Heavy Quarks at Threshold: Recent Developments

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I. Motivation
II. pNRQCD
III. Applications
IV. Conclusions/next steps

[in collaboration with B.A. Kniehl, A.Pineda, A.A. Penin, V.A. Smirnov]

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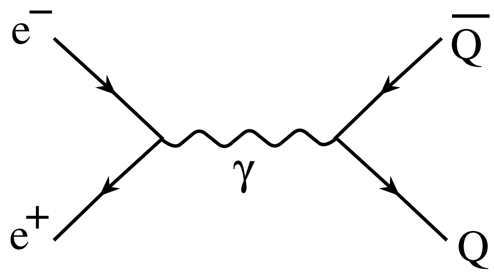
Non-relativistic regime of QCD — Why interesting?

- $(Q\bar{Q})$ ➪ quark masses, strong coupling, . . .
- experimentally and theoretically "clean"
- first principles of QCD
- highly non-trivial multi-scale dynamics:

  m_Q, $|\boldsymbol{p}| \sim m_Q v$, $E \sim m_Q v^2$ [v: velocity of quark Q]

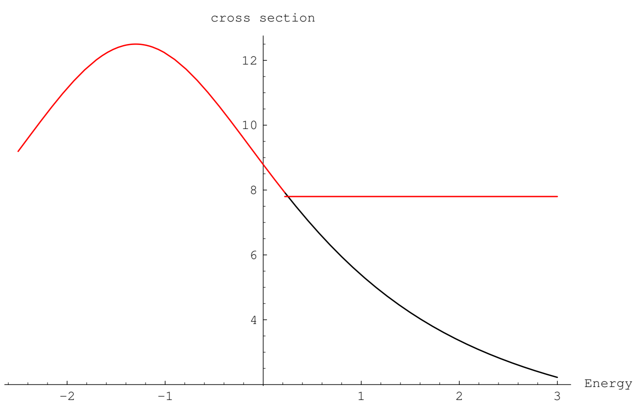
$(t\bar{t})$	$m_t v^2 \approx 20$ GeV $\gg \Lambda_{\rm QCD}$	perturbative
$(b\bar{b})$	$m_b v^2 \approx \Lambda_{\rm QCD}$	border of application
$(c\bar{c})$	$m_c \gg \Lambda_{\rm QCD} \gg m_c v$	non-perturbative quarkonium

$e^+e^- \to t\bar{t}$ close to threshold

- 

  Energy: $\sqrt{s} \approx 2\times\ m_t$
  to be measured at a furture LC
  ($\sqrt{s} \approx 350$ GeV)

- cross sections (schematic):



