Introduction to QCD

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Lecture 1: Basics of QCD

- Feynman rules
- Running coupling
  - Beta function
  - Lambda parameter
- Renormalization schemes
- Non-perturbative QCD
  - Infrared divergences

Feynman rules of QCD

- Feynman rules follow from QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F_A^{\alpha\beta} + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m_q)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

  $F^A_{\alpha\beta}$ is field strength tensor for spin-1 gluon field $\mathcal{A}^A_\alpha$,

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

  Capital indices A, B, C run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is $\alpha_{\rm s} \equiv g^2/4\pi$. Numbers f^{ABC} $(A, B, C = 1, \ldots, 8)$ are structure constants of the SU(3) colour group. Quark fields q_a $(a = 1, 2, 3)$ are in triplet colour representation, while gluon fields $\mathcal{A}^A_\alpha$ are in adjoint representation.

- D is covariant derivative:

$$\begin{aligned} (D_\alpha)_{ab} &= \partial_\alpha \delta_{ab} + ig\left(t^C \mathcal{A}^C_\alpha\right)_{ab} \\ (D_\alpha)_{AB} &= \partial_\alpha \delta_{AB} + ig(T^C \mathcal{A}^C_\alpha)_{AB} \end{aligned}$$

- t and T are matrices in the fundamental and adjoint representations of SU(3), respectively:

$$[t^A, t^B] = i f^{ABC} t^C, \quad [T^A, T^B] = i f^{ABC} T^C$$

where $(T^A)_{BC} = -i f^{ABC}$. We use the metric $g^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ and set $\hbar = c = 1$. $\not{D}$ is symbolic notation for $\gamma^\alpha D_\alpha$. Normalisation of the t matrices is

$$\mathrm{Tr}\, t^A t^B = T_R\, \delta^{AB}, \quad T_R = \frac{1}{2}.$$

- SU(N) matrices obey the relations:

$$\begin{aligned} \sum_A t^A_{ab} t^A_{bc} &= C_F\, \delta_{ac}\,, \quad C_F = \frac{N^2-1}{2N} \\ \mathrm{Tr}\, T^C T^D &= \sum_{A,B} f^{ABC} f^{ABD} = C_A\, \delta^{CD}\,, \quad C_A = N \end{aligned}$$

Thus $C_F = \frac{4}{3}$ and $C_A = 3$ for SU(3).

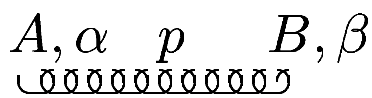
- Use free piece of QCD Lagrangian to obtain inverse quark and gluon propagators.
  - Quark propagator in momentum space obtained by setting $\partial^\alpha = -ip^\alpha$ for an incoming field. Result is in Table 1. The $i\varepsilon$ prescription for pole of propagator is determined by causality, as in QED.
  - Gluon propagator impossible to define without a choice of gauge. The choice

$$\mathcal{L}_{\text{gauge}-\text{fixing}} = -\frac{1}{2\,\lambda}\left(\partial^\alpha \mathcal{A}_\alpha^A\right)^2$$

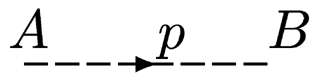
defines *covariant gauges* with gauge parameter λ. Inverse gluon propagator is then

$$\Gamma^{(2)}_{\{AB,\,\alpha\beta\}}(p) = i\delta_{AB}\left[p^2 g_{\alpha\beta} - (1-\frac{1}{\lambda})p_\alpha p_\beta\right].$$

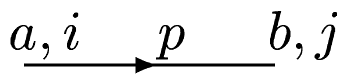
(Check that without gauge-fixing term this function would have no inverse.) Resulting propagator is in Table 1. $\lambda = 1$ (0) is *Feynman* (*Landau*) gauge.



