Theory of Parton Distributions

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1. **The Roadmap to Precision**
2. **The Three-loop Splitting Functions***
3. **The Summary**

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The Roadmap to Precision

Structure of the proton

- Structure functions F_2, F_3, F_L in deep-inelastic scattering
  - scaling violations $\longrightarrow$ precision test of perturbative QCD

- Parton distributions
  - gluon distribution at small x, quark valence and sea distribution
  - important input for hard scattering reactions at hadron colliders
    $\longrightarrow$ precise parton luminosity at LHC for Higgs or SUSY searches

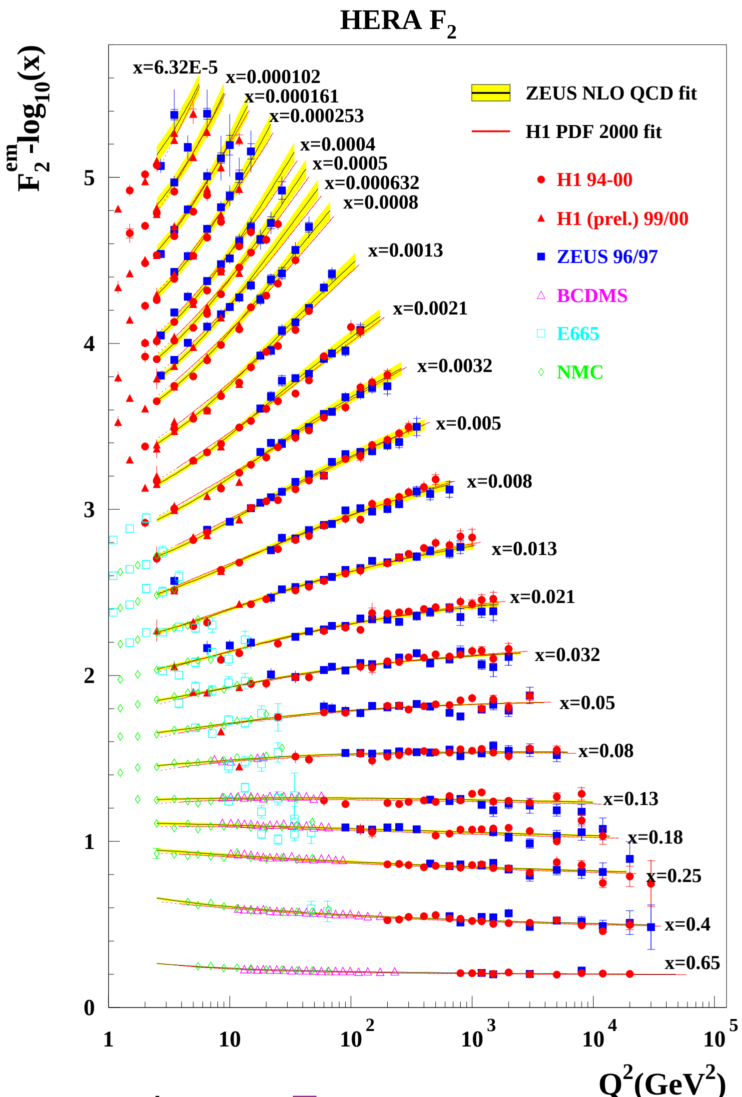
How well do we know (un)-polarized parton distributions ?

α_s from DIS

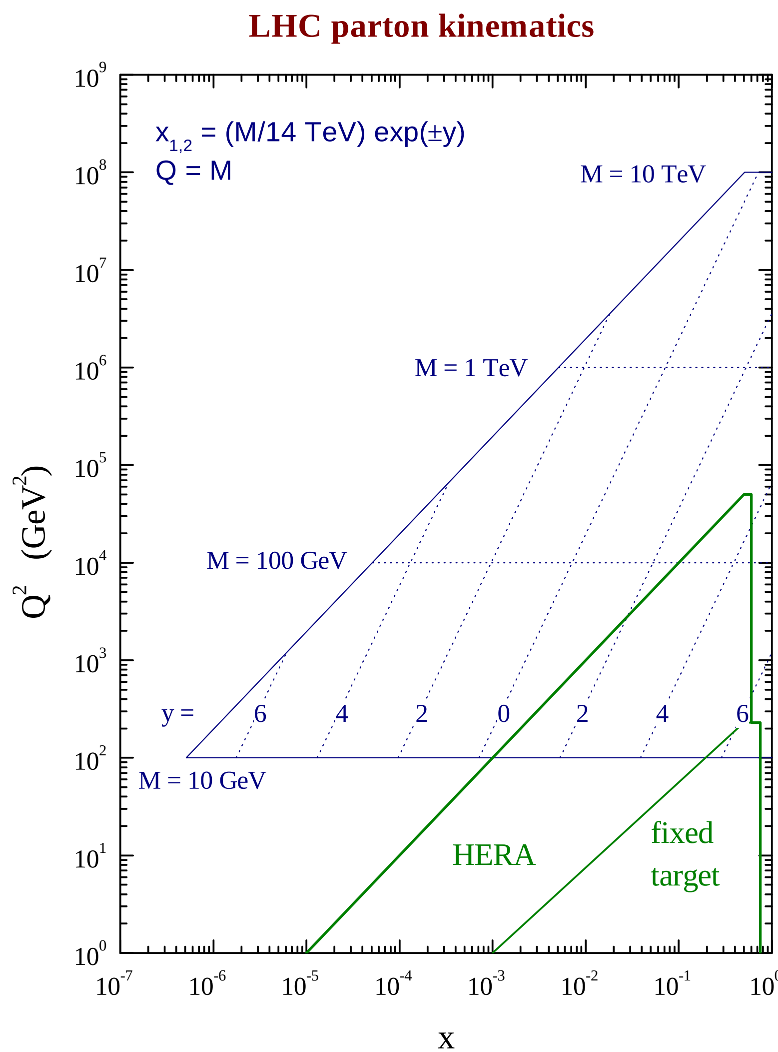
- Fundamental parameter of Standard Model
  - determination in inclusive DIS independent of hadronic final state $\longrightarrow$ ideal case

How well do we know α_s ?

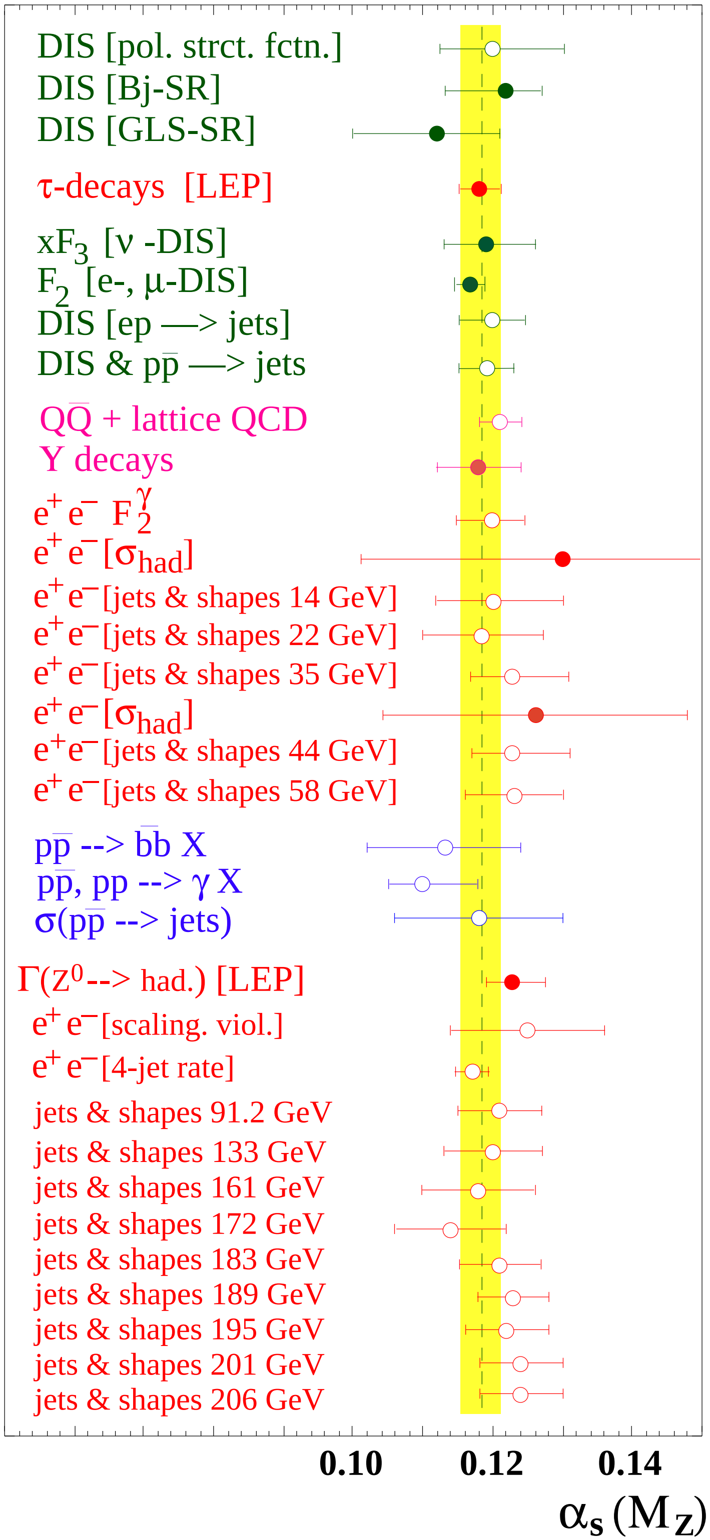
PDFs from HERA to LHC



– HERA data on F_2



– LHC parton luminosity



Recent determinations of α_s

Bethke `hep-ex/0211012`

– **NLO** QCD analysis of HERA data for $F_2(x, Q^2)$
H1 coll. `hep-ph/0012053`

$$\alpha_s(M_Z^2) = 0.115 \pm 0.002(\text{exp}) \pm 0.005(\text{theo})$$

Future

– **NNLO** QCD analysis of HERA data for $F_2(x, Q^2)$ in 2006

$$\alpha_s(M_Z^2) = x \pm 0.001(\text{exp}) \pm 0.001(\text{theo})$$

PDF uncertainties

- QCD analyses require many choices, should be reflected in PDF uncertainties $\longrightarrow$ Böttcher

