Comparison of Exact Results for the Virtual Correction to Bremsstrahlung in $e^+e^- \rightarrow f\bar{f} + \gamma$ Annihilation at High Energies

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We have compared the virtual corrections to $e^+e^- \rightarrow f\bar{f} + \gamma$ as calculated by S. Jadach, M. Melles, B.F.L. Ward and S.A. Yost to several other expressions. The most recent of these comparisons is to the Leptonic tensor calculated by J.H. Kühn and G. Rodrigo for radiative return. Agreement is found to within $10^{-5}$ or better, as a fraction of the Born cross section. The massless limits of the results are all found to be analytically identical at NLL order.
In the 1990's, S. Jadach, M. Meller, B. F. L. Ward & S. A. Yost calculated all 2 photon real and virtual corrections to the $e^+e^- \rightarrow e^+e^-$ small angle Bhattachar Scattering process used in the Luminosity Monitor at LEP (and SLC). These calculations were incorporated in a Monte Carlo, and used to achieve a 0.006% precision tag for the program BHEM1, which calculated the luminosity process.
These processes were calculated in a small-mass approximation, which treats the external $e^+$ as massless, and then adds the most important mass corrections.

Once the Bethe-Heitler processes were obtained, the same functions could be used for another important LEP process, $e^+e^- \rightarrow f\bar{f}$ ($f\neq e$), which is related to Bethe-Heitler scattering by “crossing”:

Thus we have all 2-$\gamma$ radiative corrections for this process $e^+e^- \rightarrow f\bar{f}$, at small angles. Only the “box diagrams” are omitted. They become important at larger angles, only.

(A calculation is in progress)
For the luminosity process, the graphs of interest were:

Basic
Bludke
Scattering

Order $a$ corrections (Single-$\gamma$-photon)

Order $a^2$ corrections (Z-$\gamma$-photon)
It is important to cross-check results with other relevant calculations. In this presentation, we will concentrate on the \( e^+e^- \) annihilation; specifically, the initial state radiation graphs.

\[
\begin{align*}
\text{(Diagram 1)} & & \text{(Diagram 2)} \\
\text{(Diagram 3)} & & \text{(Diagram 4)} \\
\text{(Diagram 5)} & & \text{(Diagram 6)}
\end{align*}
\]
These graphs were calculated by Jadach, Melles, Ward & Yost in Phys. Rev. D 55, 073001 (1997),
based on earlier results for the corresponding Bhabha graphs, in Jadach, Melles, Ward & Yost, Phys. Lett. B 377, 168 (1996). The results were obtained using

- Helicity Spinor methods
- The algebraic manipulation program FORM
- The FF package of scalar 1-loop Feynman integrals.

The sum of these amplitudes could be expressed as

\[ M_{1}^{\text{ISR}(1)} = \frac{e^{2}}{16\pi^{2}} M_{1}^{\text{ISR}(0)} (f_{0} + f_{1} \Gamma_{1} + f_{2} \Gamma_{2}) \]

where \( M_{1}^{\text{ISR}(0)} \) is the 1-real-photon amplitude, \( \Gamma_{1}, \Gamma_{2} \) are combinations of helicity spinors, and \( f_{0,1,2} \) are scalar form factors.
In detail,

\[ M_1^{\text{ISR}} = i e^2 \delta_{\lambda_1, \lambda_3} (s') I_0 \frac{2 \langle p_3, s | p_4, a' \rangle}{\langle p_3, s | k_5 \rangle \langle k_5 | p_2, a' \rangle} \]

with kinematic variables:

<table>
<thead>
<tr>
<th>Momentum</th>
<th>Helicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>incoming ( e^- )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>incoming ( e^+ )</td>
<td>( p_2 )</td>
</tr>
<tr>
<td>outgoing ( + )</td>
<td>( p_3 )</td>
</tr>
<tr>
<td>outgoing ( - )</td>
<td>( p_4 )</td>
</tr>
<tr>
<td>redirected ( \gamma )</td>
<td>( k )</td>
</tr>
</tbody>
</table>