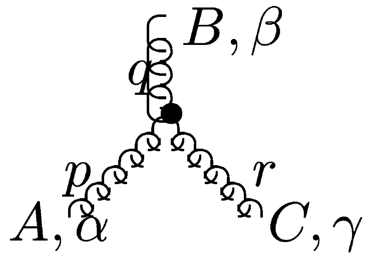
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^\alpha p^\beta}{p^2+i\varepsilon}\right]\frac{i}{p^2+i\varepsilon}$$



$$\delta^{AB}\frac{i}{p^2+i\varepsilon}$$

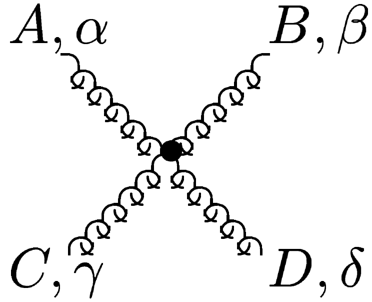


$$\delta^{ab}\frac{i}{(\not{p}-m+i\varepsilon)}_{ji}$$

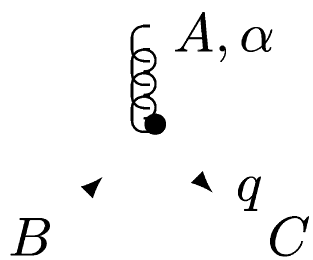


$$-gf^{ABC}\Big[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\Big]$$

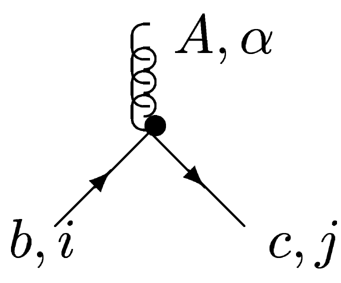
(all momenta incoming)



$$\begin{aligned}&-ig^2 f^{XAC}f^{XBD}\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)\\&-ig^2 f^{XAD}f^{XBC}\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta}\right)\\&-ig^2 f^{XAB}f^{XCD}\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)\end{aligned}$$



$$gf^{ABC}q^\alpha$$



$$-ig\left(t^A\right)_{cb}\left(\gamma^\alpha\right)_{ji}$$

- Gauge fixing explicitly breaks gauge invariance. However, in the end physical results will be independent of gauge. For convenience we usually use Feynman gauge.
- In non-Abelian theories like QCD, covariant gauge-fixing term must be supplemented by a *ghost term* which we do not discuss here. Ghost field, shown by dashed lines in Table 1, cancels unphysical degrees of freedom of gluon which would otherwise propagate in covariant gauges.

Running coupling

- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.

- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_S = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling α_S also depends on μ.

- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling α_S. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0 \, .$$

- Introducing

$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right) , \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} ,$$

we have

$$\left[- \frac{\partial}{\partial \tau} + \beta(\alpha_S) \frac{\partial}{\partial \alpha_S} \right] R = 0.$$

This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:

$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)} , \quad \alpha_S(\mu) \equiv \alpha_S .$$

Then

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} .$$

and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.

- We shall see QCD is asymptotically free: $\alpha_S(Q) \to 0$ as $Q \to \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

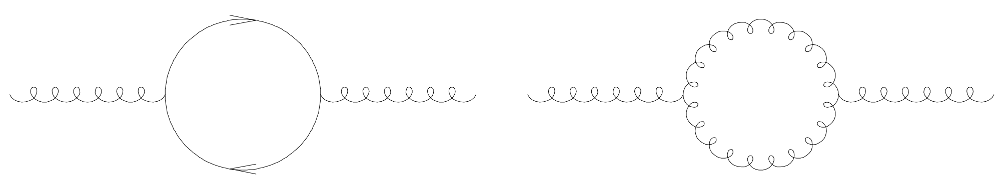
Beta function

- Running of of the QCD coupling α_s is determined by the β function, which has the expansion

$$\beta(\alpha_s) = -b\alpha_s^2(1 + b'\alpha_s) + \mathcal{O}(\alpha_s^4)$$
$$b = \frac{(11C_A - 2N_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A N_f - 3C_F N_f)}{2\pi(11C_A - 2N_f)},$$

where N_f is number of “active” light flavours. Terms up to $\mathcal{O}(\alpha_s^5)$ are known.