- Treatment of experimental uncertainties $\longrightarrow$ talks in parallel session of WG1

- Allowed functional form of PDF $xf\left(x,Q_0^2\right) = Ax^b(1-x)^c(1+dx+\ldots)$

- Treatment of heavy quarks
  - Fixed flavor number scheme
    $\longrightarrow$ n_f light flavors + heavy quark of mass m at low scales
    $\longrightarrow$ n_f+1 light flavors at high scales
  - Variable flavor number schemes
    $\longrightarrow$ matching of two distinct theories through NNLO
    Aivazis, Collins, Olness, Tung '94 ; Thorne, Roberts '98 ; Chuvakin, Smith, van Neerven '00

- Scale dependence $\longrightarrow$ variation of renormalization / factorization scale
  - gluons : stat. $\oplus$ syst. $\simeq$ input $\simeq$ scale error ; quarks : scale error already dominates
  - **NNLO** improvement of theory needed $\longrightarrow$ three-loop splitting functions

- . . .

The Three-loop Splitting Functions

Evolution of parton distribution

– Non-singlet and singlet distributions $q^{\pm}$, q^{v} and q_{s}, g

$$q^{\pm}_{\mathrm{ns},ik} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$$ flavour asymmetries

$$q^{\mathrm{v}}_{\mathrm{ns}} = \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$ total valence distribution

$$q_{\mathrm{s}} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r)$$ flavour singlet distribution, $f_{\mathrm{s}} = \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$

– Splitting function combinations

$$P^{\pm}_{\mathrm{ns}}, \qquad P^{\mathrm{v}}_{\mathrm{ns}} = P^{-}_{\mathrm{ns}} + P^{\mathrm{s}}_{\mathrm{ns}}$$ non-singlet

$$P_{\mathrm{s}} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix}, \qquad P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$$ singlet

– Evolution equations : $2n_f - 1$ scalar non-singlet equations and 2×2 singlet equations

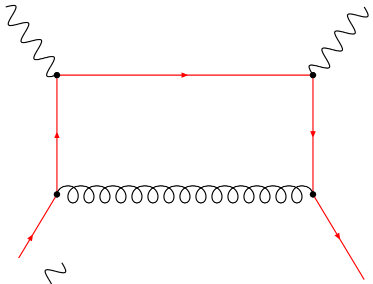
$$\frac{d}{d \ln \mu_f^2} f(x, \mu_f^2) = \left[P(\alpha_s(\mu_f^2)) \otimes f(\mu_f^2) \right](x)$$

The calculation (in a nut shell)

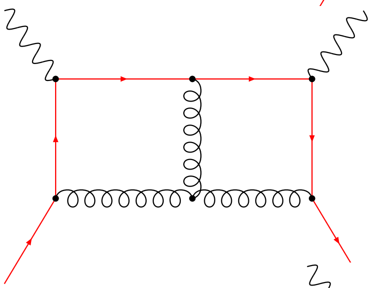
- Calculate anomalous dimensions (Mellin moments of splitting functions)
  $\longrightarrow$ divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\varepsilon$

$$\gamma_{\rm ij}^{(n)}(N) \;=\; -\int_0^1 dx\, x^{N-1} P_{\rm ij}^{(n)}(x)$$

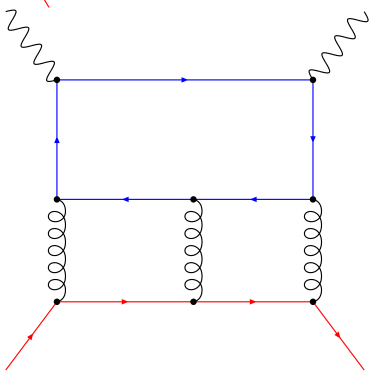
- **One-loop** Feynman diagrams
  $\longrightarrow$ in total 18 for $\gamma_{\rm ij}^{(0)}$ / $P_{\rm ij}^{(0)}$
  (pencil + paper)



- **Two-loop** Feynman diagrams
  $\longrightarrow$ in total 350 for $\gamma_{\rm ij}^{(1)}$ / $P_{\rm ij}^{(1)}$
  (simple computer algebra)



- **Three-loop** Feynman diagrams
  $\longrightarrow$ in total 9607 for $\gamma_{\rm ij}^{(2)}$ / $P_{\rm ij}^{(2)}$
  (cutting edge technology $\longrightarrow$ computer algebra system FORM Vermaseren '89-'04)



Mathematical intermezzo

- Mellin N-space : **harmonic sums** $S_{m_1,\dots,m_k}(N)$

  Gonzalez-Arroyo, Lopez, Ynduráin '79 ; Vermaseren '98 ; Blümlein, Kurth '98

  - recursive definition $S_{m_1,\dots,m_k}(N) = \sum_{i=1}^{N} \frac{1}{i^{m_1}} S_{m_2,\dots,m_k}(i)$
  - algebra of multiplication $S_j(N) S_k(N) \longrightarrow S_{\{j,k\}}(N)$

- Bjorken x-space : **harmonic polylogarithms** $H_{m_1,\dots,m_k}(x)$

  Goncharov '98 ; Borwein, Bradley, Broadhurst, Lisonek '99 ; Remiddi, Vermaseren '99

  - Basic functions of lowest weight

$$H_0(x) = \ln x\,, \quad H_1(x) = -\ln(1-x)\,, \quad H_{-1}(x) = \ln(1+x)$$

  - Higher functions defined by recursion

$$H_{m_1,\dots,m_w}(x) = \int_0^x dz\; f_{m_1}(z)\; H_{m_2,\dots,m_w}(z)$$

$$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$

  - Inverse Mellin transformation of harmonic sums $\longrightarrow$ harmonic polylogarithms in x space
  - Unique mapping :

$$H_{m_1,\dots,m_w}(x)/(1 \pm x) \longleftrightarrow S_{n_1,\dots,n_{w+1}}(N)$$

LO and NLO singlet splitting functions

$$
\begin{aligned}
P^{(0)}_{\rm ps}(x) &= 0\\
P^{(0)}_{\rm qg}(x) &= 2\,n_f\,p_{\rm qg}(x)\\
P^{(0)}_{\rm gq}(x) &= 2C_F\,p_{\rm gq}(x)\\
P^{(0)}_{\rm gg}(x) &= C_A\Big(4p_{\rm gg}(x)+\frac{11}{3}\delta(1-x)\Big)-\frac{2}{3}n_f\,\delta(1-x)
\end{aligned}
$$

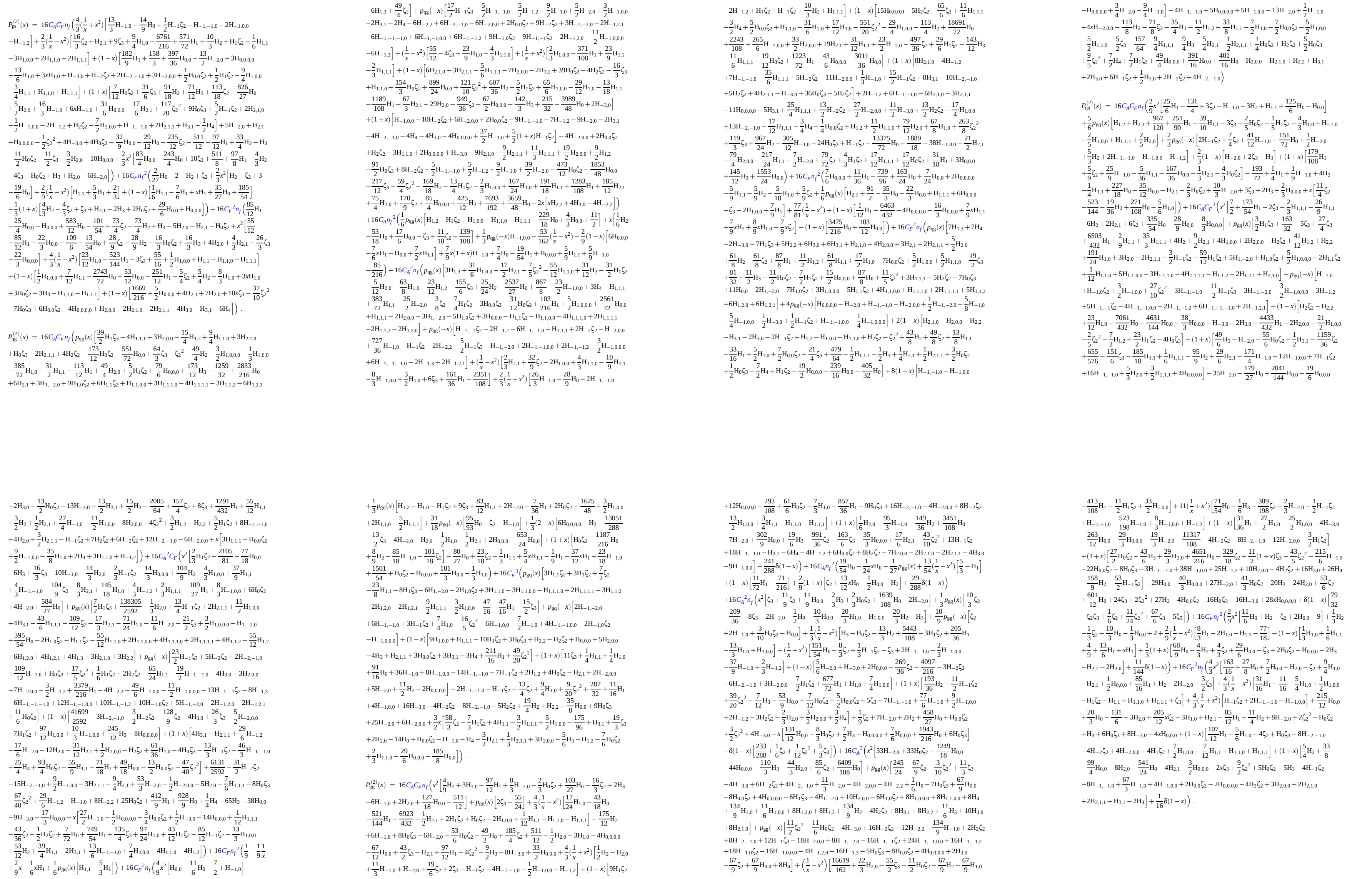
$$
P^{(1)}_{\rm ps}(x) = 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x}-2+6x-4{\rm H}_0+x^2\Big[\frac{8}{3}{\rm H}_0-\frac{56}{9}\Big]+(1+x)\Big[5{\rm H}_0-2{\rm H}_{0,0}\Big]\Big)
$$

