COMPLEX SYSTEMS, CHAOS and MEASUREMENTS

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The goal of these lectures will be to describe the complexity of dynamical systems, the theoretical concepts which were developed to estimate quantitatively this complexity and some of the techniques/algorithms to measure these quantities.

Complexity is still a rather vague notion (with hot debates about a rigorous definition) and I will limit to the discussion of dynamical systems although there are some attempts to discuss complexity in other situations (noticeably in large networks). The notions and concepts I will discuss are sufficiently general to apply or at least inspire the study of other situations.

Virtually every model of Physical phenomena is a dynamical system, often a Hamiltonian system (with friction).

Typical questions of Physical interest concern long time behaviour (stationary states).

We will only discuss classical systems. Quantum chaos is another rather different subject.

We will also restrict to systems with a small number of degrees of freedom excited.
This does not mean that the system has a small number of degrees of freedom, it may have infinitely many.
This notion of number of degrees of freedom excited will be clarified later on.

The case of many degrees of freedom excited is still in its infancy although some ideas can be borrowed from the study of the simple case. The study of networks for example is a very active subject.

Overview of the lectures.

- A short introduction to dynamical systems.
- Statistical approach to chaos.
- Sensitive Dependence on Initial Conditions, Lyapunov Exponents.
- Reconstruction of Attractors from Time Series.
- Dimension of Attractors.
- Entropies.
- Forecasting and Noise Reduction.

I will as much as possible respect the following order:

- First present the concept and the associated general results.
- Show some concrete examples.
- Present some of the simplest algorithms used to apply the concept.

A general remark:
In applications, we will not be dealing with just any time series (signal) but with one coming from a dynamical system with few degrees of freedom excited. This leads to constraints which can be exploited in various ways. Of course this may be a too strong hypothesis on the data.

A short introduction to dynamical systems.

We start by recalling some general facts and definitions about dynamical systems.
A dynamical system is given by two objects.

The first one is its phase space Ω also (called state space) which is the set of all the possible **states** of the system.

The second object is the time evolution that may come in two different flavors.

A **discrete time evolution**, that is to say a map T from Ω to itself

$$x_{n+1} = T\left(x_n\right) = T^n\left(x_0\right) = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}\left(x_0\right)$$

A **continuous time evolution** given by a vector field $\vec{Y}(\vec{y})$ on the phase space, or in other words by a system of coupled first order differential equations

$$\frac{d\vec{y}}{dt} = \vec{Y}(\vec{y}) \ .$$

One can also give the time evolution flow, that is to say the family (φ_t) of transformations of Ω which transform the initial condition (state) to the value of the solution (state of the system) at time t ($\vec{y}(t) = \varphi_t\big(\vec{y}(0)\big)$).

The vector (field) $\vec{Y}(\vec{y})$ and the mapping T may depend on parameters (very useful), they may depend on time (non autonomous systems).

There are also systems with memory (retarded) which are more complicated (their phase space is infinite dimensional).

Example of a discrete dynamical systems.

The “simplest” chaotic map:
Phase space $\Omega = [0,1]$, $T(x) = 2x \mod (1)$.

The “logistic map”:
Phase space $\Omega = [0,1]$, $T(x) = 4x(1-x)$.