- Roughly speaking, quark loop diagram (a) contributes negative N_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall.



a) b)

- QED β function is

$$\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots$$

Thus b coefficients in QED and QCD have opposite signs.

- From previous section,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\Big[1 + b'\alpha_S(Q)\Big] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives

$$\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + \alpha_S(\mu) b\tau}\,, \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$$

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q. Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b'\ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b'\ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

- What type of terms does the solution of the renormalization group equation take into account in the physical quantity R?

Assume that R has perturbative expansion

$$R = \alpha_S + \mathcal{O}(\alpha_S^2)$$

Solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} R(1, \alpha_S(Q)) &= \alpha_S(\mu) \sum_{j=0}^{\infty} (-1)^j (\alpha_S(\mu) b\tau)^j \\ &= \alpha_S(\mu) \Big[1 - \alpha_S(\mu) b\tau + \alpha_S^2(\mu)(b\tau)^2 + \ldots \Big] \end{aligned}$$

Thus there are logarithms of Q^2/μ^2 which are automatically resummed by using the running coupling. Neglected terms have fewer logarithms per power of α_S.

Lambda parameter

- Perturbative QCD tells us how $\alpha_S(Q)$ varies with Q, but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at $Q = M_Z$, which is simply a convenient reference scale large enough to be in the perturbative domain.

- Also useful to express $\alpha_S(Q)$ directly in terms of a dimensionful parameter (constant of integration) Λ:

$$\ln \frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty} \frac{dx}{bx^2(1 + b'x + \dots)}.$$

  Then (if perturbation theory were the whole story) $\alpha_S(Q) \to \infty$ as $Q \to \Lambda$. More generally, Λ sets the scale at which $\alpha_S(Q)$ becomes large.

- In leading order (LO) keep only first β-function b:

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \quad \text{(LO)}.$$

- In next-to-leading order (NLO) include also b':

$$\frac{1}{\alpha_{\rm S}(Q)} + b' \ln\left(\frac{b'\alpha_{\rm S}(Q)}{1+b'\alpha_{\rm S}(Q)}\right) = b \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

This can be solved numerically, or we can obtain an approximate solution to second order in $1/\log(Q^2/\Lambda^2)$:

$$\alpha_{\rm S}(Q) = \frac{1}{b\ln(Q^2/\Lambda^2)}\left[1 - \frac{b'}{b}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right] \qquad \text{(NLO)}.$$

This is Particle Data Group (PDG) definition.

- Note that Λ depends on number of active flavours N_f. 'Active' means $m_q < Q$. Thus for $5 < Q < 175$ GeV we should use $N_f = 5$. See ESW for relation between Λ's for different values of N_f.

Renormalization schemes

- Λ also depends on renormalization scheme. Consider two calculations of the renormalized coupling which start from the same bare coupling α_S^0:

$$\alpha_S^A = Z^A \alpha_S^0 \,, \quad \alpha_S^B = Z^B \alpha_S^0$$

  Infinite parts of renormalization constants Z^A and Z^B must be same in all orders of perturbation theory. Therefore two renormalized couplings must be related by a finite renormalization:

$$\alpha_S^B = \alpha_S^A (1 + c_1 \alpha_S^A + \ldots).$$

- Note that first two β-function, coefficients, b and b', are unchanged by such a transformation: they are therefore renormalization-scheme independent.
- Two values of Λ are related by

$$\log \frac{\Lambda^B}{\Lambda^A} = \frac{1}{2} \int_{\alpha_S^A(Q)}^{\alpha_S^B(Q)} \frac{dx}{\beta(x)}$$

This must be true for all Q, so take $Q \to \infty$, to obtain

$$\Lambda^B = \Lambda^A \exp \frac{c_1}{2b}$$

Thus relations between different definitions of Λ are given by the one-loop calculation that fixes c_1.

