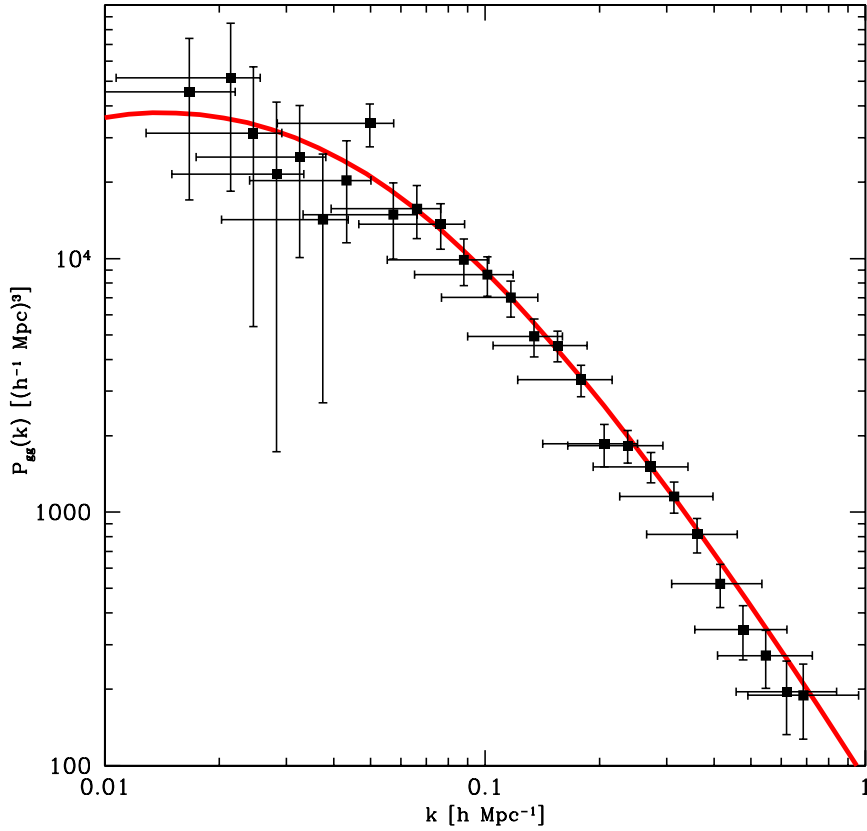
45

Figure 2.13: The Fourier spectrum of matter perturbations, reconstructed from a large galaxy redshift survey (here, the Sloan Digital Sky Survey). The data are compared with a theoretical spectrum, obtained by integrating numerically over a system of differential equations, accounting for the evolution of linear perturbations. In the calculation, the cosmological parameters were chosen in order to obtain a maximum likelihood fit to the data. Comparing the data with the linear spectrum is meaningful only for the largest wavelengths, $k < 0.2h$ Mpc^{-1}, which have not been affected by the non–linear evolution. For smaller scales, the agreement between the data and the theoretical prediction is only a coincidence.

Over the past decades, astronomers could build some very large three-dimensional maps of the galaxy distribution – even larger than the scale of non-linearity. On figure 2.12, we can see too thin "slices" of the surrounding Universe. Inside these slices, each point corresponds to a galaxy seen by the "2dF Galaxy Redshift Survey". The radial coordinate is the redshift, or the distance between the object and us. Even at redshift $z < 0.1$, we can see some darker regions, corresponding to the voids - the largest non-linear structures in the Universe. At $z > 0.1$, the average density seems to decrease smoothly, but this is simply because the survey is limited in sensitivity – at small distance it sees all the galaxies, while at large distance it sees only the brightest ones. By taking this selection into account, and by smoothing over distances of order of

50 Mpc, one recovers a very homogeneous distribution of matter.

On figure 2.13, we see the Fourier spectrum reconstructed from such a red-shift survey. Each experimental point is obtained by smoothing over smaller wavelengths. The data points with $k < 0.2h$ Mpc^{-1} can be directly compared with the linear spectrum predicted by the linear theory of cosmological perturbations. The red line shows one of the best-fit theoretical curves. From such a comparison, one can extract some information on the value of the cosmological parameters. This is similar to the parameter extraction from CMB anisotropies, but it provides some independent constraints.