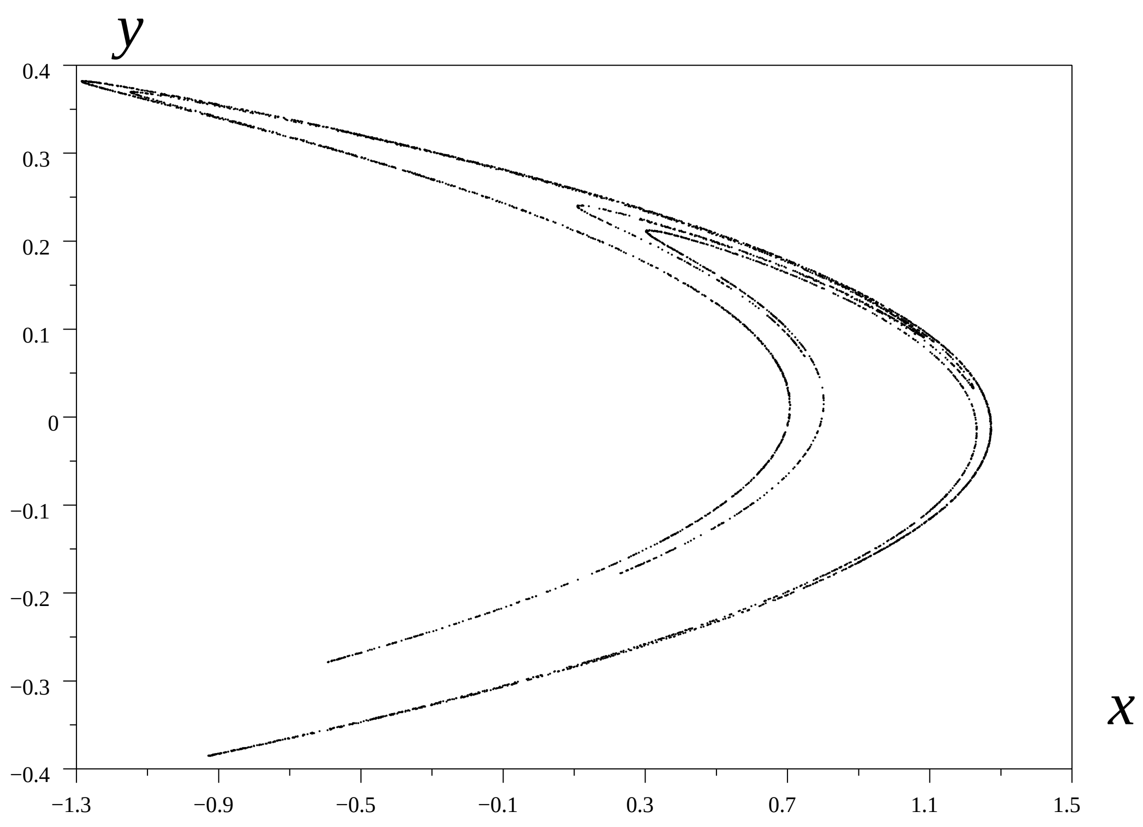
The one parameter family of quadratic maps:
Phase space $\Omega = [0,1]$, $T_\mu(x) = \mu x(1-x)$, $0 \le \mu \le 4$.

Example of a discrete dynamical system : the Hénon map.

Phase space $\Omega = \mathbb{R}^2$,

$$T_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$$

Historical values $a = 1.4$, $b = 0.3$.