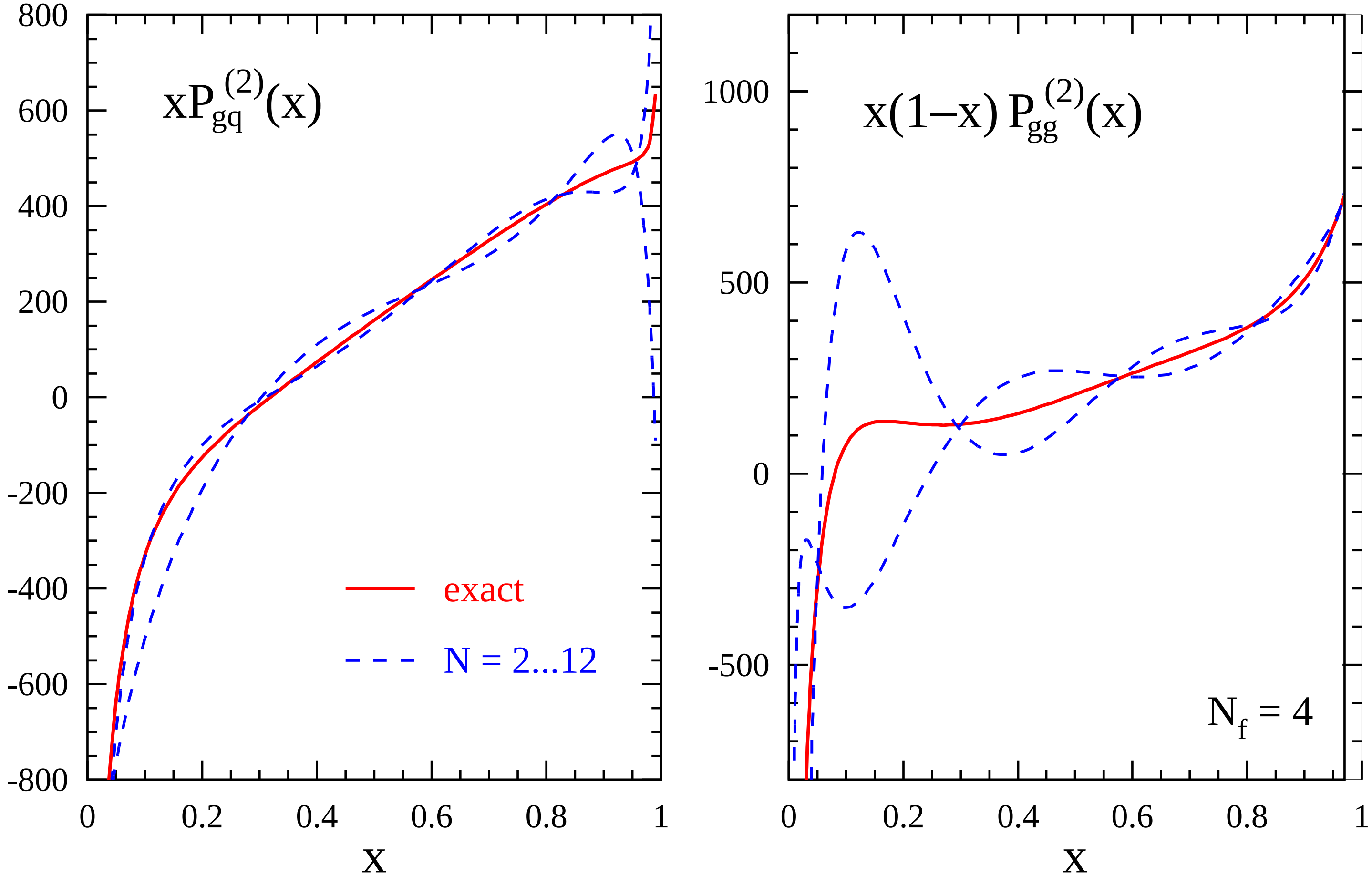
$$
\begin{aligned}
P^{(1)}_{\rm qg}(x) &= 4C_An_f\Big(\frac{20}{9}\frac{1}{x}-2+25x-2p_{\rm qg}(-x){\rm H}_{-1,0}-2p_{\rm qg}(x){\rm H}_{1,1}+x^2\Big[\frac{44}{3}{\rm H}_0-\frac{218}{9}\Big]\\
&+4(1-x)\Big[{\rm H}_{0,0}-2{\rm H}_0+x{\rm H}_1\Big]-4\zeta_2x-6{\rm H}_{0,0}+9{\rm H}_0\Big)+4C_Fn_f\Big(2p_{\rm qg}(x)\Big[{\rm H}_{1,0}+{\rm H}_{1,1}+{\rm H}_2\\
&-\zeta_2\Big]+4x^2\Big[{\rm H}_0+{\rm H}_{0,0}+\frac{5}{2}\Big]+2(1-x)\Big[{\rm H}_0+{\rm H}_{0,0}-2x{\rm H}_1+\frac{29}{4}\Big]-\frac{15}{2}-{\rm H}_{0,0}-\frac{1}{2}{\rm H}_0\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\rm gq}(x) &= 4C_AC_F\Big(\frac{1}{x}+2p_{\rm gq}(x)\Big[{\rm H}_{1,0}+{\rm H}_{1,1}+{\rm H}_2-\frac{11}{6}{\rm H}_1\Big]-x^2\Big[\frac{8}{3}{\rm H}_0-\frac{44}{9}\Big]+4\zeta_2-2\\
&-7{\rm H}_0+2{\rm H}_{0,0}-2{\rm H}_1x+(1+x)\Big[2{\rm H}_{0,0}-5{\rm H}_0+\frac{37}{9}\Big]-2p_{\rm gq}(-x){\rm H}_{-1,0}\Big)-4C_Fn_f\Big(\frac{2}{3}x\\
&-p_{\rm gq}(x)\Big[\frac{2}{3}{\rm H}_1-\frac{10}{9}\Big]\Big)+4C_F^{\,2}\Big(p_{\rm gq}(x)\Big[3{\rm H}_1-2{\rm H}_{1,1}\Big]+(1+x)\Big[{\rm H}_{0,0}-\frac{7}{2}+\frac{7}{2}{\rm H}_0\Big]-3{\rm H}_{0,0}\\
&+1-\frac{3}{2}{\rm H}_0+2{\rm H}_1x\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\rm gg}(x) &= 4C_An_f\Big(1-x-\frac{10}{9}p_{\rm gg}(x)-\frac{13}{9}\Big(\frac{1}{x}-x^2\Big)-\frac{2}{3}(1+x){\rm H}_0-\frac{2}{3}\delta(1-x)\Big)+4C_A^{\,2}\Big(27\\
&+(1+x)\Big[\frac{11}{3}{\rm H}_0+8{\rm H}_{0,0}-\frac{27}{2}\Big]+2p_{\rm gg}(-x)\Big[{\rm H}_{0,0}-2{\rm H}_{-1,0}-\zeta_2\Big]-\frac{67}{9}\Big(\frac{1}{x}-x^2\Big)-12{\rm H}_0\\
&-\frac{44}{3}x^2{\rm H}_0+2p_{\rm gg}(x)\Big[\frac{67}{18}-\zeta_2+{\rm H}_{0,0}+2{\rm H}_{1,0}+2{\rm H}_2\Big]+\delta(1-x)\Big[\frac{8}{3}+3\zeta_3\Big]\Big)+4C_Fn_f\Big(2{\rm H}_0\\
&+\frac{2}{3}\frac{1}{x}+\frac{10}{3}x^2-12+(1+x)\Big[4-5{\rm H}_0-2{\rm H}_{0,0}\Big]-\frac{1}{2}\delta(1-x)\Big)\ .
\end{aligned}
$$

NNLO singlet splitting functions





– Left : $P_{\rm gq}^{(2)}$ comparison of exact result and estimates from fixed moments

Larin, Nogueira, van Ritbergen, Vermaseren ‘97 ; Retey, Vermaseren ‘00 ; van Neerven, Vogt ‘00