\(| p, \lambda \rangle = \text{massless helicity spinor}, \quad \psi \langle p, \lambda \rangle = 0, \quad \chi_5 | p, \lambda \rangle \equiv \lambda | p, \lambda \rangle.\)

\[ G_{\lambda_1, \lambda_3}(2) = \frac{1}{2} \left\{ 1 + \frac{\bar{a} \left( \sqrt{c} + \text{Re} \left( \sqrt{c}^* \right) \right) \left( \sqrt{c} + \text{Re} \left( \sqrt{c}^* \right) \right)}{c - M^2 + i \epsilon} \right\} \]

is the propagator for \( Z, Z \) exchange,

\[ I_0 = - \bar{a} \lambda_1 \lambda_3 \left( \langle \theta, -\beta | \theta', -\beta' \rangle \right)^2 \]

where \( \bar{a} = \sum_{\lambda} \frac{p_3}{\lambda}, \quad \bar{a}' = \sum_{\lambda} \frac{p_4}{\lambda}, \quad \delta = \pm \lambda_1, \quad \delta' = \pm \lambda_3. \)
The dominant term in \( M_{\text{enh}} \) is proportional to \( M_{\text{enh}} \): the \( f_0 \) term,

\[
f_0 = 4\pi B_{\text{YFS}}(s, s\mu e) + 2(L-1-i\pi) + \frac{\Gamma_2}{1-i\pi} \\
+ \frac{2}{\pi(1-i\pi)} \left[ R(\Gamma_1, \Gamma_2) + R(\Gamma_2, \Gamma_1) \right] + \left[ 2 + \frac{2\pi}{2i(1-i\pi)} \right] \ln(1-x) \\
- \frac{\Gamma_2(2+i)}{(1-i)(1-i)} \left[ \ln \left( \frac{1-x}{\Gamma_2} \right) - i\pi \right]
\]

with

\[
4\pi B_{\text{YFS}}(s, s\mu e) = \left[ 4 \ln \left( \frac{\ln x}{\ln e} + 1 \right)(L-1+i\pi) \right] - L^2 - 1 + \frac{4\pi^2}{3} \\
+ \pi \left( 2L - 1 \right)
\]

The YFS infrared factor,

\[
L = \ln \left( \frac{s}{\mu e} \right) \text{ the "big logarithm"},
\]

\[
\Gamma_i = \frac{2\pi \alpha_k}{s} \text{ the "collision photon factor"},
\]

\[
\kappa = \Gamma_1 + \Gamma_2 = \frac{\approx}{E_{\text{beam}}} \text{ the fractional reduced energy},
\]

\[
R(x, y) = \ln^2(1-x) + 2 \ln(1-x) \left[ \ln \left( \frac{y}{1-x} \right) + i\pi \right] + 2 \text{Sp} \left( x+y \right) \\
- 2 \text{Sp} \left( \frac{y}{1-x} \right).
\]
\[ S_p(t) = \int_0^\infty \frac{d\mu_{11-t}}{t} \quad \text{is the Spence Di-logarithm function.} \]

The amplitude can be expressed in a leading log expansion, in terms of the powers of the "big logarithm" \( L = \log \frac{s}{\mu^2} \) obtained in the integrated cross section, \( L = \log \frac{s}{\mu^2} \) at \( \sqrt{s} = 2066 \).

**Leading log (LL): Order \( L \)**

**Next-to-leading log (NLL): Order \( L^2 \)**

The terms through order NLL come from collinear boundary lines \( \gamma \to 0 \) or \( \delta \to 0 \) (neglecting massless):

\[
\begin{align*}
\Phi_{\mathrm{coll}} &= 2\, \ln(1-\gamma) + 2\, \ln(1-\delta) (\ln \tau_2 + i\pi) \\
&\quad + 2\, \frac{\ln^2(1-\gamma)}{i\pi} + 3\, \frac{\ln(1-\gamma)}{1-\gamma} \\
&+ 2\, \frac{\ln^2(1-\delta)}{i\pi} + \frac{\ln(1-\delta)}{1-\delta} + \frac{6\ln(1-\gamma)}{2} \frac{\Gamma_2 - \Gamma_1}{\Gamma_1 - \Gamma_2}.
\end{align*}
\]
Mass corrections were added following the approach of the CMLC collaboration, which adds the mass terms most important in the collinear limits by the prescription

$$\left| \mu_{\mu\gamma}^{(m)} \right|^2 = \left| \mu_{\mu\gamma}^{(o)} \right|^2 - \frac{e^2 M^2}{Q \cdot \vec{k}} \left| \mu_{\mu\gamma}^{(o)} (q - k) \right|^2$$

for $k$ radiated nearly parallel to $q$ (forward line) and $\mu_{\mu\gamma}^{(o)}$ with radiated photon.