On the extreme left of figure 2.13, we see that the theoretical curve reaches a maximum (around $k \sim 0.01h$ Mpc^{-1}, i.e., $\lambda \sim 400$ Mpc). This corresponds to the scale which entered into the Hubble radius at the time of equality between radiation and matter. Future redshift surveys are expected to give better data around this scale. Observing the turnover of the matter spectrum with good precision will improve the measurement of the cosmological parameters.

2.3 Measuring the cosmological parameters

There has been some continuous progress in measuring the cosmological parameters over the past three decades; however, the most spectacular improvements have been made in the last four years. Many techniques can be employed for constraining the cosmological scenarios – probably around twenty. Instead of giving an exhaustive review, we will focus only on five important methods.

2.3.1 Abundance of primordial elements

In sections 2.1.3 and 2.1.5, we have seen that the theory of Nucleosynthesis can predict the abundance of light elements formed in the early Universe, when the energy density was of order $\rho \sim (1\text{MeV})^4$. This theory has reached today a great level of precision: it is believed that each relevant nuclear reaction are taken carefully into account. In the standard model of Nucleosynthesis, there is only one free parameter: the baryon density ρ_B, or equivalently, $\Omega_\text{B} h^2$. For each value of the baryon density, the theory predicts a given abundance of hydrogen, helium, lithium, etc. So, by observing the abundance of these elements in the Universe – for instance, using spectroscopy – it is possible to constrain the baryon density. Current observations lead to the prediction

$$\Omega_\text{B} h^2 = 0.020 \pm 0.002. \tag{2.36}$$

2.3.2 CMB anisotropies

The precise dynamics of the acoustic oscillations before decoupling depends on almost all the cosmological parameters: so, the spectrum of CMB anisotropies can give a wide variety of constraints. This type of experiment is considered as the most complete and precise cosmological test. Future CMB experiments like MAP and Planck should be able to measure each parameter with great precision. At the moment, some particular parameters or combinations of parameters are already measured unambiguously, using the CMB anisotropy map of Boomerang (and also of other experiments like Maxima and Dasi). One example is the baryon density, which is related in an interesting way (that we will not detail here) to the relative amplitude of the acoustic peaks. The Boomerang data gives

$$\Omega_\text{B} h^2 = 0.02 \pm 0.01. \tag{2.37}$$

This constraint is less precise than that from Nucleosynthesis, but they are in perfect agreement with each other – which is really striking, since these two measurements are done with completely different techniques: one uses nuclear physics, applied to the Universe when it had a density of order $(1 \text{ Mev})^4$, while the other uses fluid mechanics, applied to a much more recent era with density $(0.1 \text{ eV})^4$. The perfect overlap between the two constraints indicates that our knowledge of the Universe is impressively good, at least during the epoch between $t \sim 1$ s and $t \sim 100\ 000$ yr.

We have seen that the position of the acoustic peak is given by the Hubble radius at decoupling, which value is predicted by the theory. So, using the angular diameter–redshift relation (see section 1.3.5), it is possible to extract some information both on the spatial curvature and on the evolution of the scale factor $a(t)$. A detailed study shows that the leading effect is that of spatial curvature. So, the position of the acoustic peak gives a measurement of $\Omega_k = 1 - \Omega_0$. The result is

$$\Omega_0 = 1.0 \pm 0.1. \tag{2.38}$$

So, according to current CMB experiments, the Universe seems to be spatially flat. This result is extremely useful, but it doesn't say how the present energy density is distributed between matter and a possible cosmological constant[2].

Current CMB experiments also give other interesting constraints (for instance, on the primordial spectrum of perturbations) that we will not explain here.

2.3.3 Age of the Universe

Observations can set a reliable lower bound on the age of the Universe: if an object is seen at a given redshift, corresponding to a given age, the Universe has to be older than this object. During the 1990's, it has been established that the most distant observable quasars are at least 11 Gyr old. On the other hand, a spatially flat matter–dominated Universe would be only 9 Gyr old (see section 2.1.6). So, distant quasars seem to indicate that the Universe is either open or Λ–dominated today.

2.3.4 Luminosity of Supernovae

The evidence for a non–flat Universe and/or a non–zero cosmological constant has increased considerably in 1998, when two independent groups studied the apparent luminosity of distant type Ia supernovae (SNIa). For this type of supernovae, astronomers believe that there is a simple relation between the absolute magnitude and the luminosity decay rate. In other words, by studying the rise and fall of the luminosity curve during a few weeks, one can deduce the absolute magnitude of a given SNIa. Therefore, it can be used in the same way as cepheid, as a probe of the apparent luminosity–redshift relation. In addition, Supernovae are much brighter that cepheids, and can be observed at much larger distances (until redshifts of order one). While observable cepheids only probe short distances, where the apparent luminosity–redshift relation only gives the Hubble law (the proportionality between distance and redshift), the most distant observable SNIa's are in the region where general relativity corrections are important: so, they can provide a measurement of the spatial curvature and of the scale factor evolution.

