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"A fresh look at small x analysis"

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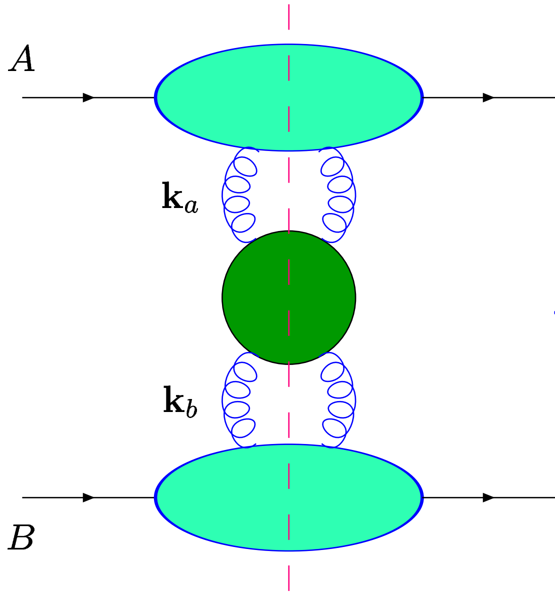
NLL BFKL Equation: $(\alpha_S\Delta)^n + \alpha_S(\alpha_S\Delta)^n$

$$\sigma(s) = \int \frac{d^2\mathbf{k}_a}{\mathbf{k}_a^2} \int \frac{d^2\mathbf{k}_b}{\mathbf{k}_b^2} \Phi_A(\mathbf{k}_a)\,\Phi_B(\mathbf{k}_b)\, f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$$



$\phi_A(\mathbf{k}_a)$ Process-dependent

$f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$ Process-independent

$\phi_B(\mathbf{k}_b)$ Process-dependent

$$f(\mathbf{k}_a, \mathbf{k}_b, \Delta) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\omega\; e^{\omega\Delta} f_\omega(\mathbf{k}_a, \mathbf{k}_b)$$

$$\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^2\mathbf{k}\; \mathcal{K}(\mathbf{k}_a, \mathbf{k}) f_\omega(\mathbf{k}, \mathbf{k}_b)$$

"Solving the NLL BFKL Equation"
Jeppe R Andersen, ASV – *Phys Lett B 567 (2003)*

"The Gluon Green's Function in the NLL BFKL Approach"
Jeppe R Andersen, ASV – *Nuclear Physics B 679 (2004)*

In dimension regularisation the equation reads

$$\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b) \;=\; \delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b)+\int d^{2+2\epsilon}\mathbf{k}'\;\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}'\right) f_\omega\left(\mathbf{k}',\mathbf{k}_b\right)$$

with kernel

$$\mathcal{K}(\mathbf{k}_a,\mathbf{k}) = 2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\,\delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k})+\mathcal{K}_r(\mathbf{k}_a,\mathbf{k})$$

$\mathbf{k}=\mathbf{k}'-\mathbf{k}_a$ shift and split the kernel:

$$\begin{aligned}\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b) \;=\;& \delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k})\,f_\omega(\mathbf{k},\mathbf{k}_b)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\,.\end{aligned}$$

To cancel the poles at $\epsilon=0$ we split the integral over transverse phase space for $\mathcal{K}_r^{(\epsilon)}$ using a phase space slicing parameter λ ...

λ appears in the ϵ–dependent real emission:

$$\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b) = \delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}\, 2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k})\, f_\omega(\mathbf{k},\mathbf{k}_b)$$
$$+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\left(\theta\left(\mathbf{k}^2-\lambda^2\right)+\theta\left(\lambda^2-\mathbf{k}^2\right)\right) f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)$$
$$+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\, f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)$$

The approximation

$$f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b) \simeq f_\omega(\mathbf{k}_a,\mathbf{k}_b) \text{ for } |\mathbf{k}|<\lambda$$

is a good one for large $|\mathbf{k}_a|$ & small λ.

We can write now

$$\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b) = \delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b)$$
$$+\left\{2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,\theta\left(\lambda^2-\mathbf{k}^2\right)\right\} f_\omega(\mathbf{k}_a,\mathbf{k}_b)$$
$$+\int d^{2+2\epsilon}\mathbf{k}\,\left\{\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,\theta\left(\mathbf{k}^2-\lambda^2\right)+\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\right\} f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)$$

What about the ϵ poles? ...