- position of peak ➪ top quark mass
- line shape ➪ m_t, α_s, Γ_t and y_t

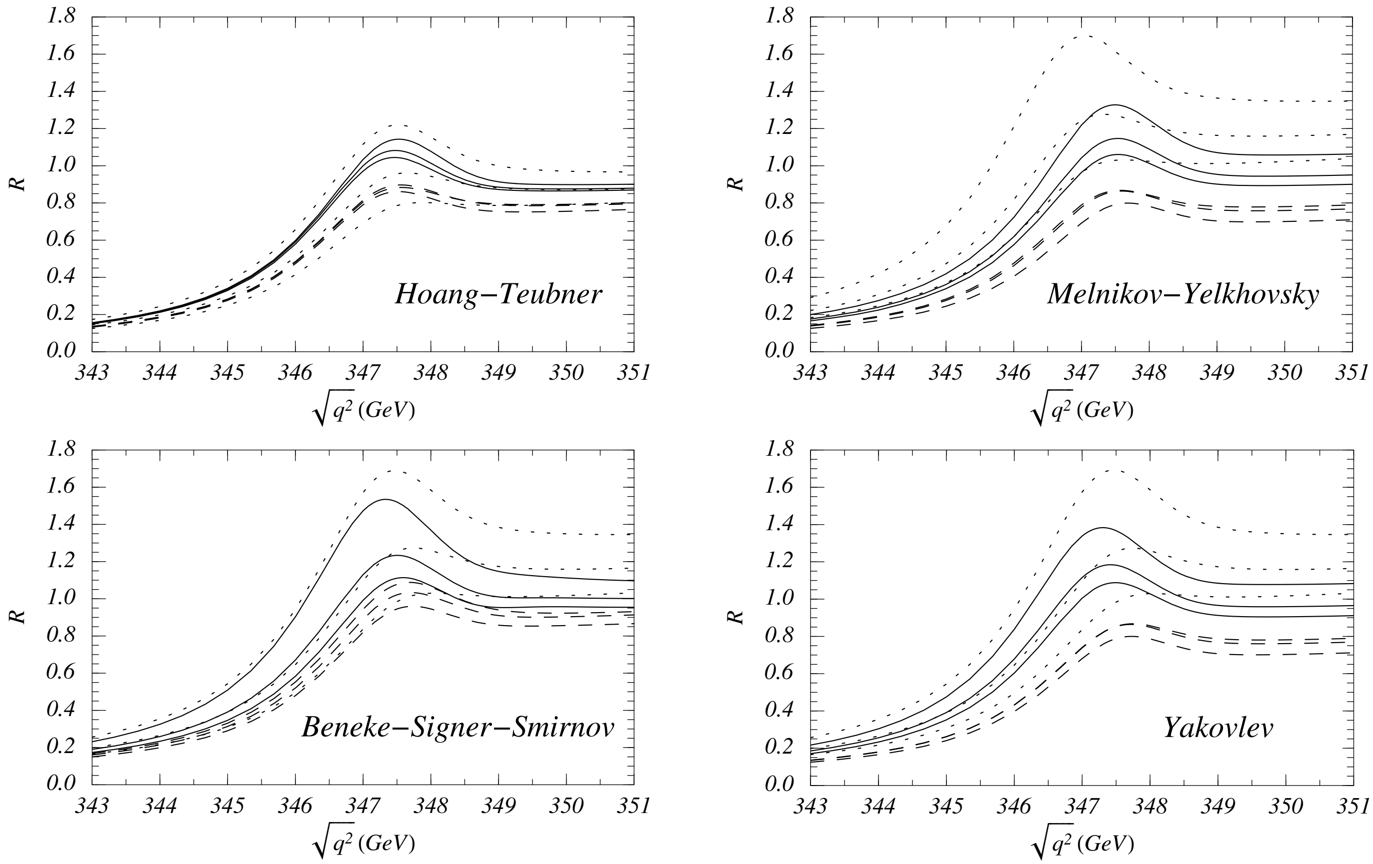
Determination of m_t, α_s, Γ_t and y_t at TESLA

[Martinez,Miquel'02]

- 9 point threshold scan; $\mathcal{L} = 300$ fb^{-1}
- observables: σ_{tot}, top quark momentum distribution, forward-backward charge asymmetry
- initial state radiation; beam smearing
- theory-error: $\pm 3\%$ assumed
- 4-parameter fit

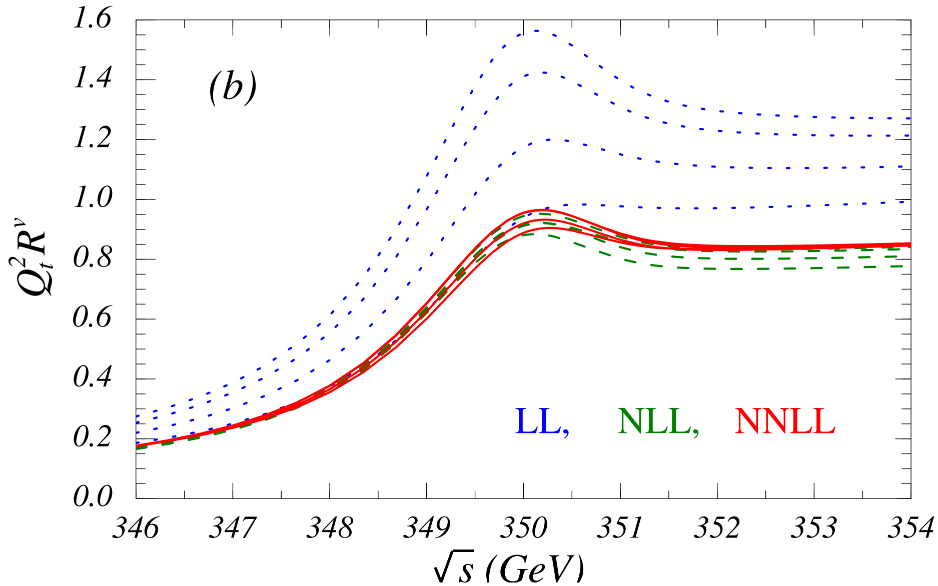
➪ $\delta m_t \sim 20$ MeV, $\delta \Gamma_t \sim 30$ MeV $\delta \alpha_s \sim 0.0012$, $\delta y_t / y_t \sim 35\%$

$e^+e^- \to t\bar{t}$ at NNLO — "threshold mass"