– Right : same for $P_{\rm gg}^{(2)}$

Easy-to-use parametrization

- Combine exact limits for $x \longrightarrow 0$ and $x \longrightarrow 1$ with smooth fit for intermediate x
  $\longrightarrow$ fit quality better than one per mille
- Notation : end-point logarithms $L_0 = \ln(x)$, $L_1 = \ln(1-x)$, +-distribution $\mathcal{D}_0 = 1/(1-x)_+$

$$
\begin{aligned}
P^{(2)}_{\rm gg}(x) \;\cong\;\; & +2643.521\,\mathcal{D}_0 + 4425.894\,\delta(1-x) + 3589\,L_1 - 20852 + 3968\,x - 3363\,x^2 \\
& + 4848\,x^3 + L_0 L_1\,(7305 + 8757\,L_0) + 274.4\,L_0 - 7471\,L_0^2 + 72\,L_0^3 - 144\,L_0^4 \\
& + 14214\,x^{-1} + 2675.8\,x^{-1} L_0 \\
+ \;\; & n_f \Big(-412.172\,\mathcal{D}_0 - 528.723\,\delta(1-x) - 320\,L_1 - 350.2 + 755.7\,x - 713.8\,x^2 \\
& \quad + 559.3\,x^3 + L_0 L_1\,(26.15 - 808.7\,L_0) + 1541\,L_0 + 491.3\,L_0^2 + 832/9\,L_0^3 \\
& \quad + 512/27\,L_0^4 + 182.96\,x^{-1} + 157.27\,x^{-1} L_0 \Big) \\
+ \;\; & n_f^2 \Big(-16/9\,\mathcal{D}_0 + 6.4630\,\delta(1-x) - 13.878 + 153.4\,x - 187.7\,x^2 + 52.75\,x^3 \\
& \quad - L_0 L_1\,(115.6 - 85.25\,x + 63.23\,L_0) - 3.422\,L_0 + 9.680\,L_0^2 - 32/27\,L_0^3 \\
& \quad - 680/243\,x^{-1} \Big) \;.
\end{aligned}
$$

The large x-limit : $x \longrightarrow 1$

– Large x-limit for diagonal splitting functions $P^{(2)}_{\rm aa}$, $\rm a = q, g$

$$P^{(2)}_{\rm aa,\to 1}(x) = \frac{A_3^{\rm a}}{(1-x)_+} + B_3^{\rm a}\,\delta(1-x) + C_3^{\rm a}\ln(1-x) + O(1)$$

– Result for $A_3^{\rm a}$ is new $\longrightarrow$ important for **threshold resummation** in soft/collinear limit

$$\text{one-loop} \qquad A_1^{\rm q} = 4C_F$$

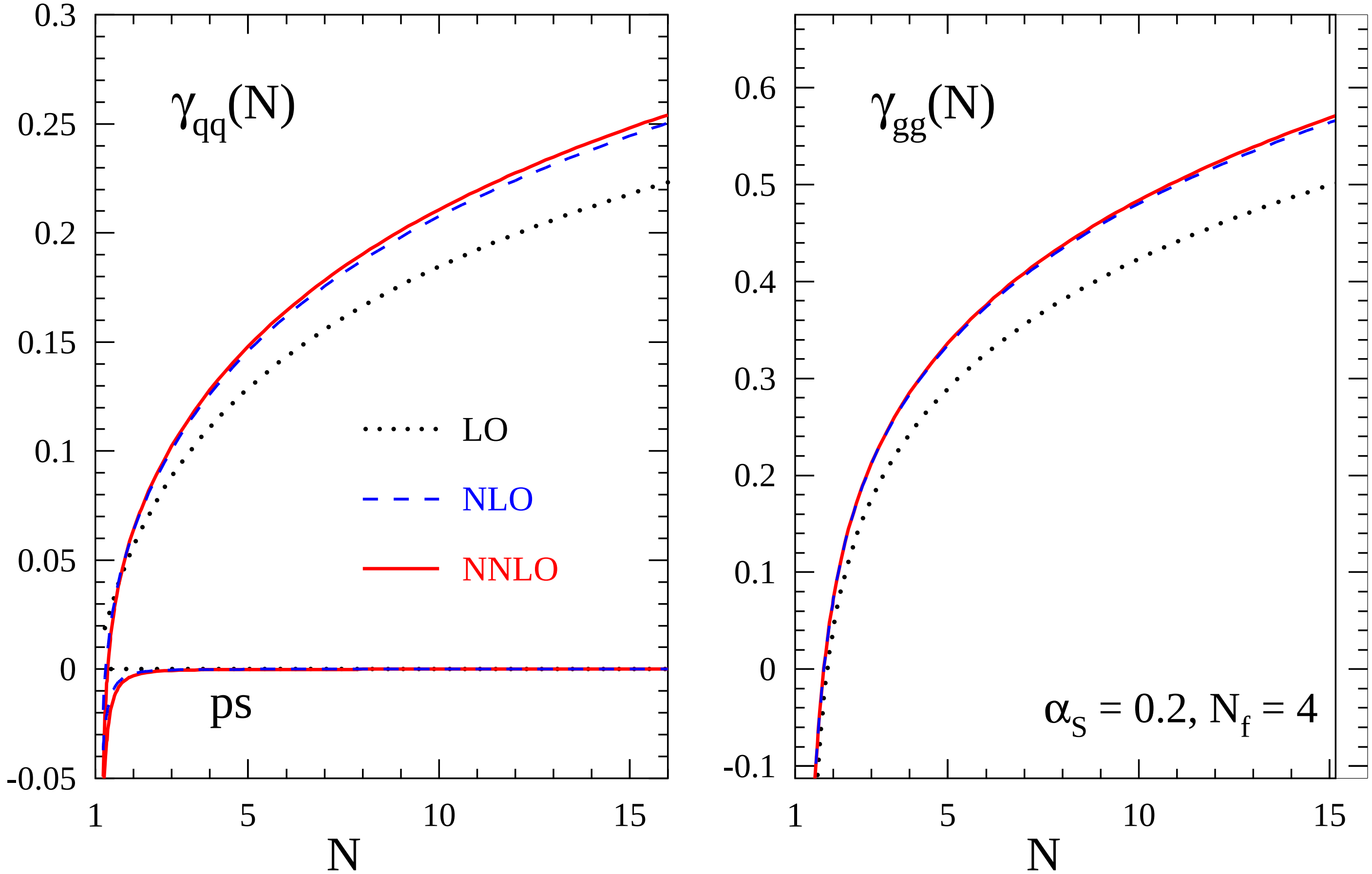
$$\text{two-loop} \qquad A_2^{\rm q} = 8C_FC_A\left(\frac{67}{18}-\zeta_2\right) - \frac{5}{9}C_Fn_f$$

$$\text{three-loop} \qquad A_3^{\rm q} = 16C_FC_A^2\left(\frac{245}{24}-\frac{67}{9}\zeta_2+\frac{11}{6}\zeta_3+\frac{11}{5}\zeta_2^{\,2}\right) + 16C_F^2n_f\left(-\frac{55}{24}+2\zeta_3\right)$$
$$+ 16C_FC_An_f\left(-\frac{209}{108}+\frac{10}{9}\zeta_2-\frac{7}{3}\zeta_3\right) + 16C_Fn_f^2\left(-\frac{1}{27}\right)$$

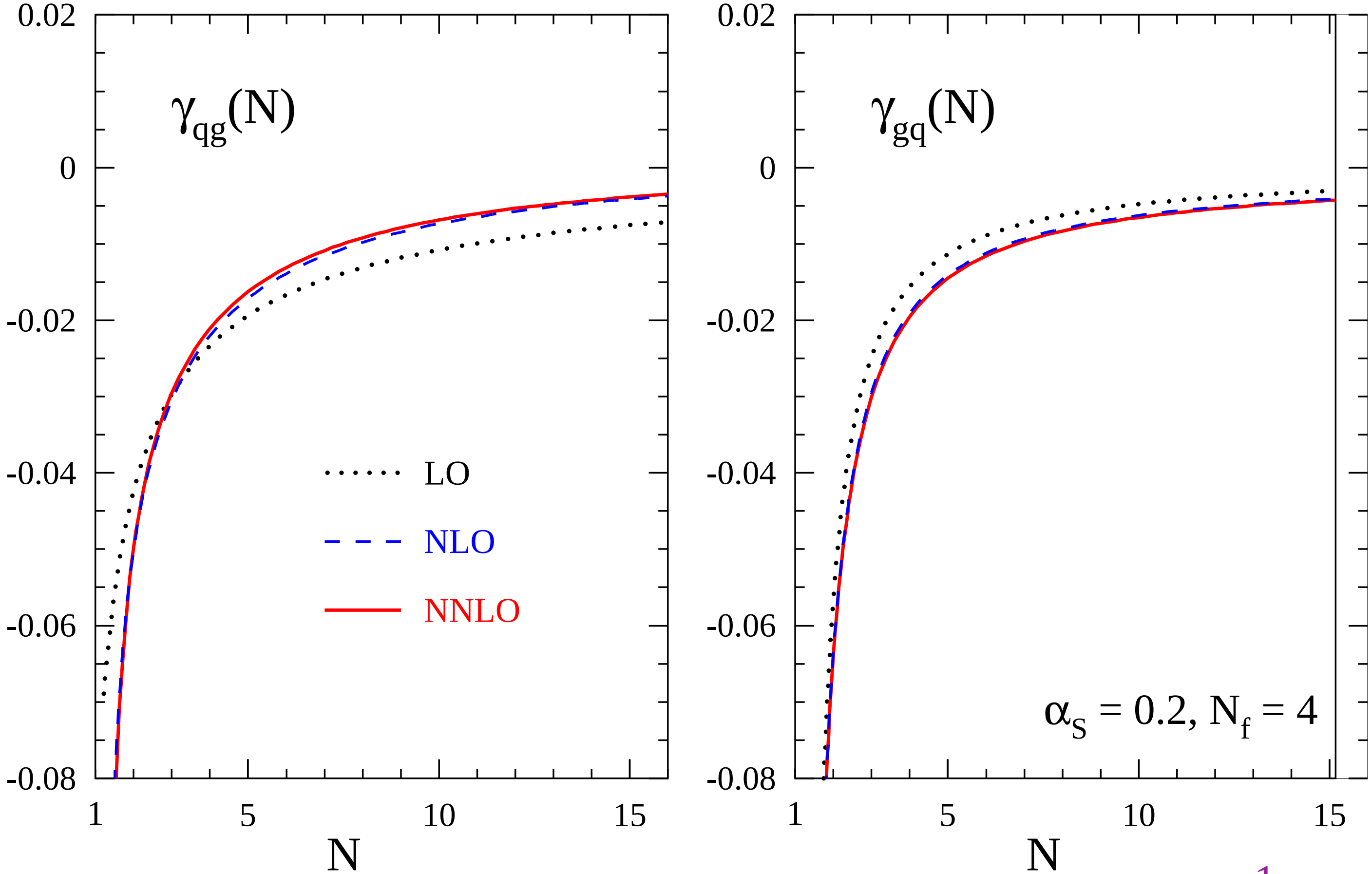
– Verify expected relation $A_3^{\rm g} = \frac{C_A}{C_F}A_3^{\rm q}$