On figure 2.14, the various curves represent the apparent magnitude–redshift relation, computed for various choices of Ω_M and Ω_Λ. For a given value of H_0,

---

[2]As far as radiation is concerned, we know that today, it is negligible with respect to matter, so that $\Omega_0 \simeq \Omega_M + \Omega_\Lambda = \Omega_B + \Omega_{CDM} + \Omega_\Lambda$.
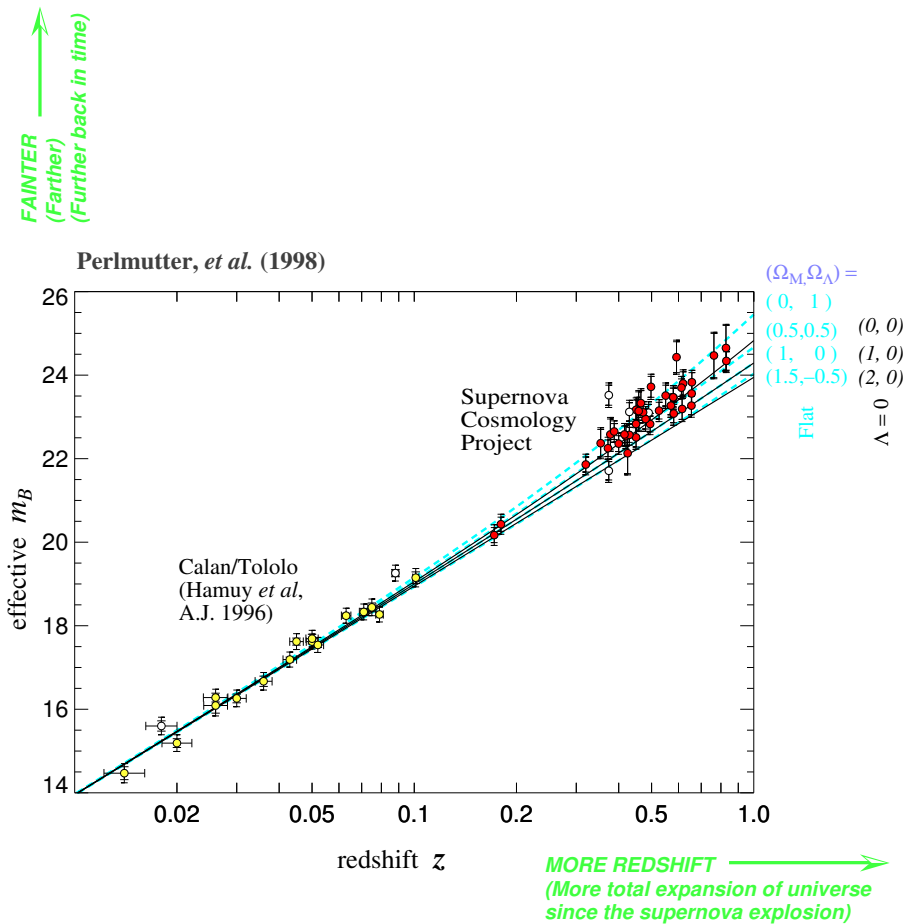
Figure 2.14: The results published by the "Supernovae Cosmology Project" in 1998. The various curves represent the apparent magnitude–redshift relation, computed for various choices of Ω_M and Ω_Λ. The solid black curves account for three examples of a closed/flat/open Universe with no cosmological constant. The dashed blue curves correspond to three spatially flat universes with different values of Ω_Λ. For a given value of H_0, all the curves are asymptotically equal at short distance, probing only the Hubble law. The yellow points are short–distance SNIa's: we can check that they are approximately aligned. The red points, at redshifts between 0.2 and 0.9, show that distant supernovae are too luminous to be compatible with a flat matter-dominated Universe $(\Omega_M, \Omega_\Lambda)$ $=(1,0)$.

