LOCAL STABLE AND UNSTABLE MANIFOLDS.

As we have seen in the last section, in the linear approximation at (almost) all points, the (tangent) space decomposes into a “sum” of stable and unstable directions. The stable directions are contracted by the forward time evolution. The unstable directions are contracted by the backward time evolution (which may not be uniquely defined).

It is natural to ask if this image persists if we take into account the nonlinearities.

Pesin has shown that to the field of Lyapunov directions $E_i(x)$ corresponds “covariant manifolds”.

Except for some special cases (uniformly hyperbolic systems, like the baker’s transformation), one can only formulate the result using an ergodic invariant measure μ.

There is a subset Γ of the phase space of full μ measure such that for all $\lambda \in]\lambda_{j+1}, \lambda_j[$, $\lambda_j < 0$, there is a real function $\alpha(x)$ defined on Γ, and for all $x \in \Gamma$ a piece of regular manifold (hyper-surface) W_x^λ containing x, tangent in x to $E_{j+1}(x)$,

and such that if y and z belong to W_x^λ, then

$$d\left(T^n(y), T^n(z)\right) \leq \alpha(x) d(y,z) e^{n\lambda} \ .$$

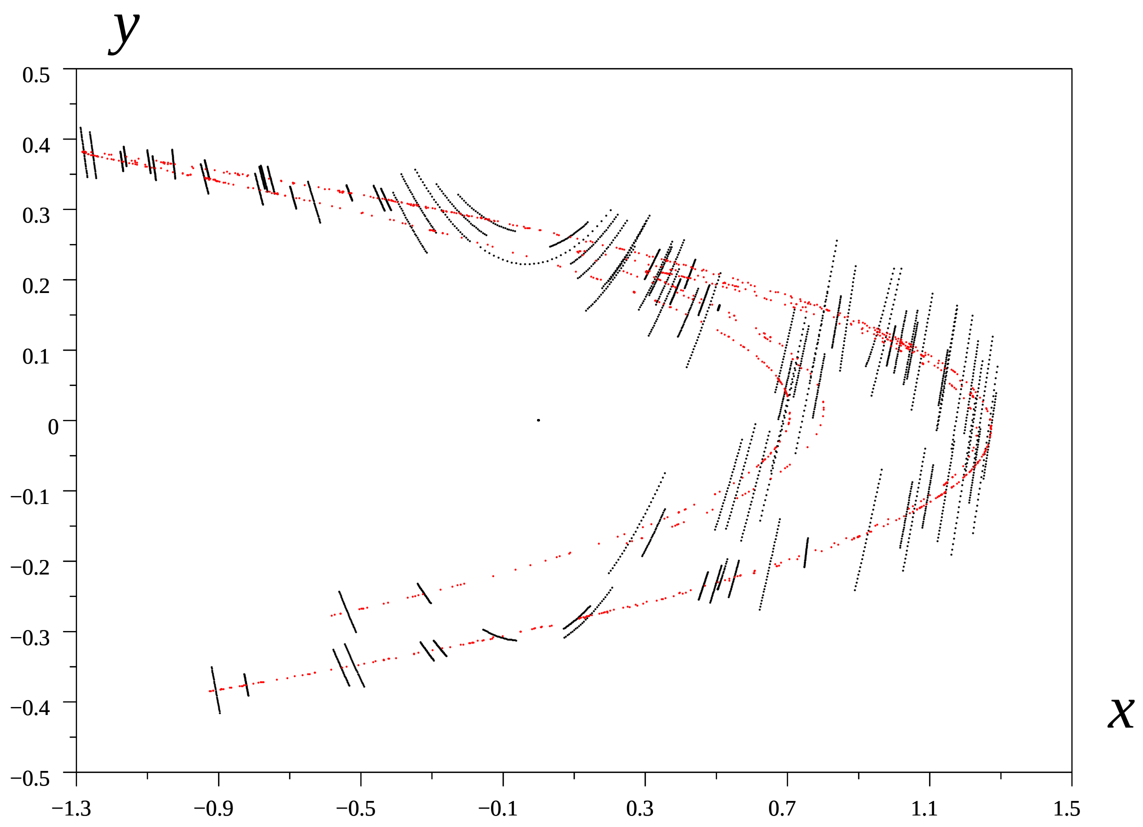
In other words, we have a stable manifold and the contraction rate is almost the linear one.

An important fact is that the manifolds W_x^λ move under T: $T\left(W_x^\lambda\right) \subset W_{T(x)}^\lambda$, unless x is a fixed point (this is the particular case of μ an invariant Dirac mass).

