Mellin space analysis for NNLO Higgs and DY cross sections

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- Introduction
- σ^{tot} for Higgs and DY Production at LHC
- x-space analysis
- Mellin N-space and nested Harmonic sums
- Conclusions

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in colloboration with J. Blümlein

A quick summary

- Hadronic cross sections involve computations of splitting functions $P_{ab}(x, \mu_F, \mu_R)$ and partonic coefficient functions $\Delta_{ab}(x, \mu_F, \mu_R)$
- Computed in pQCD to possible highest order in $\alpha_s(\mu^2)$ in order
  a) to make perturbative expansion convergent
  b) to reduce UV scale uncertainity μ_R
  c) to reduce Factorisaton scale dependence μ_F
- $\Delta_{ab}(x, \mu_F, \mu_R)$ have rich mathematical structures in terms of Nielsen integrals and/or Spence functions. At NNLO level there are about 80 such higher functions.
- Can these quantities be simplified?
  a)To get compact expressions
  b) To get a fast numerical codes for phenomenology
  c) To get insight into the structure at higher orders
- Study of such integrals in Mellin space exhibits deeper understanding of such results
- We will show that the Higgs/DY coefficient functions
  a) exhibit a Mellin convolution structure, thanks to Mellin space analysis
  b) can be expressed in terms of very few Harmonic sums or Mellin convolutions of very few functions thanks to Algebraic identities relating various Harmonic Sums

Process

$$H_1(P_1) + H_2(P_2) \rightarrow \mathrm{B}(-p_5) +' X' \,,$$

where H_1 and H_2 denote the incoming hadrons and X represents an inclusive hadronic state.

$$\sigma_{\mathrm{tot}}(x, m^2) = \sum_{a,b=q,\bar{q},g} \int_x^1 \frac{dx_1}{x_1} \int_{x/x_1}^1 \frac{dx_2}{x_2} \, f_a(x_1, \mu^2) \, f_b(x_2, \mu^2) \Delta_{ab,\mathrm{B}} \left(\frac{x}{x_1 \, x_2}, \frac{m^2}{\mu^2} \right) ,$$

with $x = \frac{m^2}{S}$, $S = (P_1 + P_2)^2$, $p_5^2 = m^2$,

$$\sigma_{\mathrm{tot}}(x, m^2) = \sum_{a,b=q,\bar{q},g} \int_x^1 \frac{dy}{y} \, \Phi_{ab}(\frac{x}{y}, \mu^2) \, \Delta_{ab} \left(y, \frac{m^2}{\mu^2} \right) ,$$

where Φ_{ab} denotes the parton luminosity.

$$\Phi_{ab}(y, \mu^2) = \int_y^1 \frac{du}{u} \, f_a(u, \mu^2) \, f_b \left(\frac{y}{u}, \mu^2 \right) .$$

Coefficient Functions $\Delta_{ab}(x)$

- At 2-loop, one encounters 4 fold integrals!

$$\int dPS_3 \quad \approx \quad \int dP_{ij} \int dP_{kl} \int d\Omega_{n-2} \qquad\qquad P_{ij} = (p_i + p_j)^2$$

$$\int dP_{ij} \int dP_{kl} \qquad \rightarrow \int_0^1 dz \int_0^1 dy$$

- Angular integrations result in Hypergeometric functions

$$F_{12}\left(\frac{\varepsilon}{2}\quad ,\frac{\varepsilon}{2},\ 1+\frac{\varepsilon}{2}, f(y,z,x)\right) = 1 + \frac{4}{\varepsilon^2}\,\mathcal{L}i_2\,(f(y,z,x))$$

$$+\frac{8}{\varepsilon^3}\Big(S_{12}(f(y,z,x)) - \mathcal{L}i_3(f(y,z,x)\Big)$$

$$+\frac{16}{\varepsilon^4}\Big(S_{13}(f(y,z,x)) - S_{22}(f(y,z,x)) + \mathcal{L}i_4(f(y,z,x)\Big)$$