LO: dotted
NLO: dashed
NNLO: full

$e^+e^- \to t\bar{t}$ at NNLL



[Hoang,Manohar,Stewart,Teubner'01] [Hoang,Stewart'02] [Hoang'03]

- idea: resum logarithms $(v \ln v)^k$, $v(v \ln v)^k$, $v^2(v \ln v)^k$
- NNLL result not complete
- velocity NRQCD
- claim: $\frac{\delta\sigma_{tt}}{\sigma_{tt}} \approx$ $\pm 3\%(2001) \to \pm 6\%(2003)$

Recent new results (within potential NRQCD)

- $\mathcal{O}(\alpha_s^3)$ corrections to position of peak
- $\mathcal{O}(\alpha_s^3 \ln \alpha_s)$ corrections to normalization of peak
- NNNLL-resummation for spin-dependent term

  ➪ NLL corr. to HFS

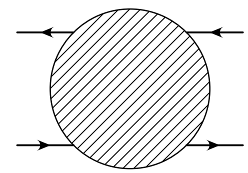
II. Non-relativistic limit of QCD

scales: mass, m: hard $\gg$ momentum, mv: soft $\gg$ energy, mv^2: ultra-soft

Aim: construct effective theory where active degrees of freedom are

potential quarks: $\begin{cases} E_{\boldsymbol{p}} \sim mv^2 \\ |\boldsymbol{p}| \sim mv \end{cases}$ $\frac{1}{E_{\boldsymbol{p}} - \frac{\boldsymbol{p}^2}{2m}}$

ultra-soft gluons: $\begin{cases} E_{\boldsymbol{k}} \sim mv^2 \\ |\boldsymbol{k}| \sim mv^2 \end{cases}$ $\frac{1}{E_{\boldsymbol{k}}^2 - \boldsymbol{k}^2}$



– $(b\bar{b})$ bound state

– $e^+e^- \to t\bar{t}$ @ threshold

QCD $\longrightarrow$ NRQCD $\longrightarrow$ pNRQCD (potential non-relativistic QCD)

$\uparrow$ integrate out hard scale "m" from QCD [Bodwin,Braaten,Lepage'95]

$\uparrow$ integrate out all scales from NRQCD except potential quarks and ultra-soft gluons [Pineda,Soto'98]

pNRQCD: $\mathcal{L}_{\rm pNRQCD} \sim \phi^\dagger \left(i\partial_0 - V(\boldsymbol{r})\right) \phi + \ldots \longrightarrow i\partial_0 \phi = \left(\frac{\boldsymbol{p}^2}{m} + V(\boldsymbol{r})\right) \phi \to$ Schrödinger equation

Computation of potentials

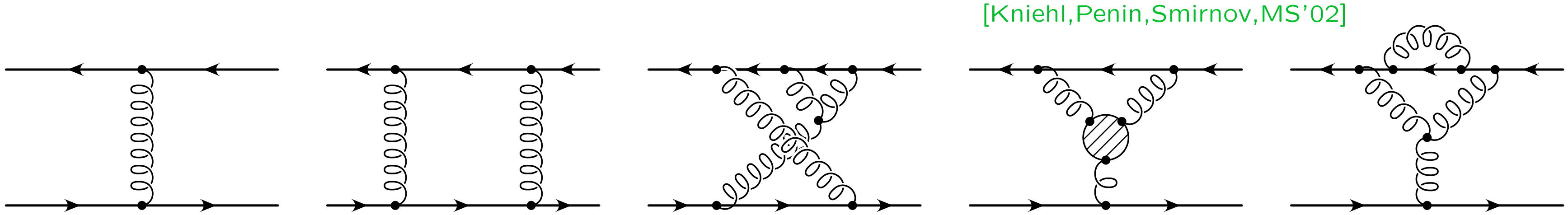
Potential $\hat{=}$ coefficient functions of pNRQCD

- can be computed in full theory ($\hat{=}$ QCD/NRQCD)
- expansion in α_s (→ counts # loops)

  and $|\boldsymbol{p}|/m \sim v$ (→ higher order operators in $\mathcal{L}_{\text{pNRQCD}}$)

Potential known to next-to-next-to-next-to leading order (N^{3}LO)

[Kniehl,Penin,Smirnov,MS'02]



$$V = V_C \Big[1 + \delta V^{\text{NLO}} + \delta V^{\text{NNLO}} + \delta V^{\text{N}^3\text{LO}} + \ldots \Big]$$