- Nowadays, most calculations are performed in *modified minimal subtraction* ($\overline{\mathrm{MS}}$) renormalization scheme. Ultraviolet divergences are 'dimensionally regularized' by reducing number of space-time dimensions to $D < 4$:

$$\frac{\mathrm{d}^4 k}{(2\pi)^4} \longrightarrow (\mu)^{2\epsilon} \frac{\mathrm{d}^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}}$$

where $\epsilon = 2 - \frac{D}{2}$. Note that renormalization scale μ still has to be introduced to preserve dimensions of couplings and fields.

- Loop integrals of form

$$\int \frac{\mathrm{d}^D k}{(k^2 + m^2)^2}$$

lead to poles at $\epsilon = 0$. The *minimal subtraction* prescription is to subtract poles and replace bare coupling by renormalized coupling $\alpha_{\mathrm{S}}(\mu)$. In practice poles always appear in combination

$$\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E,$$

(Euler's constant $\gamma_E = 0.5772\ldots$). In *modified* minimal subtraction scheme $\ln(4\pi) - \gamma_E$ is subtracted as well. From argument above, it follows that

$$\Lambda_{\overline{\rm MS}} = \Lambda_{\rm MS} e^{[\ln(4\pi) - \gamma_E]/2} = 2.66\, \Lambda_{\rm MS}$$

- Value of $\alpha_{\rm S}$ at mass of Z is [Bethke, hep-ex/0211012]

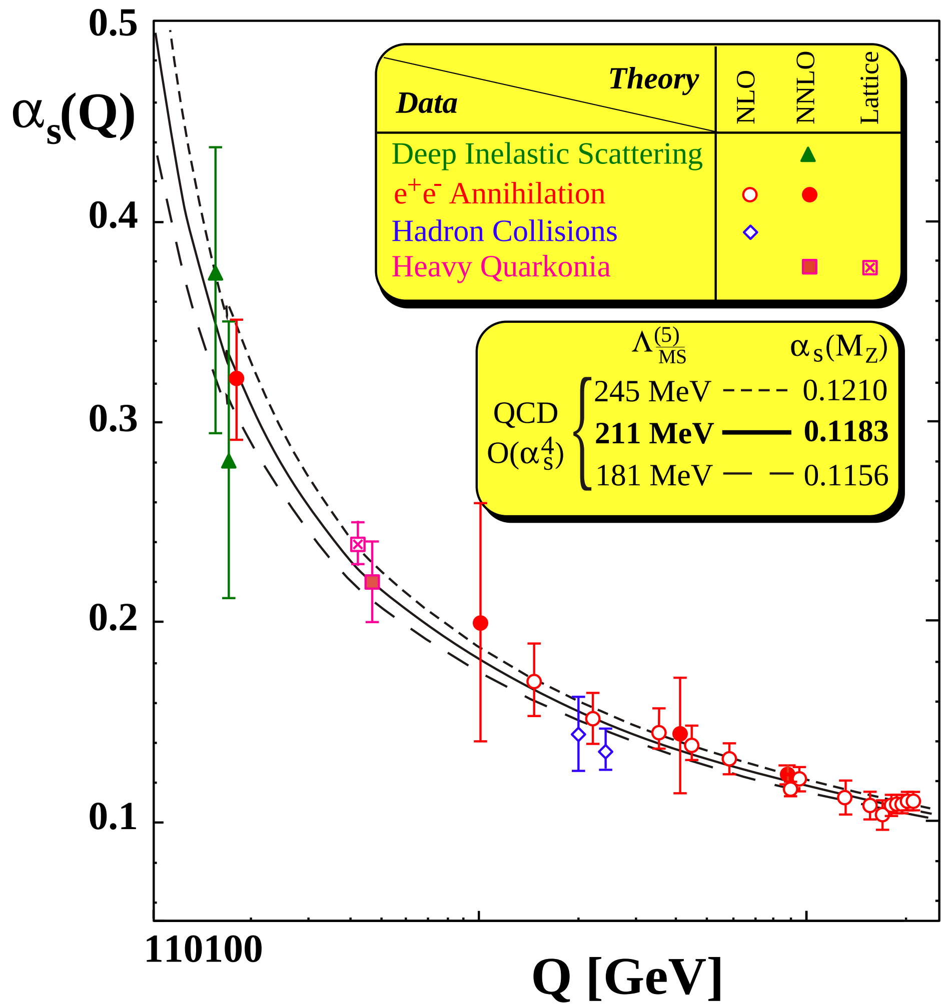
$$\alpha_{\rm S}(M_Z) = 0.1183 \pm 0.0027$$