Nielsen Integral:

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{dz}{z}\log^{n-1}(z)\log^p(1-zx)$$

$$\mathcal{L}i_n(x) = S_{n-1,1}(x)$$

- $f(y,z,x)$ are simple functions of x, y, z,

Summary of functions at 2 loops(for Higgs)

[*Catani et al,Harlander,Kilgore,Anastasiou,Melnikov, Smith,van Neerven,VR]*

Function	Coefficient
$\log^n(x)$ n=1,2,3	x^r, $\frac{1}{1+x}$, $\frac{1}{1-x}$
$\log^n(1-x)$ n=1,2,3	x^r, $\frac{1}{1-x}$
$\mathcal{L}i_3(1-x)$	x^r, $\frac{1}{1+x}$, $\frac{1}{1-x}$
$\mathcal{L}i_3(-x)$	x^r, $\frac{1}{1+x}$
$S_{12}(1-x)$	x^r, $\frac{1}{1+x}$, $\frac{1}{1-x}$
$S_{12}(-x)$	x^r, $\frac{1}{1+x}$
$\mathcal{L}i_3\left(-\frac{1-x}{1+x}\right) - \mathcal{L}i_3\left(-\frac{1-x}{1+x}\right)$	x^r, $\frac{1}{1+x}$, $\frac{1}{1-x}$
$\mathcal{L}i_2(1-x)\log(x)$	x^r, $\frac{1}{1+x}$
$\mathcal{L}i_2(1-x)\log(1-x)$	x^r
$\mathcal{L}i_2(-x)\log(x)$	x^r, $\frac{1}{1+x}$
$\mathcal{L}i_2(-x)\log(1+x)$	x^r, $\frac{1}{1+x}$
$\mathcal{L}i_2(-x)\log(1-x)$	x^r, $\frac{1}{1+x}$
$\log^n(x)\log(1-x)$ n=0,1,2	x^r, $\frac{1}{1+x}$, $\frac{1}{1-x}$
$\log^n(x)\log(1+x)$ n=0,1,2	x^r, $\frac{1}{1+x}$
$\log^n(1-x)\log(x)$ n=0,1,2	x^r, $\frac{1}{1+x}$
$\log^n(1+x)\log(x)$ n=0,1,2	x^r, $\frac{1}{1+x}$
$\log(1-x)\log(1+x)\log(x)$	x^r, $\frac{1}{1+x}$
$\delta(1-x)$	1
1,ζ_2,ζ_3	x^r, $\frac{1}{1+x}$, $\frac{1}{1-x}$

- Convolute this complicated coefficient functions with appropriate parton fluxes to get the hadronic cross sections
- The Cross section:

$$\sigma(x,Q^2) = \int_x^1 \frac{dy}{y} \Phi_{ab}\left(\frac{x}{y},\mu^2\right) \Delta_{ab}(y,\mu^2)$$

- Alternate way of getting cross section is to compute the Mellin moment of the RHS and invert back to x space

$$\sigma_{Higgs}(x) = \int_{C-i\infty}^{C+i\infty} dN e^{-Nx} \mathcal{M}\Big[\Phi_{ab}\Big](N)\ \mathcal{M}\Big[\Delta_{ab}\Big](N)$$

- Mellin Moment:

$$\mathcal{M}\Big[f\Big](N) = \int_0^1 dx\ x^{N-1}\ f(x)$$

Mellin Moments

[Blümlein,Kurth]

- Nth Mellin Moment of a function $f(x)$ is defined as

$$\mathcal{M}\Big[f\Big](N) = \int_0^1 dx\ x^{N-1}\ f(x)$$

- Mellin Moment of a convolutions of $f_i(x_i)\ i = 1...n$ is the product of Mellin moments of $f_i(x_i)$, for example,$n = 2$

$$\mathcal{M}\left[\int_x^1 \frac{dx_1}{x_1} f_1(x_1)\ f_2\left(\frac{x}{x_1}\right)\right](N)\ =\ \mathcal{M}\Big[f_1\Big](N)\ \ \mathcal{M}\Big[f_2\Big](N)$$