– **Surprising relation** for subleading logarithms

$$C_1^{\rm a} = 0\,, \quad C_2^{\rm a} = (A_1^{\rm a})^2\,, \quad C_3^{\rm a} = 2A_1^{\rm a}A_2^{\rm a}$$



- For $N \longrightarrow \infty$, diagonal anomalous dimensions grow as $\ln N$
- Left : perturbative expansion of $\gamma_{\rm qq}(N)$ (and $\gamma_{\rm ps}(N)$ seperately) with $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$
- Right : same for $\gamma_{\rm qq}(N)$



- For $N \longrightarrow \infty$, off-diagonal anomalous dimensions vanish at n-loops as $\frac{1}{N} \ln^{2n-2} N$
- Left : perturbative expansion of $\gamma_{\rm qg}(N)$ with $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$
- Right : same for $\gamma_{\rm gq}(N)$

The small x-limit : $x \longrightarrow 0$

– General structure at small x

$$P^{(2)}_{\mathrm{aa},\to 0}(x) = = E_1^{\mathrm{ab}} \frac{\ln x}{x} + E_2^{\mathrm{ab}} \frac{1}{x} + O(\ln^4 x)$$

$$E_1^{\mathrm{qq}} = -\frac{896}{27} C_A C_F n_f$$

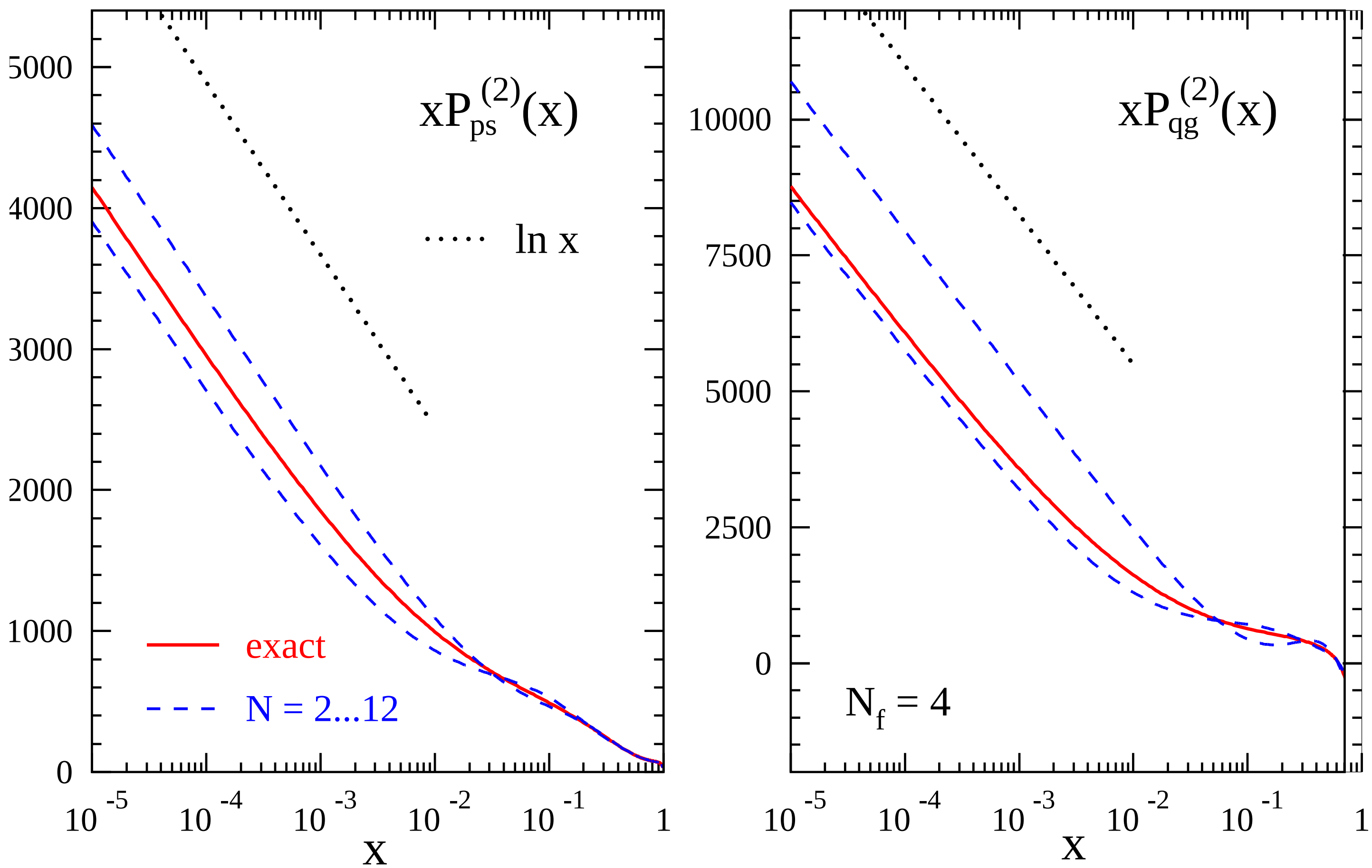
$$E_2^{\mathrm{qq}} = \left[-\frac{27044}{81} + \frac{512}{9}\zeta_2 + 96\zeta_3 \right] C_A C_F n_f + \left[\frac{220}{3} - 64\zeta_3 \right] C_F^2 n_f + \frac{64}{27} C_F n_f^2$$

$$E_1^{\mathrm{qg}} = -\frac{896}{27} C_A^2 n_f = \frac{C_A}{C_F} E_1^{\mathrm{qq}}$$

$$E_2^{\mathrm{qg}} = \left[-\frac{9404}{27} + \frac{512}{9}\zeta_2 + 96\zeta_3 \right] C_A^2 n_f + \left[\frac{220}{3} - 64\zeta_3 \right] C_A C_F n_f - \frac{424}{81} C_A n_f^2 + \frac{1232}{81} C_F n_f^2$$

– Coefficients E_1^{qq} and E_1^{qg} agree with prediction from the small-x resummation

Catani, Hautmann ‘94



– Exact result, estimates from fixed moments and leading small-x term

– Pure-singlet splitting function $P_{ps}^{(2)}$ (left) and $P_{qg}^{(2)}$ (right)

$$E_1^{gg} = \left[\frac{6320}{27} - \frac{176}{3}\zeta_2 - 32\zeta_3\right] C_A^3 + \left[\frac{1136}{27} - \frac{32}{3}\zeta_2\right] C_A^2 n_f - \left[\frac{1376}{27} - \frac{64}{3}\zeta_2\right] C_A C_F n_f$$