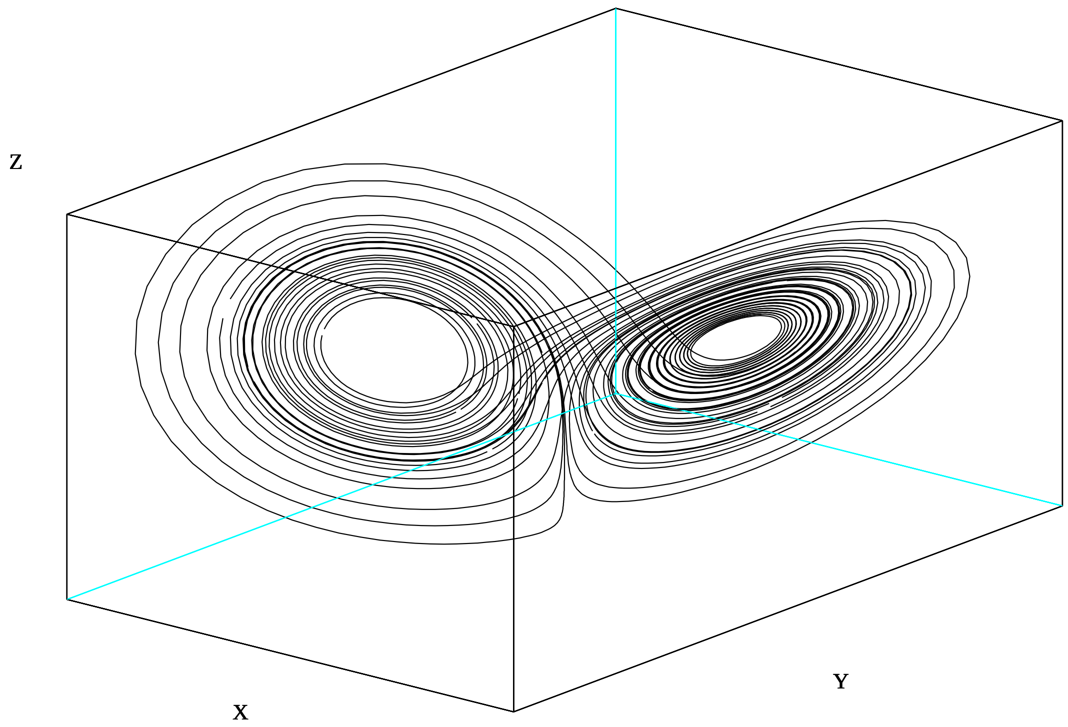


Attractor of the Hénon map.

Example of a continuous time dynamical system: The Lorenz system. Phase space $\Omega = \mathbb{R}^3$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma x + \sigma y \\ -xz + rx - y \\ xy - bz \end{pmatrix}$$

Historical values: $\sigma = 10$, $r = 28$, $b = 8/3$.



The Lorenz attractor.

An important example: Hamiltonian mechanics.

The phase space is $\mathbf{R}^{2n}$ (or a manifold), n is the number of degrees of freedom. The coordinates are $(\vec{q}, \vec{p})$.

The equations of motion are

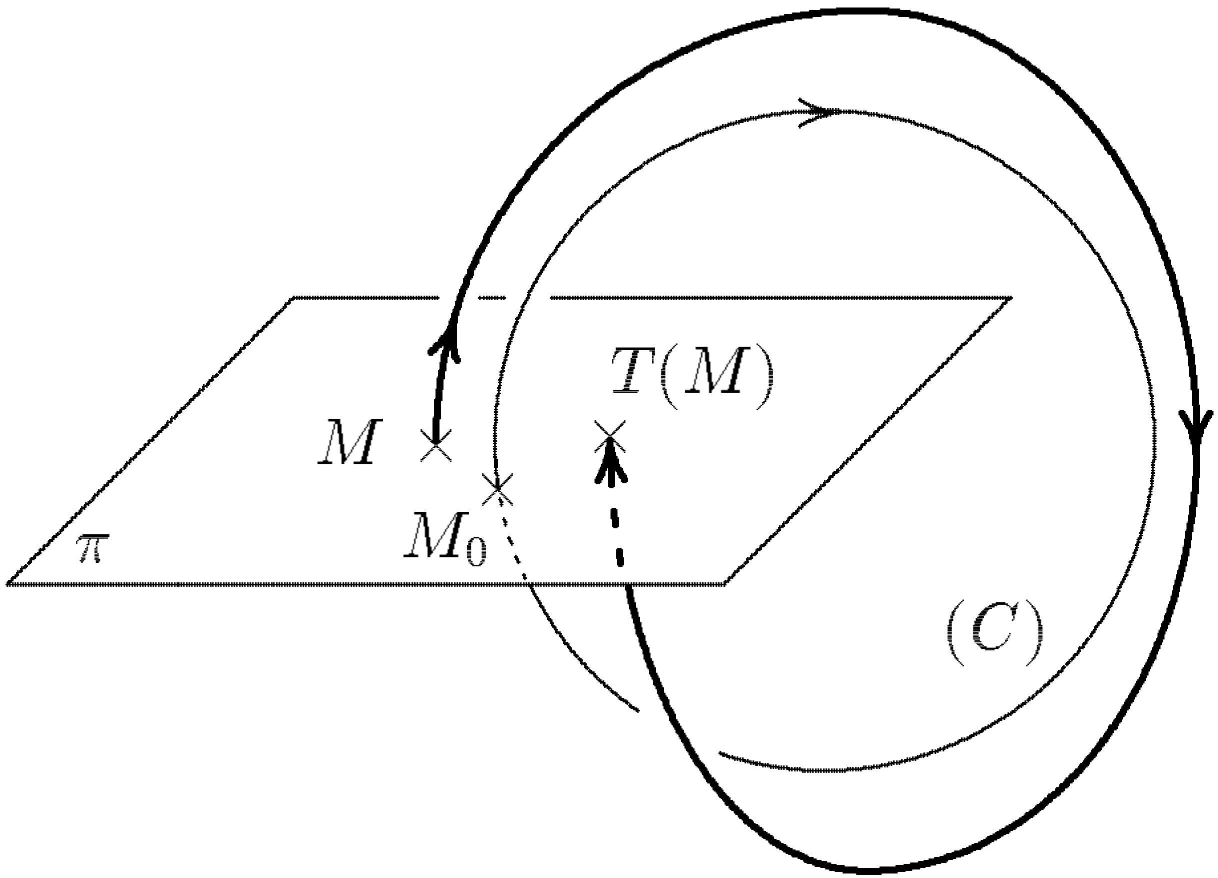
$$\frac{d\, q_j}{dt} = \partial_{p_j} H \qquad \frac{d\, p_j}{dt} = -\partial_{q_j} H \qquad (j = 1 \ldots n)\ .$$