- We can pose the following question: "Wheather the partonic cross sections factorise into Mellin convolution, hence simpler structures"
- Consider the Nth moment of Higgs total cross section

$$\mathcal{M}\Big[\sigma_{Higgs}\Big](N) = \sum_{a=q,\bar{q},g} \mathcal{M}\Big[f_a\Big](N)\ \ \mathcal{M}\Big[f_b\Big](N)\ \ \mathcal{M}\Big[\hat{\sigma}^{ab}\Big](N)$$

- Computation of $\mathcal{M}[\hat{\sigma}^{ab}](N)$ is possible with the available $\hat{\sigma}^{ab}$
- Involves the computation of Mellin moments of various Nielsen integrals
- Mellin moments of various Nielsen integrals result in Harmonic sums

Mellin Moments (2 loops)

[Blümlein,VR]

Function	Coefficients	Moments
$\log^n(x)$ n=1,2,3	$x^r, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_n(N)$
$\log^n(1-x)$ n=1,2,3	$x^r,, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_{-1,1}(N)$
$\mathcal{L}i_3(1-x)$	$x^r, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_{-1,1,2}(N)$
$\mathcal{L}i_3(-x)$	$x^r, \quad \frac{1}{1+x}$	$S_{3,-1}(N)$
$S_{12}(1-x)$	$x^r, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_{-1,3}(N)$
$S_{12}(-x)$	$x^r, \quad \frac{1}{1+x}$	$S_{2,1,-1}(N)$
$\mathcal{L}i_3\left(-\frac{1-x}{1+x}\right) - \mathcal{L}i_3\left(-\frac{1-x}{1+x}\right)$	$x^r, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_{-1,2}(N)$
$\mathcal{L}i_2(1-x)\log^n(x)$ n=0,1	$x^r, \quad \frac{1}{1+x}$	$S_{3,1}(N)$
$\mathcal{L}i_2(1-x)\log^n(1-x)$n=0,1	x^r	$S_k(N)$
$\mathcal{L}i_2(-x)\log^n(x)$n=0,1	$x^r, \quad \frac{1}{1+x}$	$S_{1,-2}(N)$
$\mathcal{L}i_2(-x)\log(1+x)$	$x^r, \quad \frac{1}{1+x}$	$S_{2,-1}(N)$
$\mathcal{L}i_2(-x)\log(1-x)$	$x^r, \quad \frac{1}{1+x}$	$S_{-1,1}(N)$
$\log^n(x)\log(1-x)$ n=0,1,2	$x^r, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_k(N)$
$\log^n(x)\log(1+x)$ n=0,1,2	$x^r, \quad \frac{1}{1+x}$	$S_{1,-1}(N)$
$\log^n(1-x)\log(x)$ n=0,1,2	$x^r, \quad \frac{1}{1+x}$	$S_k(N)$
$\log^n(1+x)\log(x)$ n=0,1,2	$x^r, \quad \frac{1}{1+x}$	$S_{1,1,-2}(N)$
$\log(1-x)\log(1+x)\log(x)$	$x^r, \quad \frac{1}{1+x}$	$S_{-1,2}(N)$
$\delta(1-x)$	1	1
1,ζ_2, ζ_3	$x^r, \quad \frac{1}{1+x}, \quad \frac{1}{1-x}$	$S_k(N)$

[*Blümlein,Kurth]*

- Mellin moments of Nielsen integrals can be expressed interms of linear combination of finite Harmonic sums

$$S_{k_1,\ldots,k_m}(N) = \sum_{n_1=1}^{N} \frac{[sign(k_1)]^{n_1}}{n_1^{|k_1|}} \sum_{n_2=1}^{n_1} \frac{[sign(k_2)]^{n_2}}{n_2^{|k_2|}} \cdots$$
$$\sum_{n_m=1}^{n_{m-1}} \frac{[sign(k_m)]^{n_m}}{n_m^{|k_m|}} \cdots \qquad N \epsilon \mathrm{N}, \forall, k_l \neq 0$$