The gluon Regge trajectory reads Fadin–Lipatov

$$2\,\omega^{(\epsilon)}\left(\mathbf{q}^2\right) \;=\; -\bar{\alpha}_s\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\left(\frac{1}{\epsilon}+\ln\frac{q^2}{\mu^2}\right)$$

$$-\frac{\bar{\alpha}_s^2}{8}\frac{\Gamma^2(1-\epsilon)}{(4\pi)^{2\epsilon}}\left\{\frac{\beta_0}{N_c}\left(\frac{1}{\epsilon^2}+\ln^2\frac{q^2}{\mu^2}\right)\right.$$

$$\left.+\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}\right)\left(\frac{1}{\epsilon}+2\ln\frac{q^2}{\mu^2}\right)-\frac{32}{9}+2\zeta(3)-\frac{28}{9}\frac{\beta_0}{N_c}\right\}$$

$\beta_0 \equiv \frac{11}{3}N_c - \frac{2}{3}n_f$, $\bar{\alpha}_s \equiv \frac{\alpha_s(\mu)N_c}{\pi}$, μ is the $\overline{\text{MS}}$ scale.

Integrating ϵ-dependent real emission ...

$$\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{q},\mathbf{q}+\mathbf{k}\right)\theta\left(\lambda^2-\mathbf{k}^2\right) \;=\; \frac{1}{\Gamma(1+\epsilon)}\frac{\bar{\alpha}_s}{(4\pi)^\epsilon}\frac{1}{\epsilon}\left(\frac{\lambda^2}{\mu^2}\right)^\epsilon$$

$$\left\{1+\frac{\bar{\alpha}_s}{4}\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\left[\frac{\beta_0}{N_c}\frac{1}{\epsilon}\left(1-\frac{1}{2}\left(\frac{\lambda^2}{\mu^2}\right)^\epsilon\left(1-\epsilon^2\frac{\pi^2}{6}\right)\right)\right.\right.$$

$$\left.\left.+\frac{1}{2}\left(\frac{\lambda^2}{\mu^2}\right)^\epsilon\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}+\epsilon\left(-\frac{32}{9}+14\zeta(3)-\frac{28}{9}\frac{\beta_0}{N_c}\right)\right)\right]\right\}$$

We can now combine these two results ...

Poles in ϵ cancel & λ dependence is

$$\omega_0\left(\mathbf{q}^2,\lambda\right) \equiv$$
$$\lim_{\epsilon\to 0}\left\{2\,\omega^{(\epsilon)}\left(\mathbf{q}^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{q},\mathbf{q}+\mathbf{k}\right)\theta\left(\lambda^2-\mathbf{k}^2\right)\right\} =$$
$$-\bar{\alpha}_s\left\{\ln\frac{\mathbf{q}^2}{\lambda^2}+\frac{\bar{\alpha}_s}{4}\left[\frac{\beta_0}{2N_c}\ln\frac{\mathbf{q}^2}{\lambda^2}\ln\frac{\mu^4}{\mathbf{q}^2\lambda^2}+\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}\right)\ln\frac{\mathbf{q}^2}{\lambda^2}-6\zeta(3)\right]\right\}$$

Using the notation

$$\omega_0\left(\mathbf{q}^2,\lambda\right) \equiv -\xi\left(|\mathbf{q}|\,\lambda\right)\ln\frac{\mathbf{q}^2}{\lambda^2}+\eta$$
$$\xi\left(\mathrm{X}\right) \equiv \bar{\alpha}_s+\frac{\bar{\alpha}_s^2}{4}\left[\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}-\frac{\beta_0}{N_c}\ln\frac{\mathrm{X}}{\mu^2}\right]$$
$$\eta \equiv \bar{\alpha}_s^2\frac{3}{2}\zeta(3)$$

we can write

$$\left(\omega-\omega_0\left(\mathbf{k}_a^2,\lambda\right)\right)f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) = \delta^{(2)}\left(\mathbf{k}_a-\mathbf{k}_b\right)$$
$$+\int d^2\mathbf{k}\left(\frac{1}{\pi\mathbf{k}^2}\xi\left(\mathbf{k}^2\right)\theta\left(\mathbf{k}^2-\lambda^2\right)+\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)$$

where ...