From continuous time to discrete time:

- Sampling. In periodically forced systems it is useful to sample at a frequency equal to the forcing frequency: stroboscope. In general for a continuous time evolution one can use the time one map of the flow φ_1.

- Poincaré section (section at the maximum of an observable etc).

Poincaré section close to a cycle.

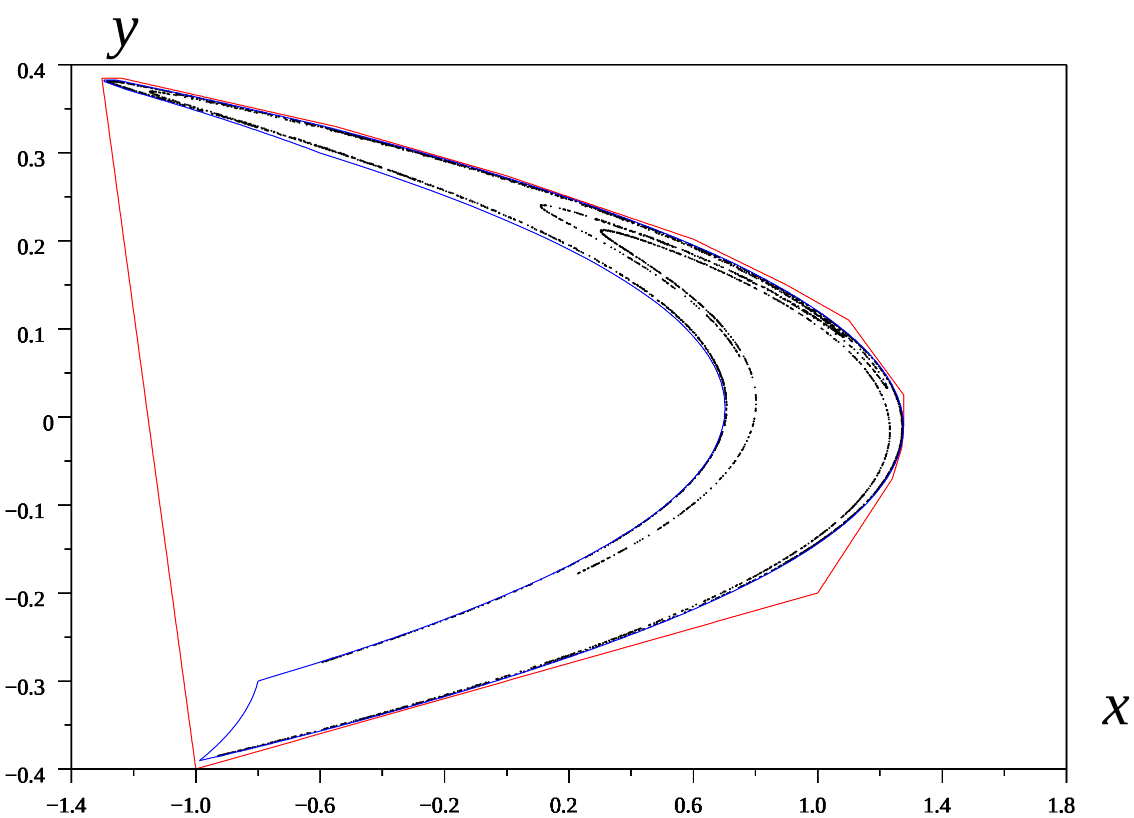


DEFINITION.