all the curves are asymptotically equal at short distance. Significant differences show up only at redshifts $z > 0.2$. Each red data point corresponds to a single supernovae, from the 1998 results of the "Supernovae Cosmology Project". Even if it is not very clear visually from the figure, a detailed statistical analysis reveals
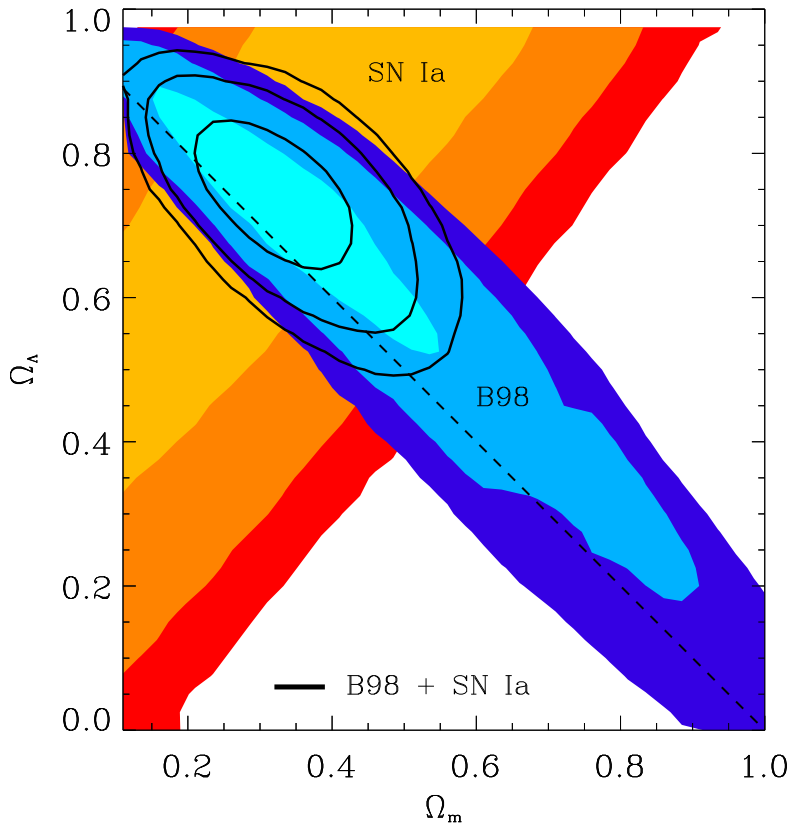
Figure 2.15: Combined constraints on $(\Omega_M, \Omega_\Lambda)$ from CMB and Supernovae experiments. The simple case of a spatially flat Universe with no cosmological constant corresponds to the lower right corner of the plot. In blue, the CMB allowed region favors a spatially flat Universe with $\Omega_M + \Omega_\Lambda \simeq 1$. In orange, the Supernovae constraint is orthogonal to the CMB one. By combining the two results, one obtains the favored region limited by the black ellipses, centered near $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$.

that a flat matter–dominated Universe (with $\Omega_M = 1$, $\Omega_\Lambda = 0$) is excluded. This result has been confirmed by more recent data at larger redshift. According to statistical analysis, the allowed region in the $(\Omega_M, \Omega_\Lambda)$ parameter space is

$$-1 < \Omega_M - \Omega_\Lambda < 0. \qquad (2.39)$$

When this observation is combined with the CMB results (figure 2.15), it appears that our Universe is spatially flat and has a non–zero cosmological constant – a case corresponding to accelerated expansion today! The favored values are $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$.

2.3.5 Large Scale Structure

We have seen that the Fourier spectrum of matter perturbations on very large scales (figure 2.13) depends on the cosmological parameters. Current galaxy

redshift surveys provide some constraints which are essentially independent from those of the previous sections. It is remarkable that they converge toward the same values as the previous techniques.

In conclusion, current experiments agree on the following matter budget in the present Universe:

$$\boxed{\begin{aligned} \Omega_0 = \Omega_{\mathrm{B}} + \Omega_{\mathrm{CDM}} + \Omega_\Lambda \simeq 1 \\ \Omega_\Lambda \simeq 0.7 \\ \Omega_{\mathrm{B}} \simeq 0.04 \end{aligned}}$$

So, the Universe seems to be accelerating today, and 96% of its energy density is non–baryonic and of essentially unknown nature! The measured value of the cosmological constant shows that the Universe started to be Λ–dominated only recently: at redshifts $z \geq 2$, the Universe was still matter–dominated. We seem to be leaving precisely during the period of transition between matter–domination and Λ–domination!