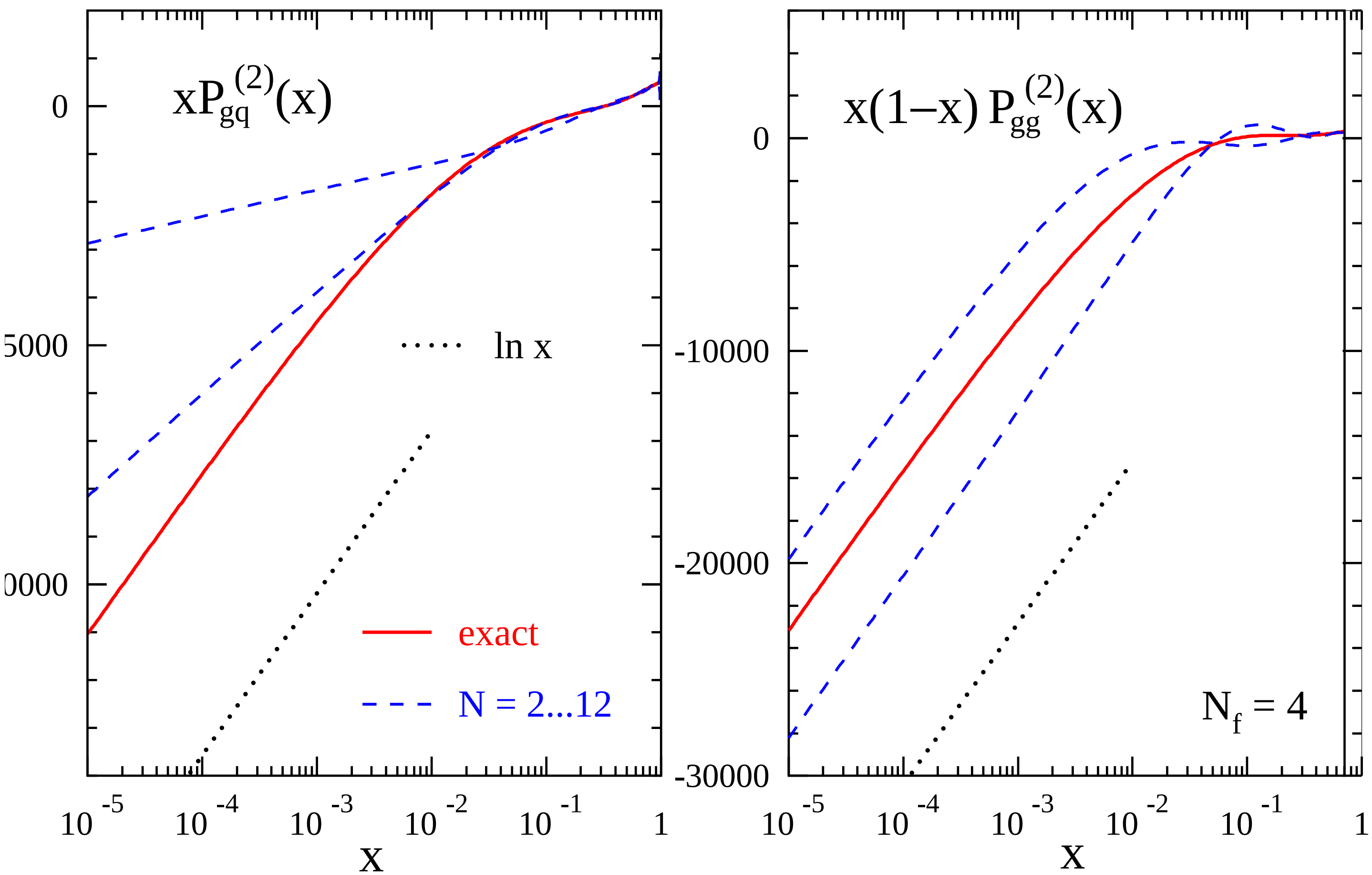
$$E_2^{gg} = \left[\frac{146182}{81} - \frac{3112}{9}\zeta_2 - \frac{1144}{3}\zeta_3 - \frac{464}{5}\zeta_2^2\right] C_A^3 + \left[\frac{19264}{81} - \frac{128}{3}\zeta_2 - \frac{176}{3}\zeta_3\right] C_A^2 n_f$$
$$- \left[\frac{30662}{81} - \frac{608}{9}\zeta_2 - \frac{224}{3}\zeta_3\right] C_A C_F n_f - \left[\frac{44}{3} - \frac{64}{3}\zeta_3\right] C_F^2 n_f + \frac{472}{81} C_A n_f^2 - \frac{1232}{81} C_F n_f^2$$

– No logarithm $\ln^2 x/x$ in $P_{gg}^{(2)}$ as predicted by leading logarithmic BFKL equation
  Kuraev, Lipatov, Fadin ‘77 ; Balitsky, L.N. Lipatov ‘78 ; Jaroszewicz ‘82

– Coefficient E_1^{gg} agrees with prediction from next-to-leading logarithmic BFKL equation
  Fadin, Lipatov ‘98

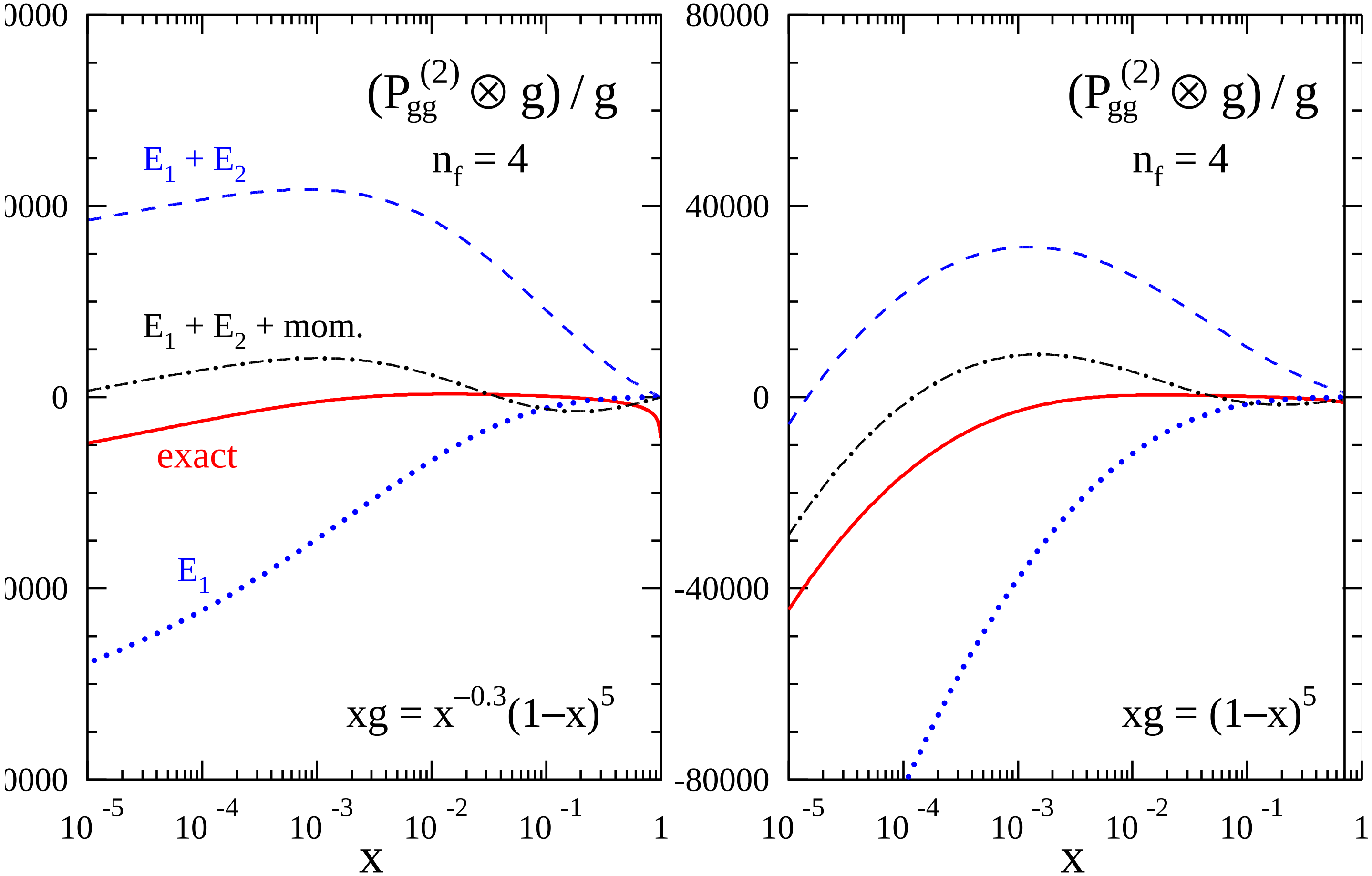
– New coefficients E_1^{gq}, E_2^{gq} with interesting relation

$$E_1^{gg} = \frac{C_A}{C_F} E_1^{gq} - \frac{8}{3} C_A n_f (C_A - 2C_F) \rightarrow \frac{C_A}{C_F} E_1^{gq} - \frac{8}{3} n_f \quad \text{for SU(N)}$$



– Exact result, estimates from fixed moments and leading small-x term

– Splitting function $P_{gq}^{(2)}$ (left) and $P_{gg}^{(2)}$ (right)



– Convolution of $P_{gg}^{(2)}$ with schematic ‘steep’ (left) and ‘flat’ gluon distributions

– Comparison of exact result for $P_{gg}^{(2)}$ with various approximations of small-x terms

Numerical implications for singlet distributions

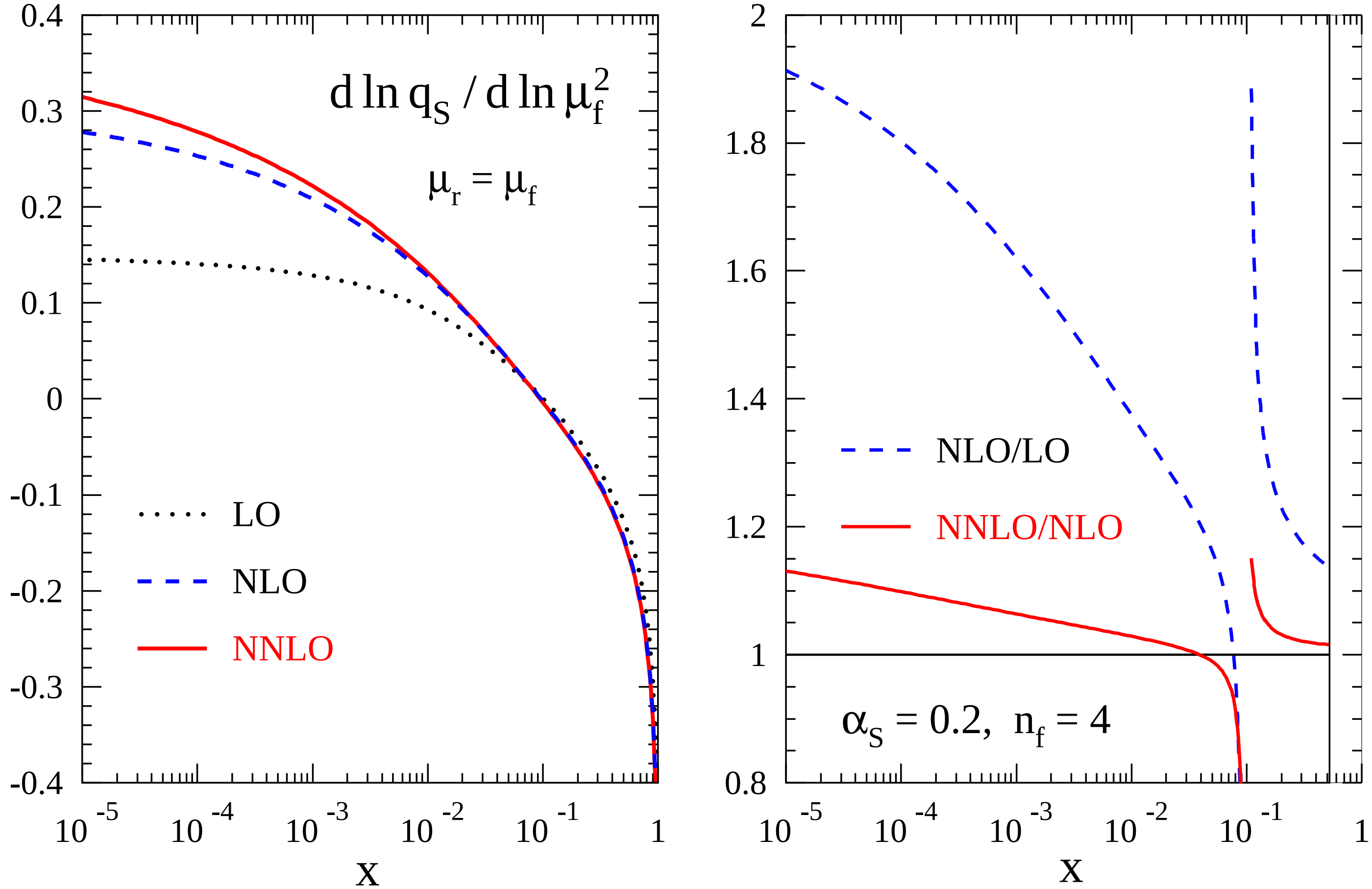
– Perturbative expansion of logarithmic scale derivative