The net effect is that $f_0$ is replaced by $f_0 + f_m$ with

$$f_m = \frac{2 e^2}{5} \left( \frac{5}{12} + \frac{2}{7} \right) \frac{2}{(1 - \gamma^2)^2 + (1 - \gamma^2)^2} \times$$

$$\left\{ \begin{array}{c}
\int
\end{array} \sum \right\} \frac{\omega}{\pi} \left[ \frac{4}{3} \omega (L + \frac{1}{2} \omega^2 - 1) - L - \omega^2 + 1 \right]$$

*(To be precise, $f_0$ is the complete massless virtual photon correction factor, but just the term displayed here...*)
The $f_1, f_2$ terms are complicated, but do not contain axial contributions. They can be found in the Phys. Rev. D65 paper.

For cross-checks, this result has been compared to various other results:

   - spin-averaged cross section, fully differential
     in photon variables $T_{1,2}$, no mass corrections

   - spin-averaged cross section, differential
     only in $x=T_{1,2}$, has mass corrections

   - spin averaged leptonic tensor, fully differential
     with mass corrections
The comparison is a neat one, which should be equivalent in all respects to our calculation, except that it was computed for a different purpose.

Rodrigo & Kuhn (et al) were interested in radiative return, a way to explore a wide range of energies in a single experiment

\[ e^+(p_1) + e^-(p_2) \rightarrow \gamma(k) + \gamma^*(q) \rightarrow \text{Hadron} \]

CMS energies \( \sqrt{s} = 1.4 \text{GeV} - 2 \text{GeV} \) could be reached by radiating most of the original CMS energy into the photon. Small angle radiation is especially useful due to the rate enhancement.

Example: \( \pi^+\pi^- \) final state investigated by DA\( \bar{E} \)NE. The cross sections were implemented in a Monte Carlo program Ekalor.
The calculation is done using a "leptonic tensor" for the initial state: For pure photon exchange,

\[ |M_{\text{initial}}|^2 = \frac{1}{4s_{\tau\bar{\tau}}^2} L_{\mu\nu} \tilde{L}_{\mu\nu} \]

where \( L_{\mu\nu} \) represents the initial state contribution, and has the general form

\[ L_{\mu\nu} = a_{\mu\nu} g_{\mu\nu} + \tilde{a}_{\mu\nu} p_{\mu}^\tau p_{\nu}^\bar{\tau} + \tilde{a}_{\mu\nu} p_{\mu}^\bar{\tau} p_{\nu}^\tau \\
+ \tilde{a}_{\mu\nu} (p_{\mu}^\tau p_{\nu}^\bar{\tau} - p_{\mu}^\bar{\tau} p_{\nu}^\tau) + i\pi a_{\mu\nu} (p_{\mu}^\tau p_{\nu}^\bar{\tau} - p_{\mu}^\bar{\tau} p_{\nu}^\tau) \]

For pure real photon emission (no virtual corrections),

\[ a_{\mu\nu}^{(0)} = -\tau(1-\tau)^2 + (1-\tau)^2 \]
\[ a_{\mu\nu}^{(1)} = a_{\mu\nu}^{(2)} = -4\tau^2 \]
\[ a_{\mu\nu}^{(3)} = 0 \]
with \( \tau = 1 - x \)

(assuming masses)

\[ H_{\mu\nu} = J_{\mu} J^{\nu*} \] in terms of the final state current \( J_{\mu} \). For \( \pi^+ \pi^- \) final state,

\[ J_{\pi^+ \pi^-} = \sigma F_{\pi^+}(q^2)(q_{\pi^+}^\mu - q_{\pi^-}^\mu). \]

But we would like to compare with \( e^+ e^- \rightarrow \pi^+ \pi^- \).
For the final state, with $f(P_3, \lambda_3)$, $f(P_4, \lambda_4)$,

\[ J^{\mu} = i e <P_3\lambda_3|G^\mu|P_4\lambda_4>, \]

\[ H_{\mu\nu} = J_\mu J_\nu^* = e^2 \text{tr} (\rho_3 \rho_3^\dagger \rho_4 \rho_4^\dagger) \]

\[ = 4e^2 (\rho_{3\mu} \rho_{4\nu} + \rho_{4\mu} \rho_{3\nu} - \rho_{3\mu} \rho_{3\nu} + \rho_{3\mu} \rho_{3\nu}) \text{g}_{\mu\nu}. \]