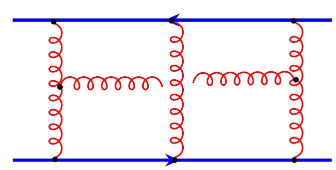
α_s, $\frac{|\boldsymbol{p}|}{m}$

α_s^2, $\alpha_s \frac{|\boldsymbol{p}|}{m}$, $(\frac{|\boldsymbol{p}|}{m})^2$

$\boxed{\alpha_s^3}$, $\alpha_s^2 \frac{|\boldsymbol{p}|}{m}$, $\alpha_s (\frac{|\boldsymbol{p}|}{m})^2$, $(\frac{|\boldsymbol{p}|}{m})^3$

$$V_C(\boldsymbol{q}) = -\frac{16\pi\alpha_s}{3\boldsymbol{q}^2}$$
$$V_C(\boldsymbol{r}) = -\frac{4\alpha_s}{3|\boldsymbol{r}|}$$

$$V_{1/m}(\boldsymbol{q}) = \frac{4\pi^2\alpha_s^2}{3m|\boldsymbol{q}|}(b_1 + \frac{\alpha_s}{\pi} b_2)$$
$$V_{1/m}(\boldsymbol{r}) = \frac{4\alpha_s}{6m\boldsymbol{r}^2}(\bar{b}_1 + \frac{\alpha_s}{\pi} \bar{b}_2)$$



Rayleigh-Schrödinger perturbation theory

Energy level: $E_n = E_n^C + \delta E_n^{(1)} + \delta E_n^{(2)} + \delta E_n^{(3)} + \ldots$ $\qquad E_n^C = -\frac{4\alpha_s^2 m}{9n^2}$

$\delta E_1^{(3)}$ gets contributions from

- $\langle \psi_1^C | \delta \mathcal{H}^{\mathrm{N^3LO}} | \psi_1^C \rangle$
- iteration of $\delta\mathcal{H}^{\mathrm{NLO}}$ and $\delta\mathcal{H}^{\mathrm{NNLO}}$; 3 iterations of $\delta\mathcal{H}^{\mathrm{NLO}}$
- $\langle \psi_1^C | \delta^{\mathrm{us}} \mathcal{H} | \psi_1^C \rangle$
- retarded ultra-soft contribution

➪ analytical result for $\delta E_1^{(3)}$

Numerical result for $\delta E_1^{(3)}$

$$\delta E_1^{(3)} = \alpha_s^3(\mu_s) E_1^C \Bigg[\begin{pmatrix} 70.590|_{n_l=4} \\ 56.732|_{n_l=5} \end{pmatrix} + 15.297 \ln(\alpha_s(\mu_s)) + 0.001\, a_3 + \begin{pmatrix} 34.229|_{n_l=4} \\ 26.654|_{n_l=5} \end{pmatrix} \Bigg|_{\beta_0^3}$$

$\delta E_1^{(1)}$ and $\delta E_2^{(1)}$

$$\delta E_1^{(1)} = E_1^C \frac{\alpha_s}{\pi}\left[4\beta_0 L_\mu + 4\beta_0 + \frac{a_1}{2}\right]$$

$L_\mu = \ln(\mu/(C_F \alpha_s m))$
β_0, β_1, β_2: QCD β function
C_A, C_F, T_F: colour factors
a_1, a_2, a_3: coef. of static pot.
S: Spin

$$\delta E_1^{(2)} = E_1^C \left(\frac{\alpha_s}{\pi}\right)^2 \Big[12\beta_0^2 L_\mu^2 + \left(16\beta_0^2 + 3a_1\beta_0 + 4\beta_1\right) L_\mu + \left(4 + \frac{2\pi^2}{3} + 8\zeta(3)\right)\beta_0^2 + 2a_1\beta_0 + 4\beta_1 + \frac{a_1^2}{16} + \frac{a_2}{8} + \pi^2 C_A C_F + \left(\frac{21\pi^2}{16} - \frac{2\pi^2}{3}S(S+1)\right) C_F^2\Big]$$

[Pineda,Yndurain'98; Penin,Pivovarov'98; Melnikov,Yelkhovsky'99]

$$\delta E_1^{(3)} = \delta E_1^{(3)}\Big|_{\beta(\alpha_s)=0} + \delta E_1^{(3)}\Big|_{\beta(\alpha_s)}$$