2.4 The Inflationary Universe

2.4.1 Problems with the Standard Cosmological Model

According to the modern version of the standard cosmological model, called the ΛCDM model, the Universe is spatially flat, with three stages: radiation–domination before $z \sim 10^4$, matter–domination, and now, transition to Λ–domination. This scenario is impressively successful in explaining a wide variety of observations, as we could see in section 2.3. However, it also raises a few problems.

The flatness problem.
The present contribution of the spatial curvature to the expansion is given by $\Omega_k = \rho_0(t_0)/\rho_c(t_0) - 1$. But this contribution has evolved with time. From the Friedmann law, it is straightforward to see that

$$|\Omega_k(t)| = |\rho_0(t)/\rho_c(t) - 1| = \frac{c^2|k|}{a^2 H^2} = \frac{c^2|k|}{\dot{a}^2}. \tag{2.40}$$

During the domination of radiation (resp. matter), the factor $\dot{a}^{-2}$ grows like t (resp. $t^{2/3}$). Therefore, a spatially flat Universe is unstable: if initially, k is *exactly* zero, Ω_k will remain zero; but if it is only *close* to zero, Ω_k will increase with time! In order to have no significant curvature–domination today ($|\Omega_k(t_0)| \leq 0.1$), we find that near the Planck time, the curvature was incredibly small, roughly $|\Omega_k(t)| \leq 10^{-60}$... The standard cosmological model provides no explanation for obtaining such a small number in the early Universe: the fact that the Universe is spatially flat – even approximately – appears like the most unnatural possibility! This is called the "flatness problem".

The horizon problem.
We have seen that during radiation and matter domination, the causal horizon is equal to the Hubble radius (times a factor 2 or 4). From the observation of CMB anisotropies, we know that the Hubble radius at the time of decoupling is seen today under an angle of one degree in the sky. So, on CMB maps, two points seen under an angle bigger than one degree seem to be be causally disconnected. In other words, the CMB maps seem to be composed of approximately 10^3 independent regions, which could not exchange any information in the past.

But then, how can they be at the same temperature – up to 10^{-5} ? Given these theoretical arguments, we would rather expect each of these regions to have its own temperature. Then, today we would observe inhomogeneities of order one. Measuring approximately the same temperature $T_0 = 2.73$ K all over the sky appears like a big mystery called the "horizon problem".

The origin of fluctuations.
The horizon problem can be reformulated in a slightly different way. We have seen on figure 2.5 that during radiation and matter domination, the Hubble radius (and the causal horizon) grow faster than the physical wavelength of each perturbation. So, all the wavelengths observable today on cosmological scales were larger than the horizon in the early Universe. So far, we did not propose any mechanism for the generation of primordial perturbations. But we have the following problem: if primordial perturbations are acausal, how can they be generated without violating causality (i.e., the fact that no information can travel faster than the speed of light)?

The Cosmic Coincidence problem.
We have seen that the cosmological constant starts to dominate the energy density of the Universe only in the very near past. Why are we leaving just at the time of the transition? If Ω_Λ is of order one, then $\rho_\Lambda^{1/4}$ is of order 10^{-3} eV. Any vacuum energy density resulting from a spontaneous symmetry breaking in the standard model of particle physics should be at least of the order of the electroweak scale, near 1 TeV. Eventually, some unknown symmetry could force the vacuum energy to vanish exactly. But why should it be as small as 10^{-3} eV?

So, particle physics seems to be easily compatible with a Universe that would be either completely matter-dominated or completely Λ–dominated today. But the subtle balance observed between ordinary matter and the cosmological constant appears as extremely unnatural and mysterious.

2.4.2 An initial stage of inflation

So, the standard cosmological model raises a series of problem – the previous list is not exhaustive. We will see that many of them can be solved in an elegant way.

Let us assume that in the early Universe, before radiation domination, there has been a stage of accelerating expansion: $\ddot{a} > 0$. So far, we don't say which mechanism could be responsible for this period, called *inflation*. If $\ddot{a}$ is positive, $\dot{a}$ increases, and according to equation (2.40), $|\Omega_k|$ decreases. The contribution of the spatial curvature to the expansion is driven to zero. This provides a possible solution to the **flatness problem**: during the initial inflationary stage, $|\Omega_k|$ becomes extremely small. Then, it grows again, but today it might still be very far from order one.