$$
\begin{aligned}
&\widetilde{\mathcal{K}}_r\left(\mathbf{q},\mathbf{q}'\right) = \\
&\frac{\bar{\alpha}_s^2}{4\pi}\left\{-\frac{1}{(\mathbf{q}-\mathbf{q}')^2}\ln^2\frac{\mathbf{q}^2}{\mathbf{q}'^2} + \left(1+\frac{n_f}{N_c^3}\right)\left(\frac{3(\mathbf{q}\cdot\mathbf{q}')^2-2\mathbf{q}^2\mathbf{q}'^2}{16\mathbf{q}^2\mathbf{q}'^2}\right)\right. \\
&\qquad\qquad \times\left(\frac{2}{\mathbf{q}^2}+\frac{2}{\mathbf{q}'^2}+\left(\frac{1}{\mathbf{q}'^2}-\frac{1}{\mathbf{q}^2}\right)\ln\frac{\mathbf{q}^2}{\mathbf{q}'^2}\right) \\
&-\left(3+\left(1+\frac{n_f}{N_c^3}\right)\left(1-\frac{(\mathbf{q}^2+\mathbf{q}'^2)^2}{8\mathbf{q}^2\mathbf{q}'^2}-\frac{(2\mathbf{q}^2\mathbf{q}'^2-3\mathbf{q}^4-3\mathbf{q}'^4)}{16\mathbf{q}^4\mathbf{q}'^4}(\mathbf{q}\cdot\mathbf{q}')^2\right)\right) \\
&\qquad\qquad \times\int_0^\infty dx\frac{1}{\mathbf{q}^2+x^2\mathbf{q}'^2}\ln\left|\frac{1+x}{1-x}\right| \\
&+\frac{2(\mathbf{q}^2-\mathbf{q}'^2)}{(\mathbf{q}-\mathbf{q}')^2(\mathbf{q}+\mathbf{q}')^2} \\
&\qquad \times\left(\frac{1}{2}\ln\frac{\mathbf{q}^2}{\mathbf{q}'^2}\ln\frac{\mathbf{q}^2\mathbf{q}'^2(\mathbf{q}-\mathbf{q}')^4}{(\mathbf{q}^2+\mathbf{q}'^2)^4}+\left(\int_0^{-\frac{\mathbf{q}^2}{\mathbf{q}'^2}}-\int_0^{-\frac{\mathbf{q}'^2}{\mathbf{q}^2}}\right)dt\frac{\ln(1-t)}{t}\right) \\
&\left.-\left(1-\frac{(\mathbf{q}^2-\mathbf{q}'^2)^2}{(\mathbf{q}-\mathbf{q}')^2(\mathbf{q}+\mathbf{q}')^2}\right)\left(\left(\int_0^1-\int_1^\infty\right)dz\frac{1}{(\mathbf{q}'-z\mathbf{q})^2}\ln\frac{(z\mathbf{q})^2}{\mathbf{q}'^2}\right)\right\}
\end{aligned}
$$

There is no angular averaging. In this way it is possible to obtain the full angular information.

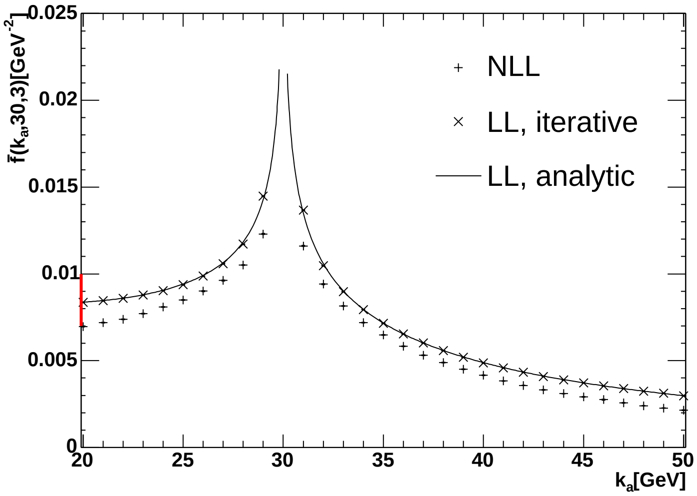
Now, how to solve it? ...