$$\delta E_1^{(3)}\Big|_{\beta(\alpha_s)=0} = -E_1^C \frac{\alpha_s^3}{\pi}\Bigg\{-\frac{a_1 a_2 + a_3}{32\pi^2} + \left[-\frac{C_A C_F}{2} + \left(-\frac{19}{16} + \frac{S(S+1)}{2}\right) C_F^2\right] a_1 + \left[-\frac{1}{36} + \frac{\ln 2}{6} + \frac{L_{\alpha_s}}{6}\right] C_A^3$$
$$+ \left[-\frac{49}{36} + \frac{4}{3}(\ln 2 + L_{\alpha_s})\right] C_A^2 C_F + \left[-\frac{5}{72} + \frac{10}{3}\ln 2 + \frac{37}{6} L_{\alpha_s} + \left(\frac{85}{54} - \frac{7}{6} L_{\alpha_s}\right) S(S+1)\right] C_A C_F^2$$
$$+ \left[\frac{50}{9} + \frac{8}{3}\ln 2 + 3 L_{\alpha_s} - \frac{S(S+1)}{3}\right] C_F^3 + \left[-\frac{32}{15} + 2\ln 2 + (1 - \ln 2) S(S+1)\right] C_F^2 T_F$$
$$+ \frac{49 C_A C_F T_F n_l}{36} + \left[\frac{11}{18} - \frac{10}{27} S(S+1)\right] C_F^2 T_F n_l + \frac{2}{3} C_F^3 L_1^E\Bigg\}$$

[Kniehl,Penin,Smirnov,MS'01]

$$\delta E_1^{(3)}\Big|_{\beta(\alpha_s)} = E_n^C \left(\frac{\alpha_s}{\pi}\right)^3 \Bigg\{32\beta_0^3 L_\mu^3 + \left[40\beta_0^3 + 12 a_1 \beta_0^2 + 28\beta_1\beta_0\right] L_\mu^2 + \Bigg[\left(\frac{16\pi^2}{3} + 64\zeta(3)\right)\beta_0^3 + 10 a_1 \beta_0^2$$
$$+ \left(40\beta_1 + \frac{a_1^2}{2} + a_2 + 8\pi^2 C_A C_F + \left(\frac{21\pi^2}{2} - \frac{16\pi^2}{3} S(S+1)\right) C_F^2\right)\beta_0 + 3 a_1 \beta_1 + 4\beta_2\Bigg] L_\mu$$
$$+ \left(-8 + 4\pi^2 + \frac{2\pi^4}{45} + 64\zeta(3) - 8\pi^2\zeta(3) + 96\zeta(5)\right)\beta_0^3 + \left(\frac{2\pi^2}{3} + 8\zeta(3)\right) a_1 \beta_0^2$$
$$+ \Bigg(\left(8 + \frac{7\pi^2}{3} + 16\zeta(3)\right)\beta_1 - \frac{a_1^2}{8} + \frac{3}{4} a_2 + \left(6\pi^2 - \frac{2\pi^4}{3}\right) C_A C_F$$
$$+ \left(8\pi^2 - \frac{4\pi^4}{3} + \left(-\frac{4\pi^2}{3} + \frac{4\pi^4}{9}\right) S(S+1)\right) C_F^2\Bigg)\beta_0 + 2 a_1 \beta_1 + 4\beta_2\Bigg\}$$

[Penin,MS'02]

$$L_{\alpha_s} = -\ln(C_F \alpha_s),\ L_\mu = \ln(\mu/(C_F \alpha_s m))$$

Perturbation theory for wave function

Wave function: $|\psi_n^C(0)|^2 \left(1 + \delta\psi_n^{(1)} + \delta\psi_n^{(2)} + \delta\psi_n^{(3)} + \ldots\right)$ $\quad |\psi_n^C(0)|^2 = \frac{8\alpha_s^3 m^3}{27\pi n^3}$

$\delta\psi_n^{(3)} = \left(\frac{\alpha_s}{\pi}\right)^3 \left[K_2 \ln^2 \alpha_s + K_1 \ln \alpha_s + K_0\right]$ $\quad$ "present limit" : K_1

K_2: [Kniehl,Penin'00, Manohar,Stewart'01]

[Kniehl,Penin,Smirnov,MS'02, Hoang'03]

$$
\begin{aligned}
K_1 \;=\; & \left[\left(-3+\frac{2\pi^2}{3}\right)C_AC_F + \left[\frac{4\pi^2}{3} - \left(\frac{10}{9}+\frac{4\pi^2}{9}\right)S(S+1)\right]C_F^2\right]\beta_0 \\
& + \left[-\frac{3}{4}C_AC_F + \left(-\frac{9}{4}+\frac{2}{3}S(S+1)\right)C_F^2\right]a_1 + \frac{1}{4}C_A^3 + \left(\frac{59}{36} - 4\ln 2\right)C_A^2C_F \\
& + \left[\frac{143}{36} - 4\ln 2 - \frac{19}{108}S(S+1)\right]C_AC_F^2 + \left[-\frac{35}{18} + 8\ln 2\right. \\
& \left. -\frac{1}{3}S(S+1)\right]C_F^3 + \left[-\frac{32}{15} + 2\ln 2 + (1-\ln 2)\,S(S+1)\right]C_F^2T_F \\
& + \frac{49}{36}C_AC_FT_Fn_l + \left[\frac{8}{9} - \frac{10}{27}S(S+1)\right]C_F^2T_Fn_l\,, \qquad (\mu = C_F m_q \alpha_s)
\end{aligned}
$$