Often these pieces of manifolds are not of uniform size. For example for the Hénon map, there are points where they do not exist (length zero). This is a non uniformly hyperbolic situation.

One gets similar results for local unstable manifolds if the map is invertible (and even in the general case by restricting to orbits).

Local stable manifolds for several points on the attractor of the Hénon map.



Note that some of them have a rather large curvature, and some of them are almost tangent to the attractor.

Exercise

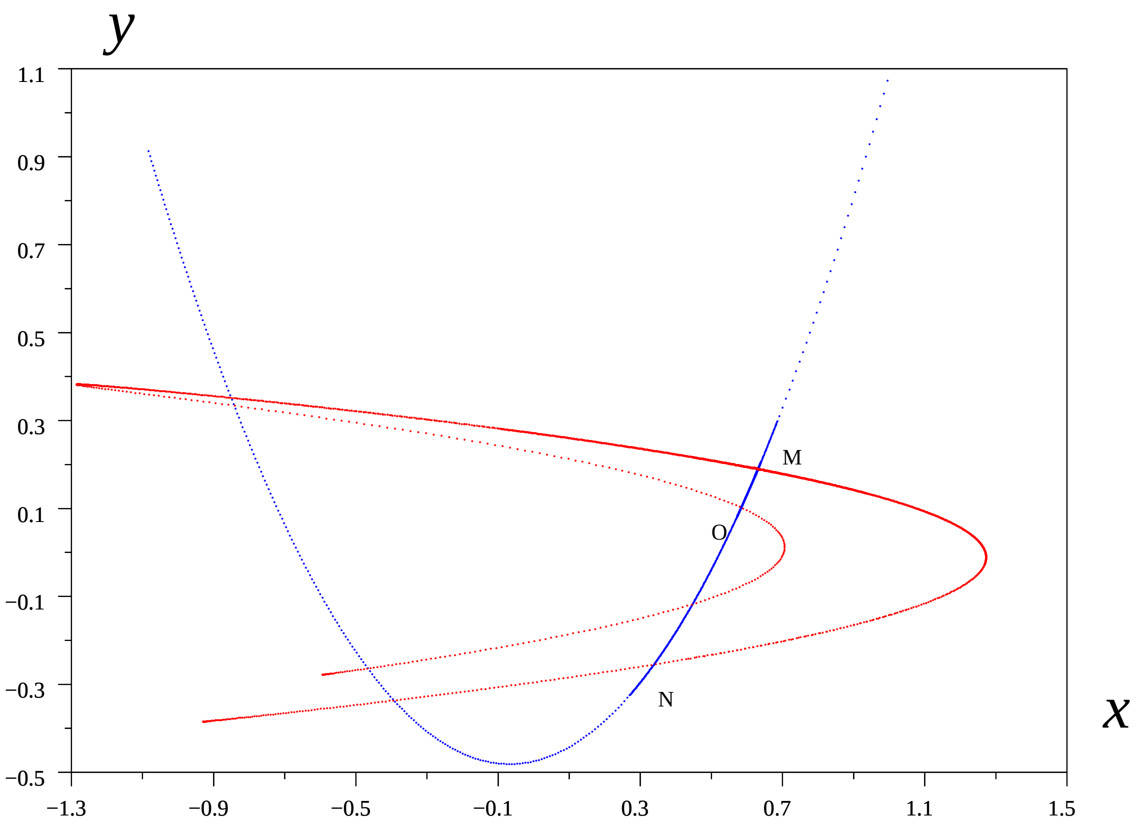
For the dissipative baker's map show that the local stable manifolds are the vertical segments and the unstable manifolds the horizontal segments.

Stable manifolds cannot cross without being identical. The same holds for the unstable manifolds.

Stable and unstable manifolds cannot be tangent. In the Hénon map, this appears as a limiting situation on a Cantor set where these manifolds have length zero.

On the other hand, stable and unstable manifolds can cross transversally as for example in the case of transverse homoclinic or heteroclinic points.

Stable and unstable manifold for the fixed point of the Hénon map.



The crossings between the stable and unstable manifolds are invariant by the transformation and its inverse. This implies a strong folding of the attractor, unless the transformation is singular as in the baker's map.

This is the famous horseshoe discovered by S.Smale on the beach of Rio de Janeiro.

Such crossings and folding are not particular to the stable and unstable manifolds of the fixed point, they can occur for any periodic orbit and more generally for any pair of points (but these points move with the time evolution). In most chaotic systems they are dense in the attractor making the folding picture very complicated.