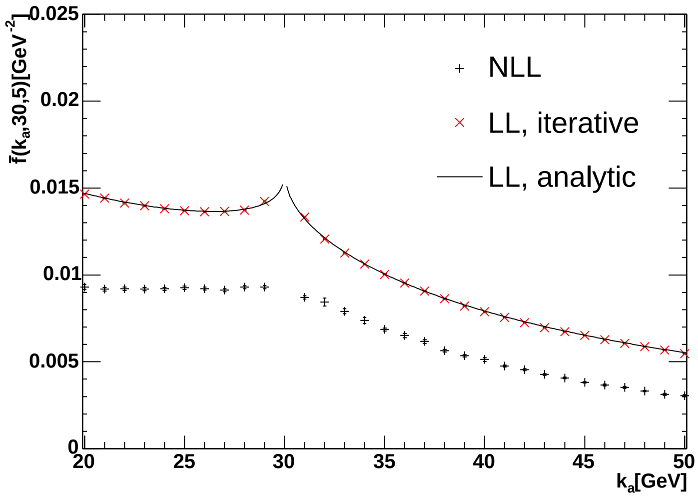
... the NLL solution reads

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) \quad &= \quad \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda\right) \Delta\right) \left\{ \delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \right. \\
&+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2\mathbf{k}_i \left[\frac{\theta\left(\mathbf{k}_i^2 - \lambda^2\right)}{\pi \mathbf{k}_i^2} \xi\left(\mathbf{k}_i^2\right) + \widetilde{\mathcal{K}}_r \left(\mathbf{k}_a + \sum_{l=0}^{i-1} \mathbf{k}_l, \mathbf{k}_a + \sum_{l=1}^{i} \mathbf{k}_l \right) \right] \\
&\times \int_0^{y_{i-1}} dy_i \, \exp\left[\left(\omega_0 \left(\left(\mathbf{k}_a + \sum_{l=1}^{i} \mathbf{k}_l \right)^2, \lambda \right) - \omega_0 \left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1} \mathbf{k}_l \right)^2, \lambda \right) y_i \right] \\
&\left. \times \delta^{(2)} \left(\sum_{l=1}^{n} \mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b \right) \right\}.
\end{aligned}
$$

where we use $y_0 \equiv \Delta$.

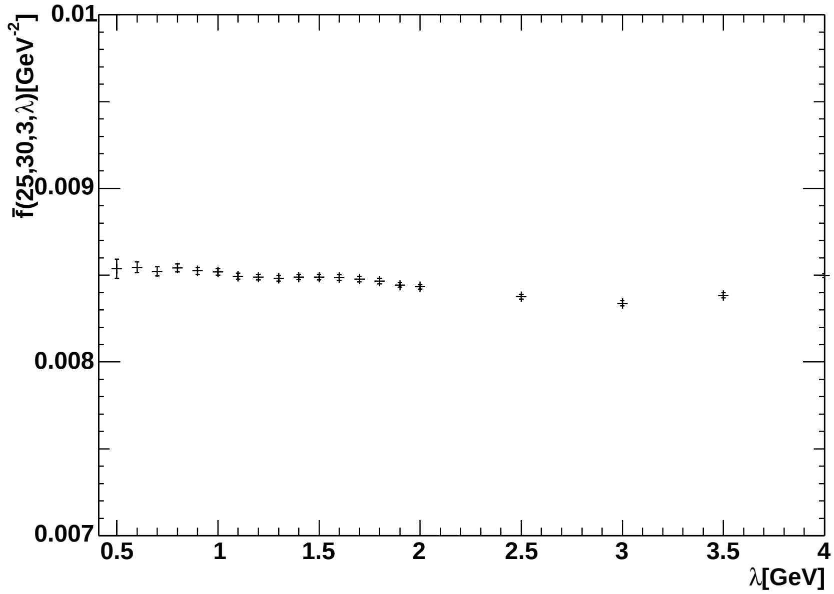
The gluon Green's function at LL and NLL with this method:



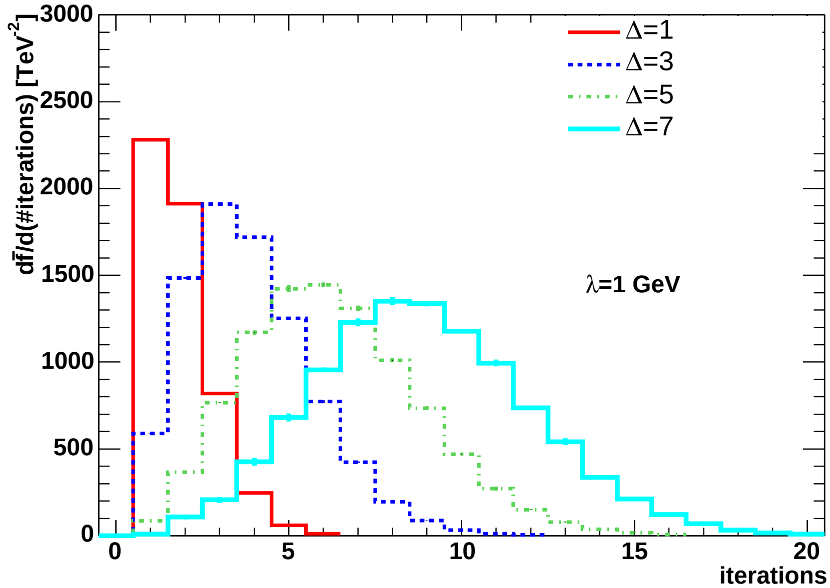


... these plots show how the initial condition evolves with energy.

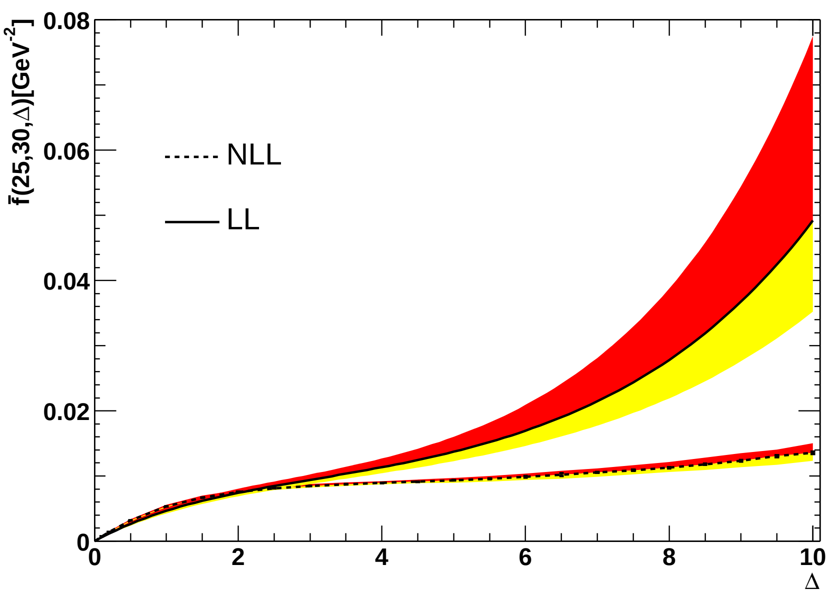
The solution is independent of λ for small λ