III.A. Top quark mass

$$E_{\text{res}} = 2m_t + E_1^{\text{p.t.}} + \delta^{\Gamma_t} E_{\text{res}}$$

$E_1^{\text{p.t.}}$: perturbative contribution to energy level (discussed above)

$\delta^{\Gamma_t} E_{\text{res}} = 100 \pm 10$ MeV: effect from (large) width, continuum and higher resonances

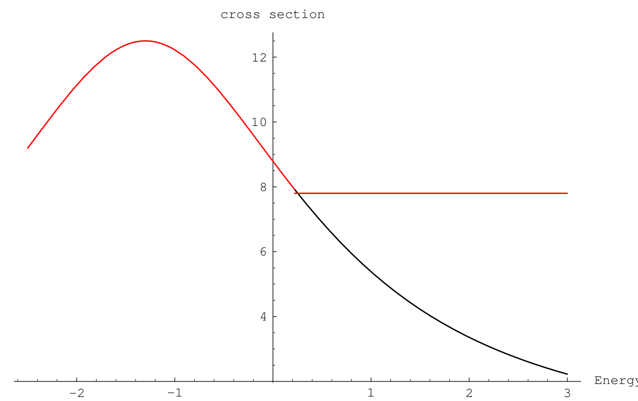
typical scale: $C_F m_t \alpha_s \approx 30$ GeV

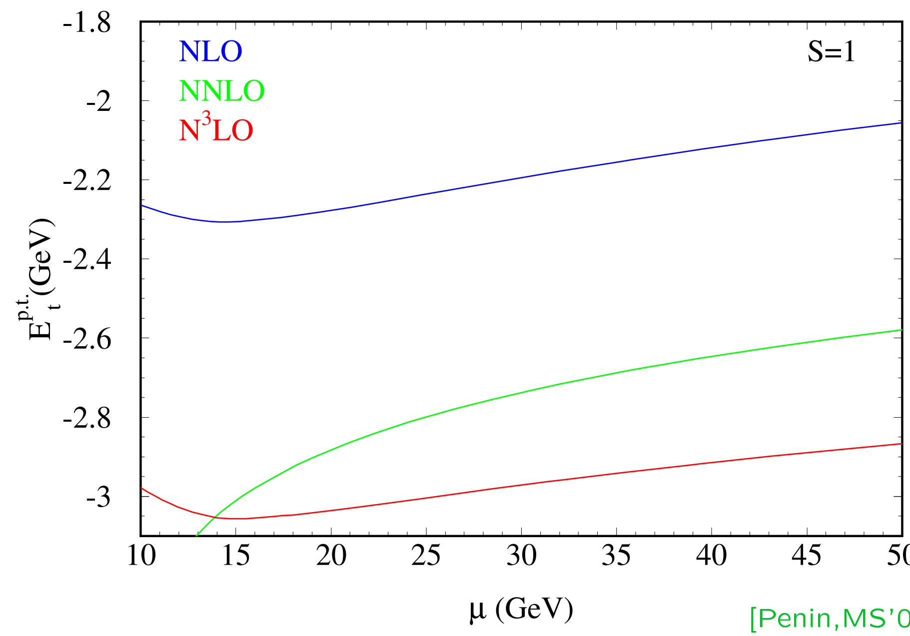
➪ $E_{\text{res}} = (1.9833 \pm 0.0009) \times m_t$

➪ measure peak position of cross section $\sigma(e^+e^- \to t\bar{t})$

➪ determine m_t with an error of 80 MeV

- convergence for the pole quark mass observed
- renormalon does not (yet) dominate
- fixed order pole quark mass
- $\overline{\text{MS}}$ mass: convergence slightly worse





[Penin,MS'02]

III.B. Peak for $\sigma(e^+e^- \to t\bar{t})$

$$R_{\rm res}(e^+e^- \to t\bar{t}) = \frac{6\pi N_c Q_t^2}{m_t^2 \Gamma_t} c_v^2 |\psi_1(0)|^2$$

c_v: hard matching coefficient

[Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]