The simple relation that we found between the causal horizon and the Hubble radius is not true anymore during inflation. As a particular example of accelerated expansion, let us assume that $a(t)$ grows exponentially:

$$a(t) = e^{H_i t}. \tag{2.41}$$

In that case, the Hubble radius is constant:

$$R_H = c \left(\frac{\dot{a}}{a}\right)^{-1} = \frac{c}{H_i}, \tag{2.42}$$

while the causal horizon grows exponentially:

$$d_H(t_1, t_2) = 2a(t_2) \int_{t_1}^{t_2} \frac{c\, dt}{a(t)} \longrightarrow \frac{c}{H_i} e^{H(t_2 - t_1)} \qquad \text{when} \qquad t_2 \gg t_1. \quad (2.43)$$

So, at the end of an inflation, the causal horizon is much bigger than the Hubble radius. After, during radiation and matter domination, they grow at the same rate, and the causal horizon *remains* larger. So, an initial stage of inflation can solve the **horizon problem**: at the time of decoupling, if the causal horizon is at least on thousand times larger than the Hubble radius, it encompasses the whole observable Universe. Then, all the points in the CMB map are in causal contact, and may have acquired the same temperature in the early Universe.

In the inflationary scenario, it is important not to make confusion between the different causal horizons. An horizon is always defined with respect to an initial time. We have seen that if the horizon is defined with respect to the initial singularity, or to the beginning of inflation, then it can be very large. But the horizon defined with respect to any time *after the end of inflation* is still close to R_H. So, the causal distance for a physical process starting *after* inflation is unchanged. In particular, the characteristic scale that appears in the evolution of cosmological perturbations is still the Hubble radius. In inflationary cosmology, one must be very careful because there are two different causal horizons, which are relevant for different problems.

Let's come back finally to the **origin of fluctuations**. We can update figure 2.5 by adding a preliminary stage of inflation. In general, the ratio of an arbitrary physical wavelength over the Hubble radius is given by

$$\frac{\lambda(t)}{R_H(t)} = \frac{2\pi a(t)}{k} \frac{\dot{a}(t)}{c\, a(t)} = \frac{2\pi \dot{a}(t)}{c\, k} \quad (2.44)$$

So, under the simple assumption that $\ddot{a} > 0$, we see that this ratio increases during inflation: each wavelength grows faster than the Hubble radius. We conclude that each Fourier mode observable today on cosmological scales started inside the Hubble radius, crossed it during inflation, and re-entered later (see figure 2.16). In addition, all these wavelengths could have been smaller than the causal horizon at any time. This helps a lot in finding a realistic mechanism for the generation of fluctuations, as we will see in the next section.

All these arguments – and also some other ones – led a few cosmologists like Guth, Starobinsky and Hawking to introduce the theory of inflation in 1979. Today, it is one of the corner stones of cosmology.

2.4.3   Scalar field inflation

So, the first three problems of section 2.4.1 can be solved under the assumption of a long enough stage of accelerated expansion in the early Universe. How can this be implemented in practice?

First, by combining the Friedman equation (1.30) in a flat Universe with the conservation equation (1.32), it is easy to find that

$$\ddot{a} > 0 \qquad \Rightarrow \qquad \rho + 3p < 0. \quad (2.45)$$

What type of matter corresponds to such an unusual relation between density and pressure? A positive cosmological constant can do the job:

$$p_\Lambda = -\rho_\Lambda \qquad \Rightarrow \qquad \rho_\Lambda + 3p_\Lambda = -2\rho_\Lambda < 0. \quad (2.46)$$
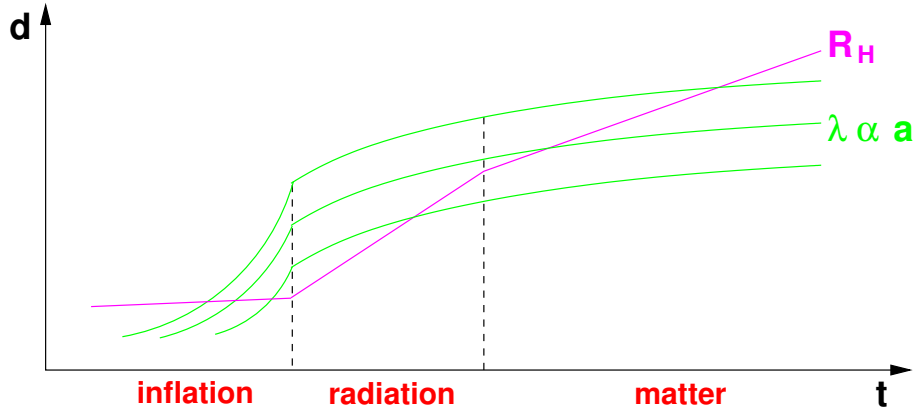
Figure 2.16: Comparison of the Hubble radius with the physical wavelength of a few cosmological perturbations. During the initial stage of accelerated expansion (called inflation), the Hubble radius grows more slowly than each wavelength. So, cosmological perturbations find their origin inside R_H. Then, each mode exits from the Hubble radius during inflation and re-enters during radiation or matter domination.