- Upto two loop level harmonic sums up to level,we have

$$\sum_{j=1}^{m} |k_j| = 4$$

- Single harmonic sum

$$S_{\pm}k(N) = \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \cdots \int_0^{x_{k-1}} \frac{(\pm x_k)^N - 1}{x_k \mp 1}$$
$$= \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 dx \log^{k-1}(x) \frac{(\pm x)^N - 1}{x \mp 1}$$

[*Blümlein,Kurth*]

- Higher harmonic sums $S_{k_1,k_2\ldots k_i}$ can be obtained using $S_k(N), S_{-k}(N)$ and

$$\sum_{k=1}^{n} \frac{(\pm x)^k}{k^l} = \frac{(-1)^{l-1}}{(l-1)!}\int_0^x dz \log^{l-1}(z)\frac{(\pm z)^n - 1}{z \mp 1}$$

- Finite harmonic sums are connected by various algebraic relations
- Full or partial permutation of the indices and the order of the summation gives various relations among finite harmonic sums
- Using Euler's identity

$$S_{m,n} + S_{n,m} = S_m S_n + S_{sign\{m\}sign\{n\}|m|+|n|} = S_m S_n + S_{m\wedge n}$$

- Few cases(Two fold and level upto four)

$$\begin{aligned} S_{1,-1} + S_{-1,1} &= S_1 S_{-1} + S_{-2} \\ S_{-1,-2} + S_{-2,-1} &= S_{-1} S_{-2} + S_3 \\ S_{-1,-3} + S_{-3,-1} &= S_{-1} S_{-3} + S_4 \end{aligned}$$

[Blümlein,Kurth]

- 3-fold Harmonic sum

$$\sum_{perm l,m,n} S_{l,m,n} = S_l S_m S_n + \sum_{invperm l,m,n} S_l S_{m\wedge n} + 2S_{l\wedge m\wedge n}$$

- Few cases(3-fold and level-4)

$$\begin{aligned}
S_{1,2,1} &= -2S_{2,1,1} + S_{3,1} + S_1 S_{2,1} + S_{2,2} \\
S_{1,1,2} &= S_{2,1,1} + \frac{1}{2}\Big(S_1(S_{1,2} - S_{2,1}) + S_{1,3} - S_{3,1}\Big) \\
S_{1,-2,1} &= -2S_{-2,1,1} + S_{-3,1} + S_1 S_{-2,1} + S_{-2,2} \\
S_{1,1,-2} &= S_{-2,1,1} + S_{-2}S_2 - S_{-2,2} - S_{-2}S_{1,1} \\
&\quad + S_1 S_{1,-2} + S_{1,-3} - S_1 S_{-3}
\end{aligned}$$

- Many complicated harmonic sums cancel among themselves leaving the Mellin moment of the coefficients functions with very few simple sums
- Sums such as $S_{1,-1,2}$ $S_{-1,-1,-2}$, $S_{1,-1}$ and permutations which appear in the intermediate stages of the computation disappear at the end, thanks to various algebraic relations

Final sums

[Blümlein,VR]

- 1-fold Harmonic sum

$$S_{-k}(N) \rightarrow \mathcal{M}\left[\frac{log^{k-1}(x)}{1+x}\right](N) \qquad n = 1,2,3,4$$

$$S_{k}(N) \rightarrow \mathcal{M}\left[\frac{log^{k-1}(x)}{1-x}\right](N) \qquad n = 1,2,3,4$$