For a discrete time dynamical system, a compact (bounded) invariant set $\mathscr{A} \subset \Omega$ is attracting if there is a neighborhood V of $\mathscr{A}$ such that for any open set U containing $\mathscr{A}$, there is an integer $N > 0$ (depending of V and U) such that for any $n > N$, $T^n\left(V\right) \subset U$.

In other words, initial conditions close to $\mathscr{A}$ give rise to orbits which accumulate on $\mathscr{A}$.

Invariant neighborhood for the Hénon map.



An attractor is an attracting set with irreducibility (minimality) property (dense orbit).

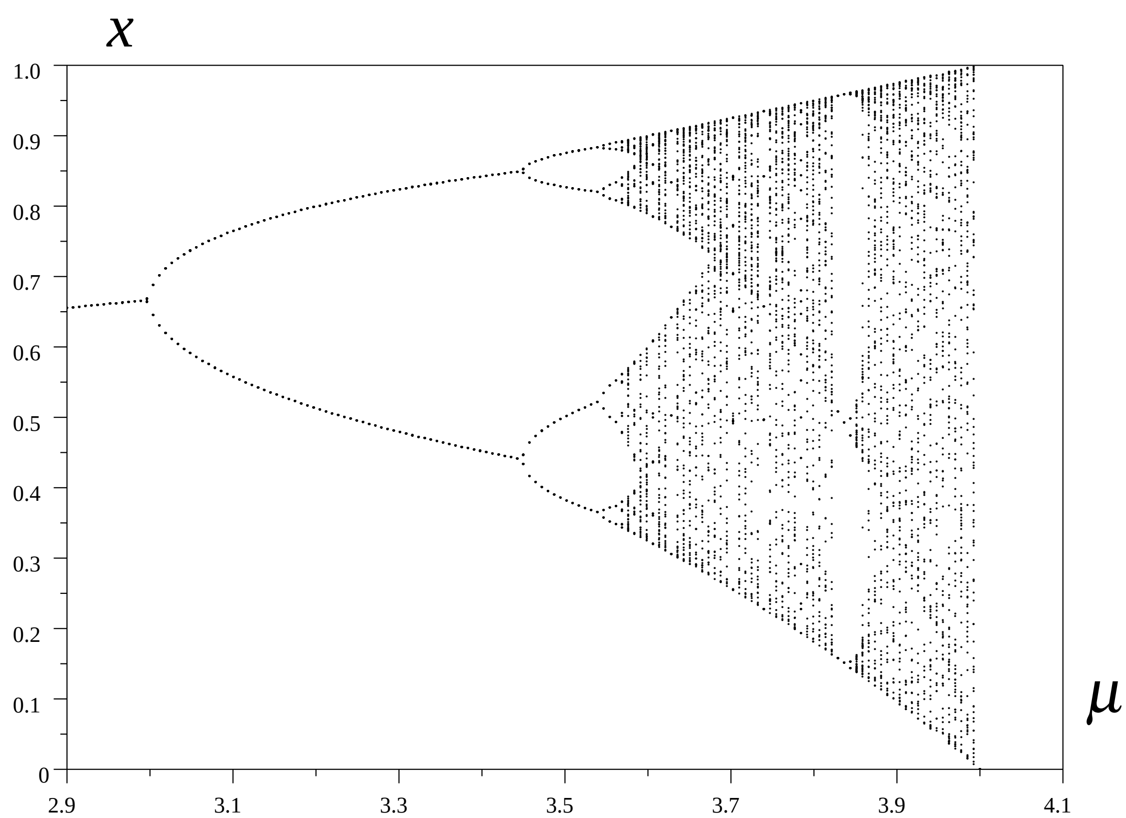
The largest neighborhood V in the above definition is called the basin of the attracting set. In a same phase space several attractors can coexist, together with repellers and invariant sets which are neither attractors nor repellers.

The concept of attractor is suited for the study of dissipative systems. Volume preserving systems do not have attractors, like the mechanical systems without friction.

Attractors can be complicated (strange, fractal...) like in the Hénon map or the Lorenz system.

The simplest attractors are the stable fixed points (stationary solutions) and the stable periodic orbits (stable invariant cycles).