corresponding to a preferred value of $\Lambda_{\overline{\rm MS}}$ (for $N_f = 5$) in the range

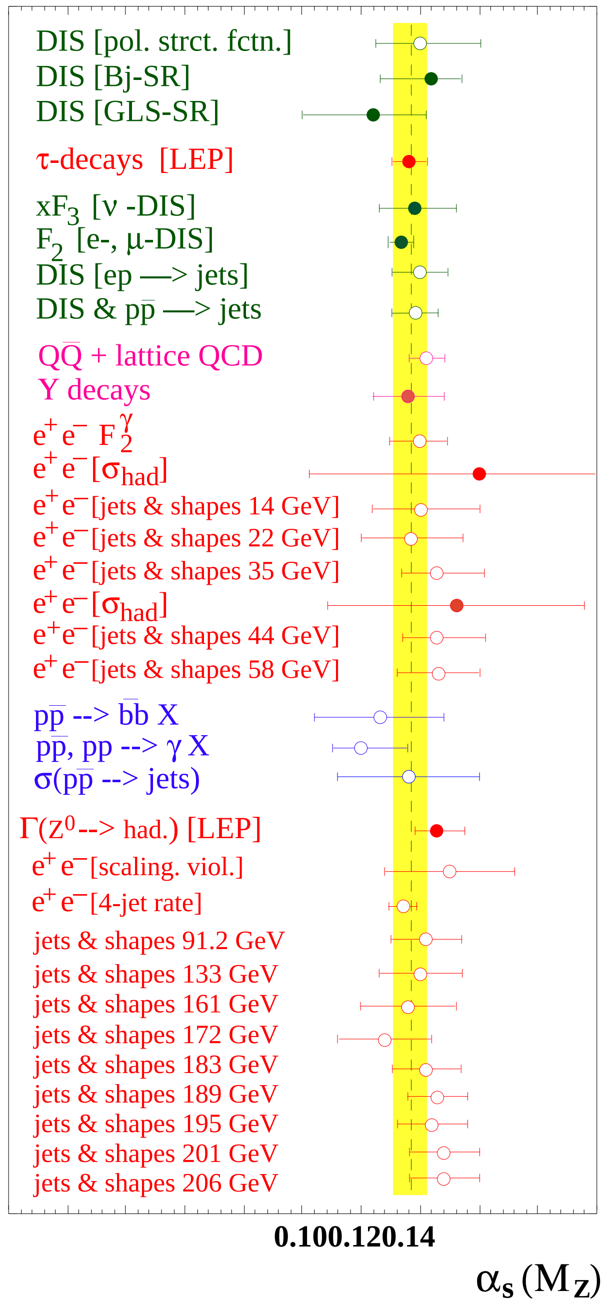
$$181 \text{ MeV} < \Lambda_{\overline{\rm MS}}(5) < 245 \text{ MeV}.$$

- Uncertainty in $\alpha_{\rm S}$ propagates directly into QCD cross sections. Thus we expect at least errors of $\sim 3\%$ in prediction of cross sections which begin in order $\alpha_{\rm S}$.

- Measurements of α_s are reviewed in ESW. A more recent compilation by Bethke is shown below. Evidence for logarithmic fall-off of $\alpha_s(Q)$ is persuasive.



- Using the formula for running $\alpha_s(Q)$ to rescale all measurements to $Q = M_Z$ gives excellent agreement.