Contracting $L$ with $H$ gives (neglecting mass corrections)

\[ |M_{\text{SE}^{(1)}(1)}|^2 = \left\{ -4 z_{\text{SM}} s' s + 2 z_1 t_1 c_1 + 2 z_2 t_2 c_2 - \frac{e^2}{s^2 z_{11} z_{22}} \right\}^2 \]

where $z_{\text{SM}} = (\rho_{1\rho}^2)^2 = 4 E_{\text{beam}}^2 / s$, $s' = (P_3 + P_4)^2 = s z'$,

\[ t_1 = (P_1 - P_3)^2, \quad t_2 = (P_1 - P_4)^2, \quad c_1 = (P_1 - P_3)^2, \quad c_2 = (P_2 - P_3)^2. \]

The tree-level result is found to be (as expected)

\[ |M_{\text{SE}^{(1)}(0)}|^2 = \frac{e^2}{s^2 z_{11} z_{22}} \left( t_1^2 + t_2^2 + c_1^2 + c_2^2 \right). \]
The real and virtual squared amplitudes can be written as a term proportional to the tree level result, plus extra terms:

\[
|\mathcal{M}_{1}^\text{ISR}(\omega)|^2 = |\mathcal{M}_{1}^{(0)}|^2 \left( \frac{a_{\omega}}{a_{\omega}^{(0)}} - \frac{r_1^2 + r_2^2}{2 \, \omega} \frac{a_{\omega}^{(0)}}{a_{\omega}} \right)
+ \frac{2 e^6}{\pi^2 \pi^2} \frac{\tilde{t}_1 r_1}{a_{\omega}^{(0)}} (a_{\omega}^{(0)} + \frac{1}{2} a_{\omega} - \frac{a_{\omega}}{a_{\omega}^{(0)}} a_{\omega}^{(0)}),
+ \frac{2 e^6}{\pi^2 \pi^2} \frac{\tilde{t}_2 r_2}{a_{\omega}^{(0)}} (a_{\omega}^{(0)} + \frac{1}{2} a_{\omega} - \frac{a_{\omega}}{a_{\omega}^{(0)}} a_{\omega}^{(0)}),
- \frac{e^6}{\pi^2 \pi^2} \frac{a_{\omega}^{(0)}}{a_{\omega}^{(0)}} \left[ (r_3 - r_1^2 + (r_2 - r_1)^2) \right]
\]

with \( r_3 = \frac{2 E_{\text{ISR}}}{\epsilon} \).

It is expected that the coefficient of \( |\mathcal{M}_{1}^{\text{ISR}(\omega)}|^2 \) should be essentially the same as Re \( \mathcal{M}_{1}^{(0)} \) in our expression, averaged over helicity, and that the remaining terms should not contribute to this sum.
The coefficient functions are

\[
\begin{align*}
a_{00}^{(1)} & = a_{00}^{(2)} \left[ 8 \pi B \gamma f s \left( 5 \pi \omega \right) + 2 \left( 1 - \gamma \right) + 3 \gamma \right] \\
& - \frac{\gamma \varepsilon}{2} - 2 \gamma \frac{\varepsilon}{1 - \gamma} - 2 \left( 1 + \frac{2 \gamma \varepsilon}{1 - \gamma} \right) \frac{\partial}{\partial \varepsilon} \\
& + \gamma \left( 4 - \frac{2 \gamma \varepsilon}{1 - \gamma} \right) \frac{\partial}{\partial \gamma} \frac{\gamma}{1 - \gamma} \\
& - \frac{\gamma}{2} \left[ 1 + \frac{1 - \gamma}{\gamma} \right] \frac{\partial}{\partial \gamma} \\
& - \frac{\gamma}{2} \left[ 1 + \frac{1 - \gamma}{\gamma} \right] \frac{\partial}{\partial \gamma} \\
\end{align*}
\]