But since a cosmological constant is... constant, it cannot be responsible for an initial stage of inflation: otherwise this stage would go on forever, and there would be no transition to radiation domination.

Let us consider instead the case of a scalar field (i.e., a field of spin zero, represented by a simple function of time and space, and invariant under Lorentz transformations). In general, for a homogeneous scalar field $\varphi(t)$, the density and pressure read

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{2.47}$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{2.48}$$

where V is the scalar potential. So, the pressure is equal to the kinetic energy *minus* the potential energy. We see that in the limit – called *slow–roll* – where the kinetic energy is subdominant, the pressure is close to $-\rho$, and the field behaves almost like a cosmological constant. However, the field is still slowly rolling toward the minimum of its potential – which should have a *small* slope, otherwise the kinetic energy would be leading.

So, a particular way to obtain a stage of accelerated expansion in the early Universe is to introduce a scalar field, with a flat enough potential. Scalar field inflation has been proposed in 1979 by Guth. During the 80's, most important aspects of inflation were studied in details, by Guth, Starobinsky, Hawking, Linde and other people. Finally, during the 90's, many ideas and models were proposed in order to make contact between inflation and particle physics. The purpose of scalar field inflation is not only to provide a stage of accelerated expansion in the early Universe, but also, a mechanism for the generation of matter and radiation particles, and another mechanism for the generation of primordial cosmological perturbations. Let us summarize how it works in a very sketchy way.

Slow-roll.
First, let us assume that just after the initial singularity, the energy density

is dominated by a scalar field, with a potential flat enough for slow–roll. In any small region where the field is approximately homogeneous and slowly–rolling, accelerated expansion takes place: this small region becomes exponentially large, encompassing the totality of the present observable Universe. Inside this region, the causal horizon becomes much larger than the Hubble radius, and any initial spatial curvature is driven almost to zero – so, some of the main problems of the standard cosmological model are solved. After some time, when the field approaches the minimum its potential, the slow-roll condition breaks, and inflation ends: the expansion becomes decelerated again (this is not true for all potentials, it depends on additional assumptions; for instance, with any simple potential of the form $V(\varphi) = \varphi^{2n}$, slow–roll has to break in a vicinity of the minimum).

Reheating.

At the end of inflation, the kinetic energy of the field is bigger than the potential energy; in general, the field is quickly oscillating around the minimum of the potential. According to the laws of quantum field theory, the oscillating scalar field will decay into fermions and bosons. This could explain the origin of all the particles of matter and radiation. Because the particles generally end up with a thermal distribution, this stage is called "reheating".

Generation of primordial perturbations.

Finally, the theory of scalar field inflation also explains the origin of cosmological perturbations – the ones leading to CMB anisotropies and large scale structure formation. Using again quantum field theory in curved space–time, it is possible to compute the amplitude of the small quantum fluctuations of the scalar field. The physical wavelength of these fluctuations grows quickly, like in figure 2.16. So, they are initially inside the Hubble radius, where quantum mechanics is important, and imposes the initial normalization of each mode; then, they grow larger than the Hubble radius. The perturbations of the scalar field also generate some perturbations of the space–time curvature – in simple words, of the gravitational potential. When the wavelength is bigger than the Hubble radius, the perturbations of the gravitational potential remain frozen: they are not affected by the decay of the scalar field during reheating. But when the radiation and matter particles are formed during reheating, they are sensitive to the gravitational potential, and more particles accumulate in the potential wells. So, the gravitational potential behaves like a mediator between the scalar field perturbations during inflation and the radiation/matter perturbations in the radiation/matter–dominated Universe. Trough this mechanism, the primordial perturbations of the photons, the baryons and the CDM originate from the quantum fluctuations of the scalar field at the beginning of inflation.