Nonperturbative QCD

- Corresponding to asymptotic freedom at high momentum scales (short distances), we have infrared slavery: $\alpha_S(Q)$ becomes large at low momenta (long distances). Perturbation theory (PT) not reliable for large α_S, so nonperturbative methods (e.g. lattice) must be used.

- Important low momentum-scale phenomena:

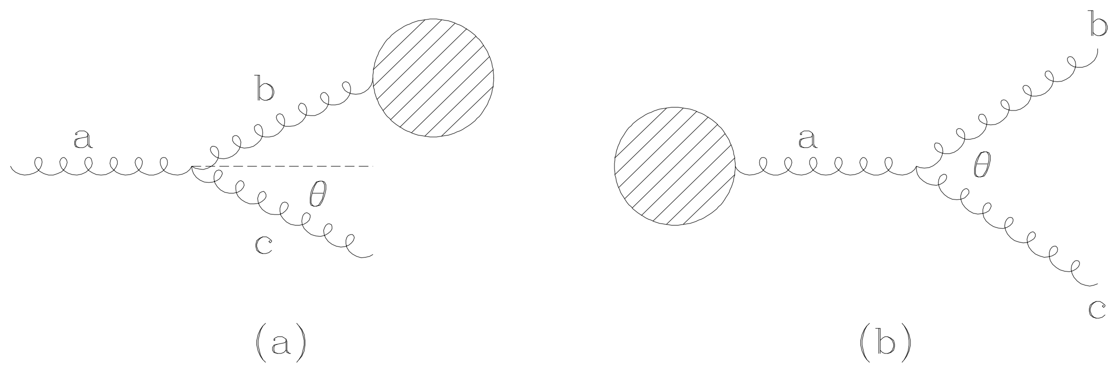
  - Confinement: partons (quarks and gluons) found only in colour-singlet bound states (hadrons), size ~ 1 fm. If we try to separate them, it becomes energetically favourable to create extra partons, forming additional hadrons.

  - Hadronization: partons produced in short-distance interactions reorganize themselves (and multiply) to make observed hadrons.

- Note that confinement is a static (long-distance) property of QCD, treatable by lattice techniques whereas hadronization is a dynamical (long timescale) phenomenon: only models are available at present (see later).

Infrared divergences

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



❖ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = -E_a E_c (1 - \cos\theta) \leq 0 \ .$$

Propagator factor $1/p_b^2$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity). If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -E_a E_c (1 - v_a \cos\theta) \leq 0 \ ,$$

Hence $E_c \to 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \to 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \to 0$ divergence.

- ❖ Timelike branching: gluon splitting on outgoing line (b)

$$p_a^2 = E_b E_c (1 - \cos\theta) \geq 0 \ .$$

Diverges when either emitted gluon is soft (E_b or $E_c \to 0$) or when opening angle $\theta \to 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \to 0$ limit. Again, soft quark divergences cancelled by vertex factor.

- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.

- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.

- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:

- Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
- Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.

- In either case, infrared divergences must be *regularized* during PT calculation, even though they cancel or factorize in the end.
  - Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, gluon mass breaks gauge invariance.
  - Dimensional regularization: analogous to that used for ultraviolet divergences, except we must *increase* dimension of space-time, $\epsilon = 2 - \frac{D}{2} < 0$. Divergences are replaced by powers of $1/\epsilon$. See example in Lecture 2.

Summary of Lecture 1

- QCD is SU(3) gauge theory of quarks (3 colours) and gluons (8 colours, self-interacting).
- Since renormalization introduces (arbitrary) scale μ, dimensionless quantities are not in general scale-independent.
- QCD is asymptotically free: running coupling $\alpha_{\rm S}(Q) \to 0$ as $Q \to \infty$.
- $\alpha_{\rm S}(M_Z) \simeq 0.118$ in 5-flavour $\overline{\rm MS}$ renormalization scheme.
- Perturbative QCD has infrared singularities due to collinear parton or soft gluon emission. Hence we can only calculate infrared safe or factorizable quantities using perturbation theory.