\[
\begin{align*}
a_{11}^{(1)} & = a_{11}^{(2)} \left[ 8 \pi B \gamma f s \left( 5 \pi \omega \right) + 2 \left( 1 - \gamma \right) + 3 \gamma \right] \\
& + 2 \left( 1 + \frac{2 \varepsilon}{\gamma} \right) \left( \frac{1}{1 - \gamma} - \frac{1}{\gamma} \right) - \frac{8 \varepsilon}{1 - \gamma} \\
& - \frac{4 \gamma \varepsilon}{\gamma} \left[ \left( 1 - \gamma \right) \left( \frac{1}{\gamma} + \frac{2 \varepsilon}{\gamma} \right) + \frac{2 \varepsilon}{x} \right] \frac{\partial}{\partial \gamma} \\
& - 2 \gamma \left( \frac{2 - 2 \gamma}{\gamma} \right) \frac{\partial}{\partial \gamma} \frac{\gamma}{1 - \gamma} - 4 \gamma \left( 1 + \frac{1}{\gamma} \right) \frac{\partial}{\partial \gamma} \\
& - \frac{4 \gamma}{\gamma} \left( 3 + \frac{2 \varepsilon}{\gamma} + \frac{\gamma}{1 - \gamma} \right) \frac{\partial}{\partial \gamma} \frac{\gamma}{1 - \gamma} \\
\end{align*}
\]

\[
\begin{align*}
a_{12}^{(1)} & = a_{12}^{(2)} \left( \gamma \left( 1 + \frac{\gamma}{1 - \gamma} \right) \right) \\
\end{align*}
\]

\[
\begin{align*}
a_{12}^{(2)} & = \frac{\gamma}{1 - \gamma} - \frac{4 \gamma}{1 - \gamma} \left( 1 - \frac{\gamma}{1 - \gamma} \right) \\
& - \frac{2}{x} \left( \frac{\gamma}{1 - \gamma} + 1 + \frac{2 \gamma \varepsilon}{1 - \gamma} \right) \frac{\partial}{\partial \gamma} \\
& - \frac{2}{x} \left( 1 + \frac{2 \gamma \varepsilon}{1 - \gamma} \right) \frac{\partial}{\partial \gamma} \\
& + \left( \gamma \left( 1 + \frac{\gamma}{1 - \gamma} \right) \right) \\
\end{align*}
\]
The $\frac{1}{z}$ and $\frac{1}{x}$ factors are apparent sources of
singularities in the collinear & soft limits. They
can turn out to be compensable in the
numeratorless, but only after a 2nd order expansion.

$$H(z) = 2y(1 - \nu) - 5y(1 - \nu) + ln(2 + y) - ln(\frac{z}{\nu})$$

- The stabilized functions were programmed
in the KK Monte-Carlo generator and compared
to our expressions. Agreement on the order
of 10^-6 of the Born cross section is found, at
much of the kinematic range, up to 10^-5 of
large vectors.

- Analytic agreement is found between $\rho_0$
and the helicity averaged version of $\rho_0$
to NLO order.
Comparisons of the YFS residual $\beta_1$ for the complete cross section $e^+e^- \to f\bar{f} + \gamma$, normalized with respect to the Born cross section, as a function of the cut $v_{\text{max}}$ on the maximum energy radiated into the photon relative to the beam energy. The functions are compared in the KK Monte Carlo for 100M events.


The order ($\alpha^2$) virtual correction to $\beta_1$ as a function of the cut $v_{\text{max}}$ on the maximum fraction of the beam energy radiated into the emitted photon. The functions are compared in the KK Monte Carlo for 100M events.


The residual part of the cross sections after subtracting common “LL” (order \(\alpha^2 L^2\)) terms is compared for four different expressions, for cut \(v_{\text{max}}\) on the maximum energy fraction carried by the emitted photon. The last data point is off-scale on this graph. The functions are compared in the KK Monte Carlo for 100M events.


The residual part of the cross sections after subtracting common “NLL” (order ($\alpha^2L$)) terms is compared for four different expressions, for cut $v_{\text{max}}$ on the maximum energy fraction carried by the emitted photon.


Closer view of the residual part of the cross sections after subtracting common “NLL” (order ($\alpha^2 L$)) terms is compared for four different expressions, for cut $v_{\text{max}}$ on the maximum energy fraction carried by the emitted photon. The last data point is off-scale on this graph.