Of course, the perturbations that we observe today are not of quantum nature, and we don't need to employ wave functions and operators when we compute the evolution CMB anisotropies. This is consistent with another nice property of inflation, which can be deduced rigorously from the laws of quantum field theory, applied to the inflationary space-time in accelerated expansion. During inflation, and after exiting the Hubble radius, each perturbation undergoes a quantum–to–classical transition. In other words, quantum mechanics is unavoidable in order to describe them initially, but later, they become gradually equivalent to random classical fluctuations.

It is far beyond the level of these notes to compute the evolution of primordial perturbations during inflation. However, we should stress that it can be studied mathematically, in a very precise way. This leads to generic predictions for the primordial perturbations, and in particular, for their Fourier spectrum,

which is related to a few free parameters – like the derivatives of the scalar potential. The biggest success of inflation is that the CMB anisotropies observed for instance by Boomerang are in perfect agreement with the predictions of inflation concerning the primordial perturbations (in technical words, they are coherent, Gaussian, adiabatic, and approximately scale–invariant: we will not explain these concepts here). A priori, such a nice agreement between theory and observations was far from obvious. It is possible to build some mechanisms with an initial stage of accelerated expansion, which solve the flatness and horizon problem, but fail in explaining the spectrum of primordial perturbations. Current observations of CMB anisotropies are the best evidence in favor of inflation, and with future CMB experiments, it should be possible to reconstruct the shape of the inflationary potential to some extent; that would be really fantastic: some observations made today of the Universe at decoupling would give us details about what happened just after the initial singularity – at energies of order $\rho^{1/4} \sim 10^{16}$ GeV !

Inflation is still a very active field of research, especially as far as the connection with particle physics is concerned. First, the scalar field invoked for inflation is *ad hoc*, it is introduced just for cosmological purposes. There are several proposals concerning the role of this field in particle physics model. For instance, it could be a Higgs boson associated with the GUT symmetry, or it could have to do with extra dimensions in string theory. Some people try to think how these assumptions could be tested, at least in some indirect way (e.g., through the shape of the primordial spectrum of perturbations). Second, the theory of reheating is still being investigated in details, in a more an more realistic way from the point of view of particle physics.

2.4.4 Quintessence ?

It is a pity that the theory of inflation, which is able to solve so many crucial issues in cosmology, doesn't say anything about the Cosmic Coincidence problem. Inflation predicts that $\Omega_M + \Omega_\Lambda = 1$, in agreement with what is observed today, but it doesn't give any hint on the splitting between matter and cosmological constant – while the hint from particle physics is either $\Omega_\Lambda = 0$ or $\Omega_\Lambda = 1$, at odds with observations.

As long as we explain the supernovae data with a cosmological constant, it appears as an incredible coincidence that ρ_Λ is of the same order of magnitude as ρ_M just in the present Universe, while in the early Universe it was incredibly smaller...

We have seen that during slow-roll inflation, the pressure of the scalar field is negative, almost like for a cosmological content. In the past five years, many people have suggested to explain the supernovae data with a slow-rolling scalar field. So, there would be two stages of scalar field inflation, one in the early Universe, and one today. A priori, they would be caused by two different scalar fields, called respectively the inflaton and the quintessence field (although some model try to unify the two). The big advantage with these models is that the energy density responsible for the cosmic acceleration today doesn't have to be constant in the past. In the early Universe, it could very well be of the same order of magnitude as that of radiation, and only slightly smaller. During radiation domination, the density of quintessence and radiation would decay at the same rate, and after radiation–matter equality, quintessence would enter into a stage of slow–roll. Then, its energy density would remain almost constant, and it would start to dominate today, leading to the acceleration of the Universe.

Such classes of models are under investigation. However, they have their

own problems, and they don't solve the Cosmic Coincidence problem in a fully convincing way. In fact, a wide variety of models have been proposed in order to explain the supernovae data, but none of them succeeds in explaining while the Universe starts to accelerate only in the very near past. Together with the nature of dark matter, the nature of the cosmological term is the biggest enigma of modern cosmology – and these two questions are far from being small details, since altogether, Λ and CDM account for 96% of the present energy density in the Universe! So, cosmology has done some very impressive progresses in the past decades, but it seems that many other exciting discoveries could be waiting for us in the near future!

Acknowledgments