$$\frac{d}{d\ln\mu_f^2} f(x,\mu_f^2) \quad = \quad \left[P(\alpha_s(\mu_f^2)) \otimes f(\mu_f^2)\right](x)$$

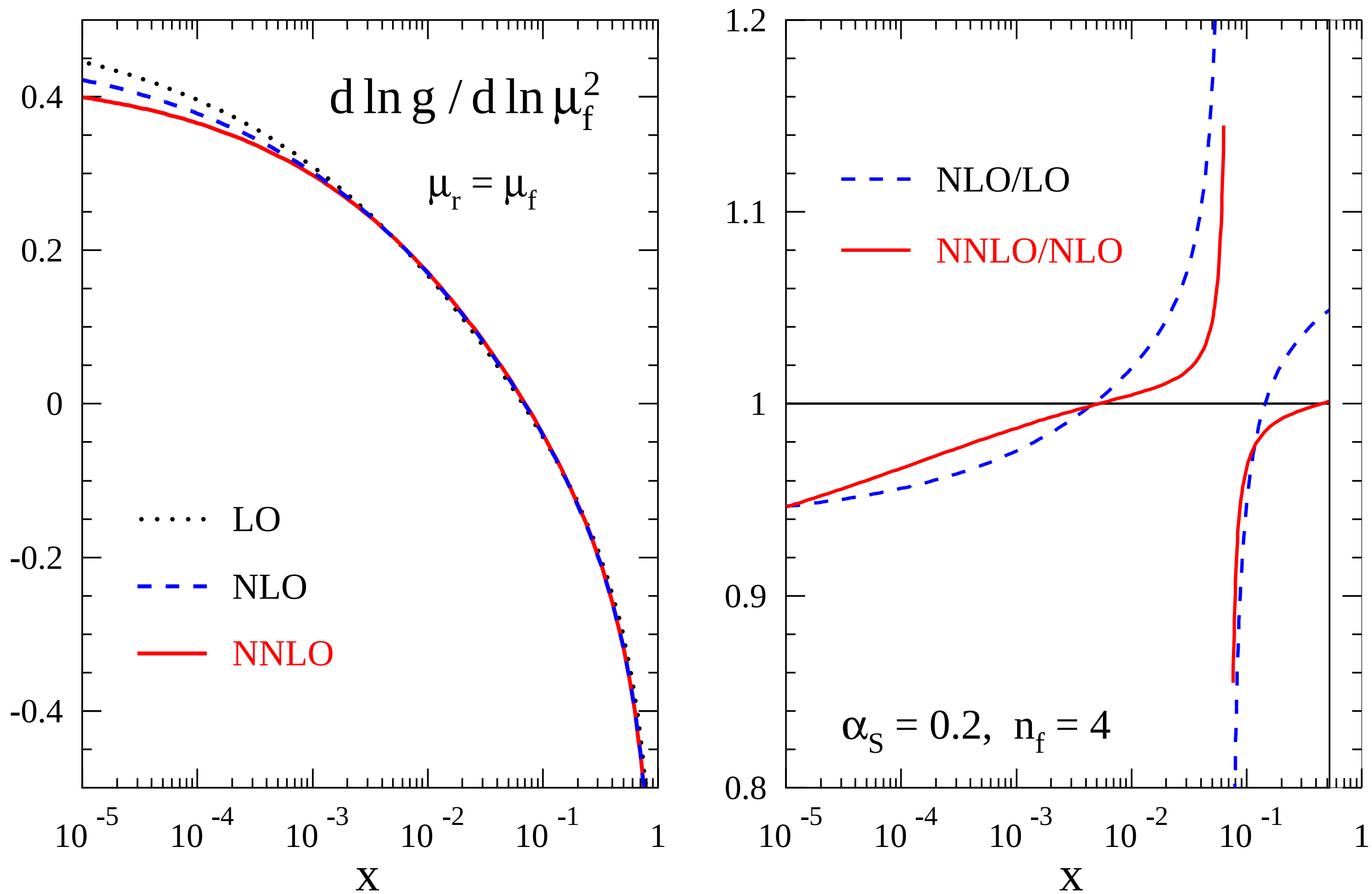
– Parametrization of singlet quark distribution

$$\begin{aligned} xq_{\mathrm{s}}(x,\mu_0^2) &= 0.6\,x^{-0.3}(1-x)^{3.5}\,(1+5.0\,x^{0.8}) \\ xg(x,\mu_0^2) &= 1.6\,x^{-0.3}(1-x)^{4.5}\,(1-0.6\,x^{0.3}) \end{aligned}$$

– Default values $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$

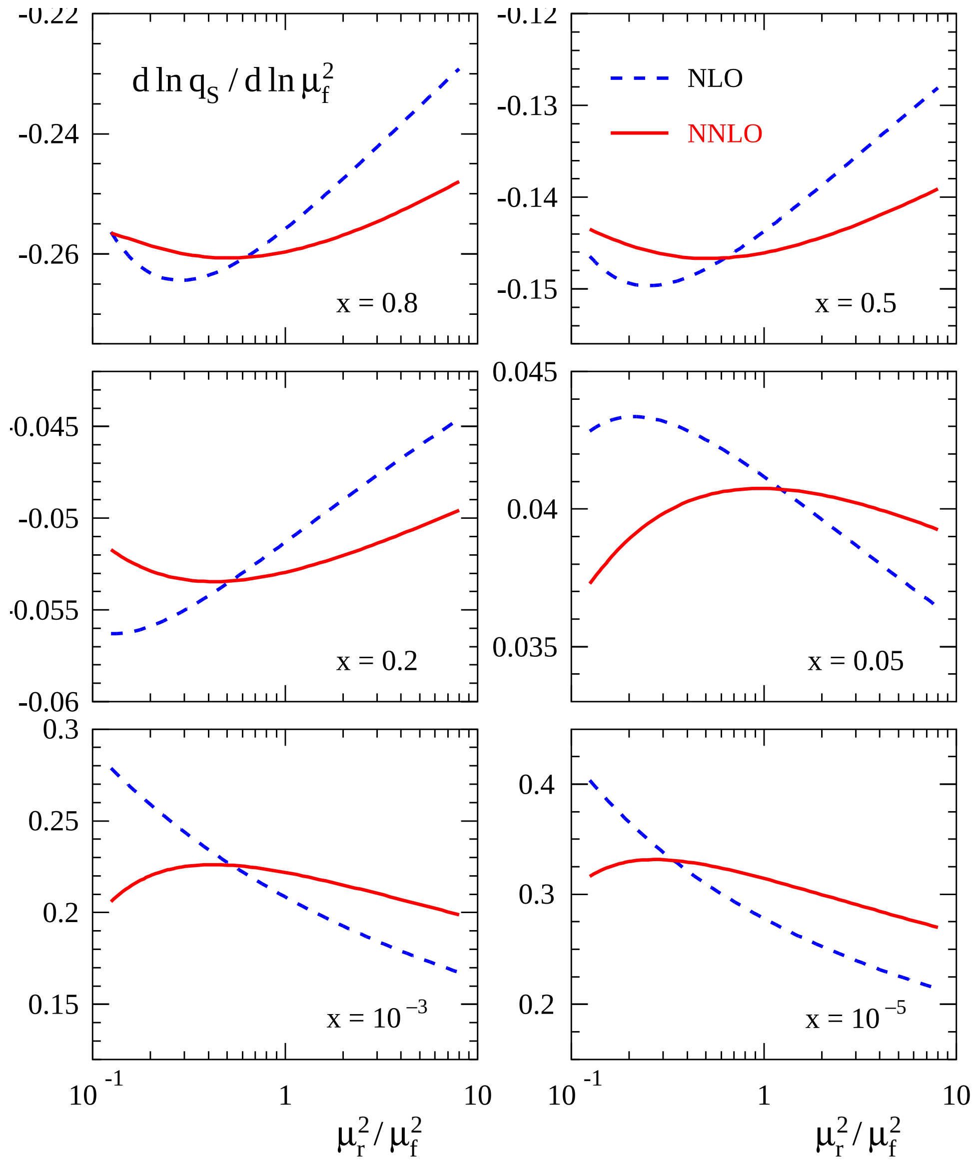


- Perturbative expansion of logarithmic scale derivative $d \ln q_s / d \ln \mu_f^2$
- Scale derivative dominated at large x by $P_{qq} \otimes q_s$, at small x by $P_{qg} \otimes g$ contribution
- $P_{qq} \otimes q_s$ contribution negligible at small x, $P_{qg} \otimes g$ contribution negligible at large x