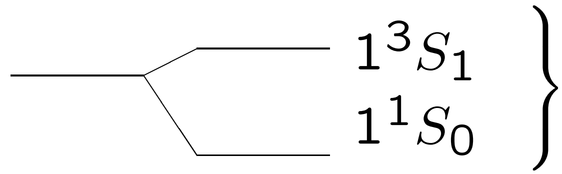
$$\frac{R_{\rm res}(e^+e^- \to t\bar{t})}{R_{\rm res}^{\rm LO}(e^+e^- \to t\bar{t})} = 1 - 0.244_{\rm NLO} + 0.438_{\rm NNLO} - 0.171_{\rm N^3LO;no\ const.}$$

- nice μ dependence
- first sign of convergence

III.C. Prediction of $M(\eta_b)$

Hyperfine splitting

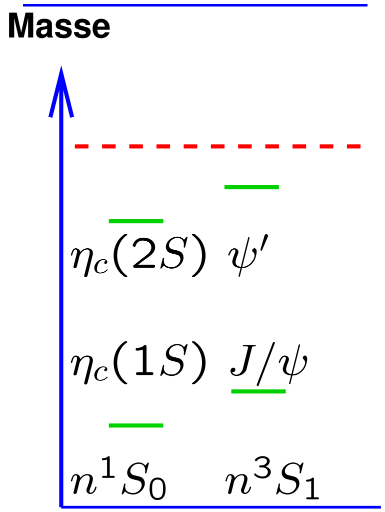
H atom: $n = 1$



⇨ „21cm-line“

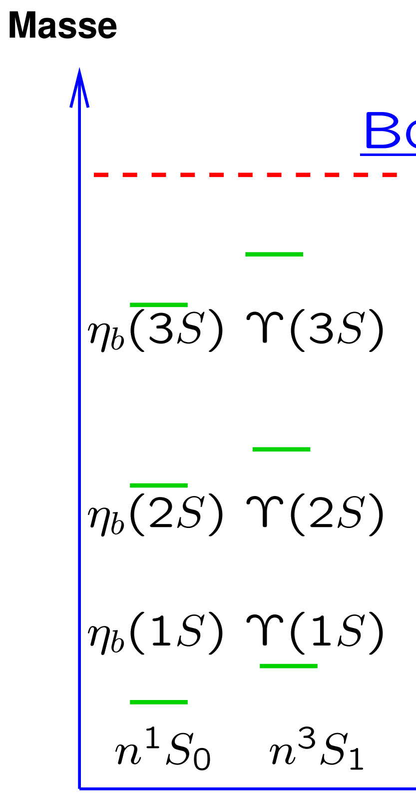
Quarkonium:

Charmonium



$M(\eta_c(1S)) = 2979.2(1.3)$ MeV
$M(J/\psi) = 3096.87(4)$ MeV

Bottomonium



$M(\Upsilon(1S)) = 9460.30(26)$ MeV
$\eta_b(1S)$ not yet observed

Nonrelativistic Renormalization Group (NRG)

$$\delta\mathcal{H}_{\text{spin}} = D^{(2)}_{S^2,s}\frac{4C_F\pi}{3m_q^2}\boldsymbol{S}^2, \qquad \boldsymbol{S} = \frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2},$$