- 2-fold Harmonic sum

$$S_{-3,1}(N) \rightarrow \mathcal{M}\left[\frac{\mathcal{L}i_3(x)}{1+x}\right](N), \qquad S_{-2,1}(N) \rightarrow \mathcal{M}\left[\frac{\mathcal{L}i_2(x)}{1+x}\right](N),$$

$$S_{-2,2}(N) \rightarrow \mathcal{M}\left[\frac{1}{1+x}\Big(2\mathcal{L}i_3(x) - \log(x)(\mathcal{L}i_2(x)+\zeta_2)\Big)\right](N), \quad \leftarrow \frac{d}{dx}$$

$$S_{2,1}(N) \rightarrow \mathcal{M}\left[\left(\frac{\mathcal{L}i_2(x)}{1-x}\right)_+\right](N), \qquad S_{3,1}(N) \rightarrow \mathcal{M}\left[\frac{\mathcal{L}i_2(x)\log(x)}{1-x}\right](N) \quad \leftarrow$$

- 3-fold Harmonic sum

$$S_{-2,1,1}(N) \rightarrow \mathcal{M}\left[\frac{S_{12}(x)}{1+x}\right](N), \qquad S_{2,1,1}(N) \rightarrow \mathcal{M}\left[\left(\frac{S_{12}(x)}{1-x}\right)_+\right](N)$$

Final

[*Blümlein,VR]*

- At the level of Mellin moment of the coefficient functions there is a delecate cancellation of complicated harmonic sums due to various algebraic identities
- Mellin moment of the coefficient function is a linear combination of few harmonic sums

$$\mathcal{M}\left[\hat{\sigma}\right](N) = \sum_{\{i,\{jk\},\{lmn\},\{pqrs\}\}} \mathcal{C}_{i,jk,lmn,pqrs} S_i\, S_{jk}\, S_{lmn}\, S_{pqrs}$$

$$= \sum_{i,j,\ldots} \mathcal{M}\left[g_i\right](N)\, \mathcal{M}\left[g_j\right] \cdots$$

- The functions $g_i(x)$ at 2 loop level

$$x^r, \qquad \delta(1-x),$$

$$\frac{\log^n(x)}{1-x}, \qquad \frac{\log^n(x)}{1+x}, \qquad n = 0,1,2,3$$

$$\frac{1}{1-x}\mathcal{L}i_2(x)\log^n(x), \qquad \frac{1}{1+x}\mathcal{L}i_2(x)\log^n(x), \quad n = 0,1$$

$$\frac{1}{1+x}\mathcal{L}i_3(x), \quad \frac{1}{1+x}S_{12}(x), \qquad \frac{1}{1-x}S_{12}(x)$$

The last step

- This implies the partonic cross sections factorise as Mellin convolutions of very few elementary functions

$$\hat{\sigma}(z) = \int \int \cdots \frac{dz_1}{z_1} \frac{dz_2}{z_2} \cdots g_1(z_1) g_2(z_2) \cdots$$

- This convolution structure is very similar to Hadronic cross section factorising into Mellin convolutions of partonic cross sections and parton distribution functions
- The last step is to invert back to x-space by inverse Mellin transformation

$$\sigma_{Higgs}(x) = \int_{C-i\infty}^{C+i\infty} dN e^{-Nx} \mathcal{M}\left[f_{a/P_i}\right](N) \mathcal{M}\left[f_{b/P_j}\right](N) \mathcal{M}\left[\hat{\sigma}^{ab}\right](N)$$

- Computation of the cross section using Mellin moments of few functions improves the computational speed. It takes only few milli-seconds compared to x-space analysis which take around 10 minutes per one point

Conclusions

- We have discussed the framework of the computation of NNLO cross sections for hard processes in perturbative QCD
- Most of the integrals can be expressed in terms of Nielsen integrals with simple arguements of the scaling variable. The number of such integrals is large
- Mellin moment of these coefficient functions exhibit a simple form with very few Harmonic sums thanks to various symmetric identities and algebraic relations.
- Hence the coefficient functions have Mellin convolution structure with very few basic functions
- We observe that only five sums appear at the level of 2 loop coefficient functions for the Higgs and DY production processes