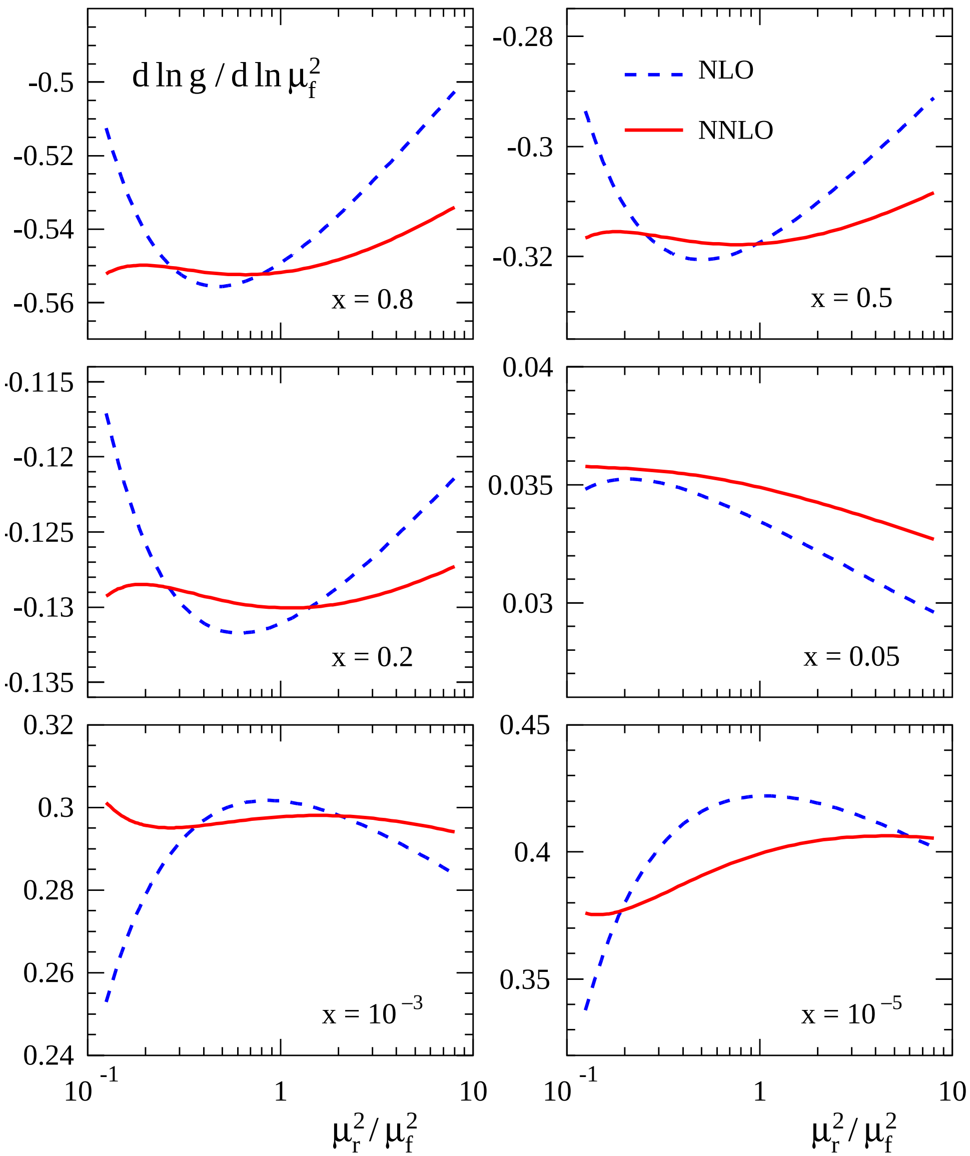
– Perturbative expansion of logarithmic scale derivative $d\ln g/d\ln\mu_f^2$

– Scale derivative dominated by $P_{\rm gg}\otimes g$ at all x, however $P_{\rm gq}\otimes q_{\rm s}$ nowhere negligible



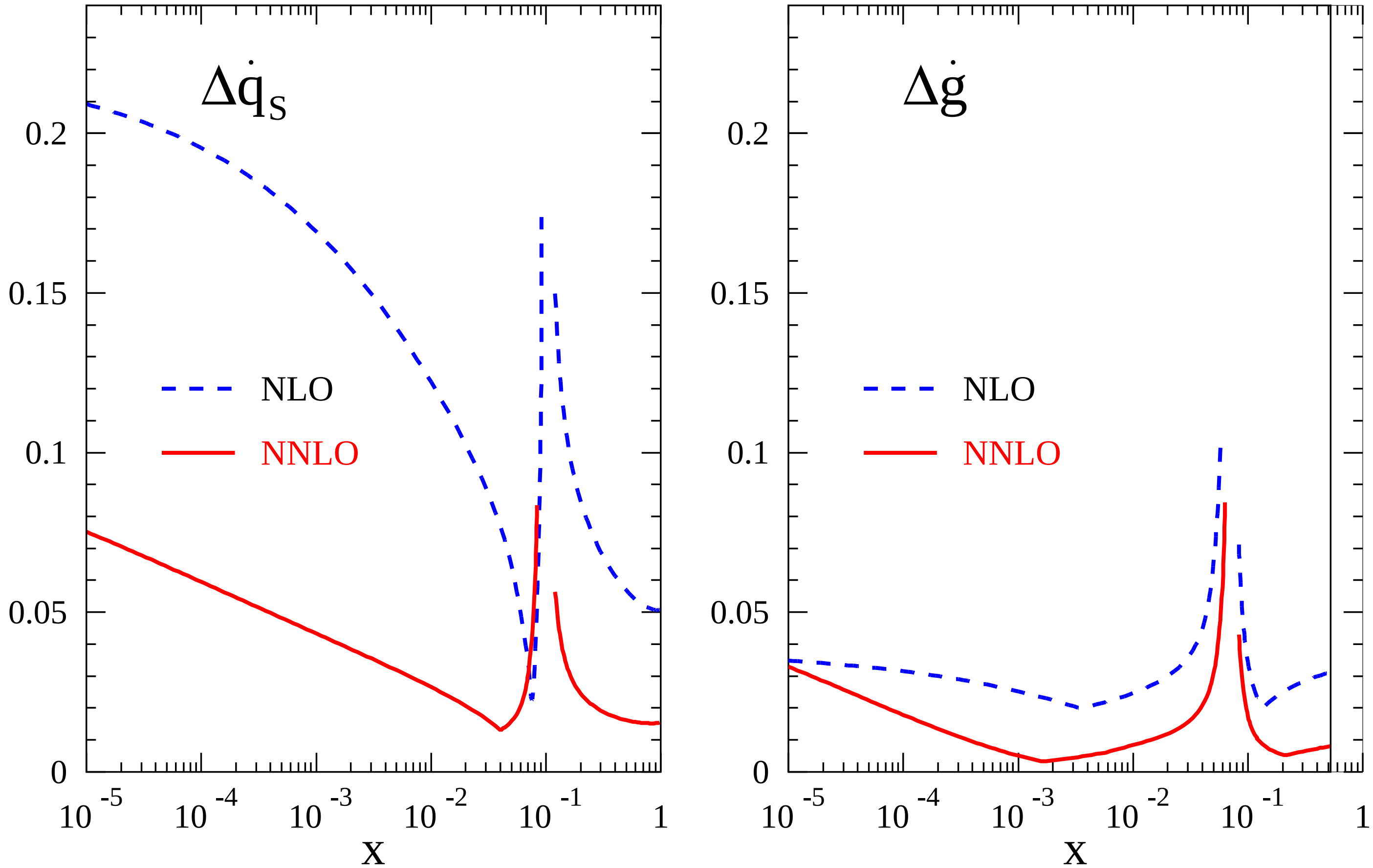
– Singlet-quark distribution :

NLO and NNLO dependence of $\dot{q}_{\rm s} \equiv d \ln q_{\rm s} / d \ln \mu_f^2$ on renormalization scale μ_r



– Gluon distribution :

NLO and NNLO dependence of $\dot{q}_{\rm s} \equiv d \ln q_{\rm s} / d \ln \mu_f^2$ on renormalization scale μ_r



– Renormalization scale uncertainty

$$\Delta\dot{f} \equiv \frac{\max\left[\dot{f}(x,\mu_r=\frac{1}{2}\mu_f\ldots2\mu_f)\right]-\min\left[\dot{f}(x,\mu_r=\frac{1}{2}\mu_f\ldots2\mu_f)\right]}{2\left|\mathrm{average}\left[\dot{f}(x,\mu_r=\frac{1}{2}\mu_f\ldots2\mu_f)\right]\right|}$$

– NLO and NNLO prediction for $\Delta\dot{q}_s$ (left) and for $\Delta\dot{g}$ (right)

The Summary

Motivation

- **NNLO** analysis of deep-inelastic structure functions $F_2, F_3 \longrightarrow$ **high precision**
  - stability under scale variations at NNLO
  - match experimental accuracy in final HERA data
  - NNLO parton distributions for LHC precision analyses

Methods

- Mellin moments and nested sums $\longrightarrow$ **powerful technology**
  - apply innovative and efficient method to solve multi-loop integrals
  - formalism with wide range of applications

Upshot

- Phenomenology for deep-inelastic scattering and hard hadronic interactions
  - reach **new level of precision**