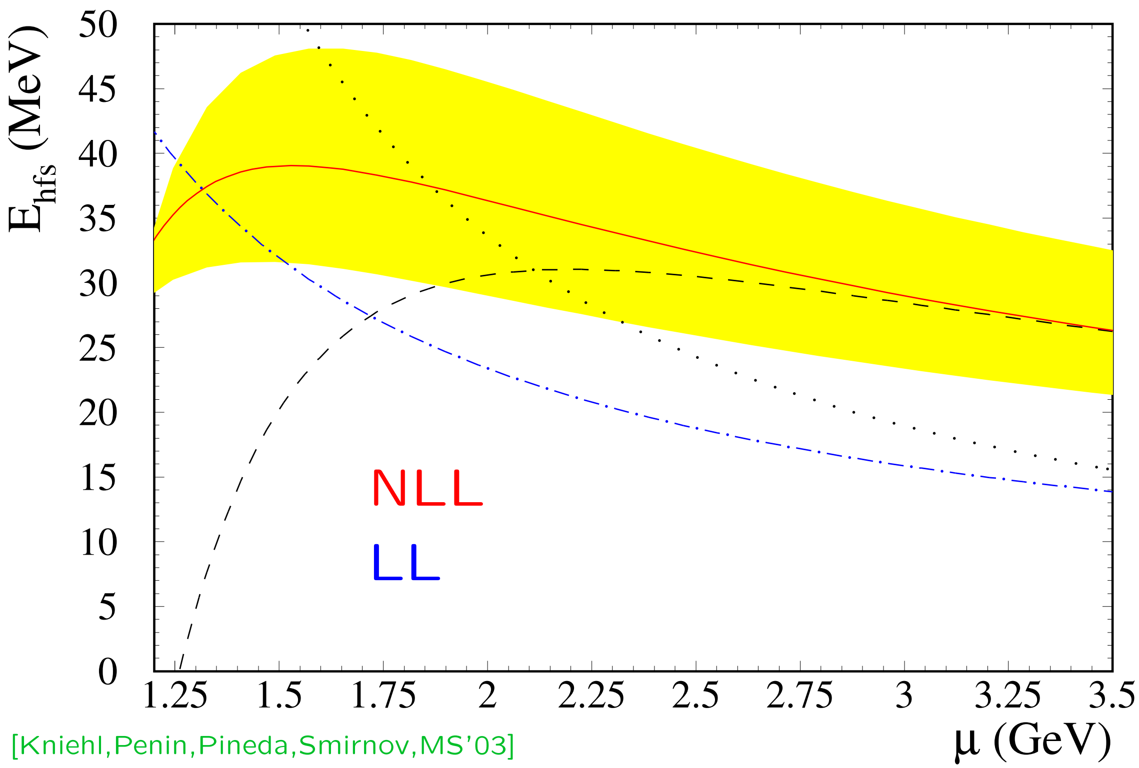
- consider soft, potential and ultra-soft running of $D^{(2)}_{S^2,s}$
  i.e.: consider corresponding UV divergences
- LL: only soft running: $\nu_s\frac{d}{d\nu_s}D^{(2)}_{S^2,s} = \alpha_s c_F^2\gamma_s$
- NLL: in addition potential and ultra-soft contributions
- matching performed at scale m_q
- analytical calculation

$$E_{\text{hfs}} = -\frac{64}{27}E_1^C\alpha_s^2\Big[1+\frac{\alpha_s}{\pi}\left(c_1^{(1)}\ln\alpha_s + c_0^{(1)}\right) + \left(\frac{\alpha_s}{\pi}\right)^2\left(c_2^{(2)}\ln^2\alpha_s + c_1^{(2)}\ln\alpha_s + c_0^{(2)}\right) + \left(\frac{\alpha_s}{\pi}\right)^3\left(c_3^{(3)}\ln^3\alpha_s + c_2^{(3)}\ln^2\alpha_s + c_1^{(3)}\ln\alpha_s + c_0^{(3)}\right) + \ldots\Big]$$

LL (blue terms) — NLL (red terms)

"Leading Log" — "Next-to-Leading Log"

E_{hfs} — $M(\eta_b)$



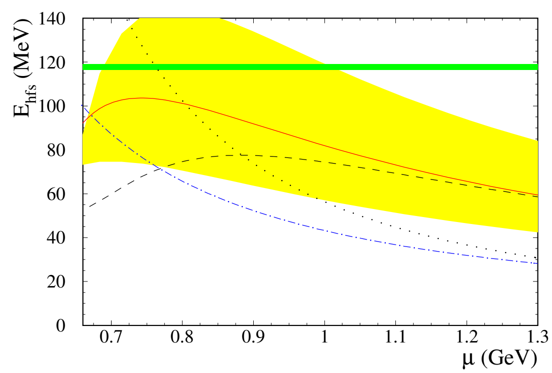
[Kniehl,Penin,Pineda,Smirnov,MS'03]

➪ $M(\eta_b) = 9420 \pm 10(\text{th})^{+9}_{-8}(\alpha_s)$ MeV

Good agreement with lattice results:

SESAM 33.4 MeV

CP-PACS 33.2 MeV



Charm system: $M(J/\psi) - M(\eta_c) \approx 104$ MeV

Experiment: $M(J/\psi) - M(\eta_c) = 117.7 \pm 1.3$ MeV

Conclusions — Next Steps

- E_1: N^{3}LO; good convergence; $\delta m_t = 80$ MeV
- $\delta\psi$: $\alpha_s^3 \ln \alpha_s$; first sign of convergence; constant = ?
- HFS: NLL, significantly reduced μ-dependence, prediction for $M(\eta_b)$

- a_3, $c_v^{(3)}$
- $\delta\psi_{\text{const}}^{(3)}$
- summation of logarithms for spin-independent terms
- . . .