If the dynamical system depends on a parameter, the attractor may depend on the parameter in a very complex manner. This occurs for the attractor of the quadratic family $x \to \mu x(1-x)$.



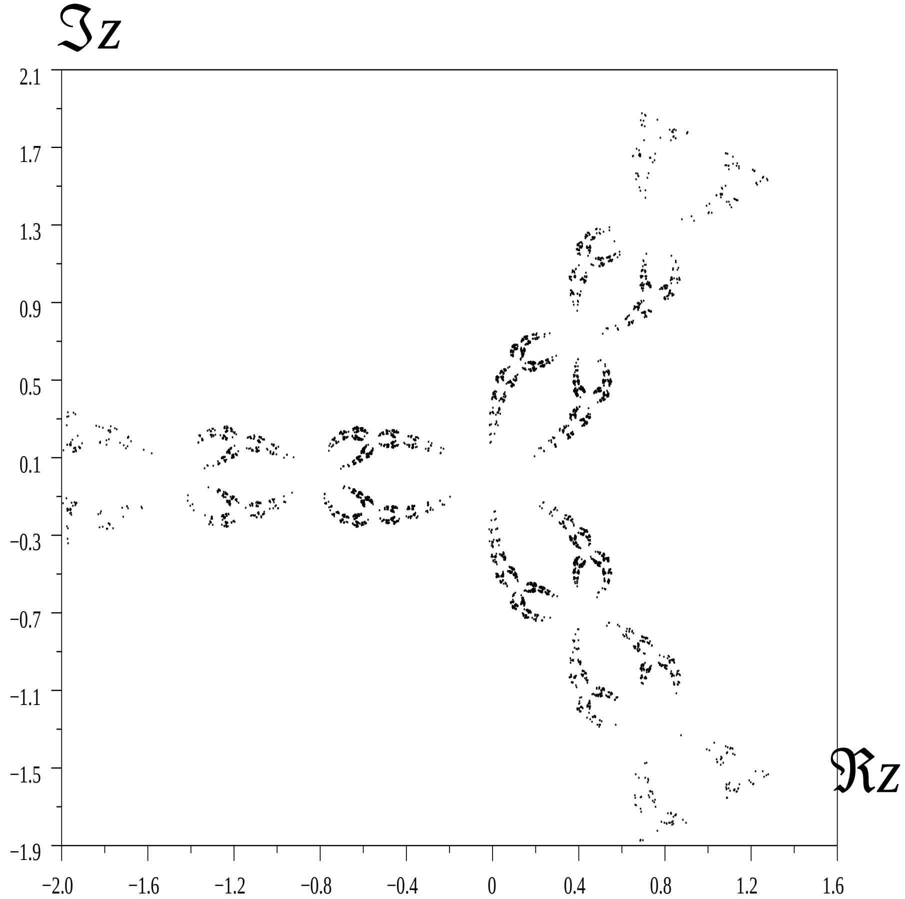
Attractor of the quadratic family as a function of the parameter.

A dynamical system may have several attractors in the same phase space.

Each attractor has its own basin: the set of initial conditions attracted by this attractor.

The boundaries of the basins are in general complicated invariant sets, repelling transversally (towards the attractors).

A well known example is provided by the Newton method applied to the equation $z^3 = 1$. $z_{n+1} = f(z_n) = (z_n + 2/z_n^2)/3$. There are three stable fixed points (attractors): 1, $j, \bar{j}$.



Change of variables.

For discrete time, if $x_{n+1} = f(x_n)$, and $y = g(x)$ with g invertible (not necessarily linear), $y_n = g(x_n)$ satisfies $y_{n+1} = h(y_n)$ with $h = g \circ f \circ g^{-1}$.

For continuous time, if $\vec{x}(t)$ satisfies $d\vec{x}/dt = \vec{X}(\vec{x})$ and $\vec{y} = g(\vec{x})$ with g invertible (not necessarily linear), then $\vec{y}(t) = g(\vec{x}(t))$ satisfies $d\vec{y}/dt = \vec{Y}(\vec{y})$ with

$$\vec{Y}(\vec{y}) = Dg_{g^{-1}(\vec{y})}\vec{X}\left(g^{-1}(\vec{y})\right)$$