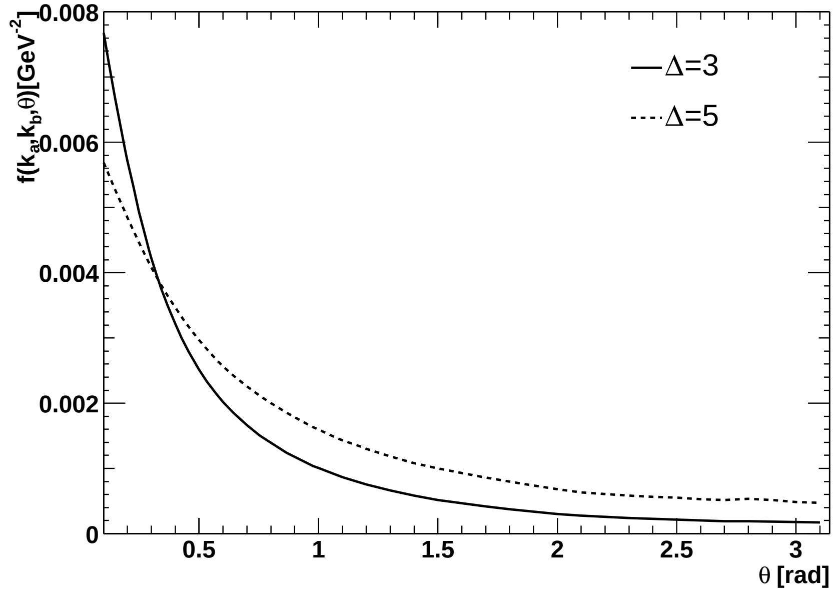
How many iterations are needed to build up the Green's function?



What is the growth with energy of the gluon Green's function?



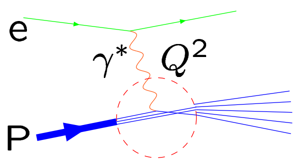
All the angular information is there ...

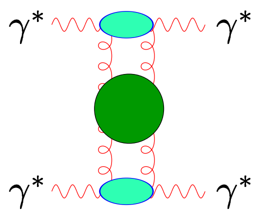


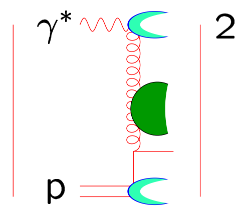
Dependence at NLL on the angle between $\mathbf{k}_a$ ($k_a = 25$ GeV) and $\mathbf{k}_b$ ($k_b = 30$ GeV) for the choice of $\Delta = 3$ and $\Delta = 5$. The renormalisation point is chosen at $\mu = k_b$.

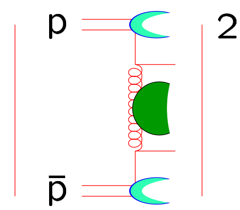
Phenomenology:

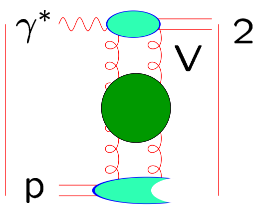
$$\sigma\left(s\right)=\int\frac{d^2\mathbf{k}_a}{2\pi\mathbf{k}_a^2}\int\frac{d^2\mathbf{k}_b}{2\pi\mathbf{k}_b^2}\Phi_A\left(\mathbf{k}_a\right)\Phi_B\left(\mathbf{k}_b\right) f\left(\mathbf{k}_a,\mathbf{k}_b,s\right)$$



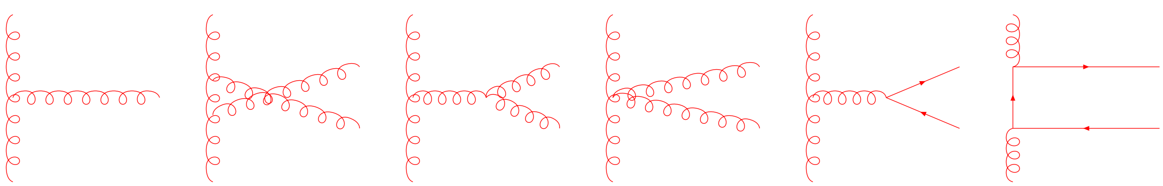








- Gluon at small $x \simeq \frac{Q^2}{s_{\gamma^* P}}$
- $\gamma^*\gamma^*$
- Forward jets
- Mueller-Navelet Jets
- High-t diffractive vector production
- Final states (Multiplicities & angular correlations).



Theoretical Excursions:

N=4 SUSY: (Kotikov–Lipatov)

No running coupling, Conformal invariance
Full eigenfunctions at NLL:

$$f\left(\vec{k}_a, \vec{k}_b, \mathrm{Y}\right) = \frac{1}{2\pi k_a k_b} \sum_{n=-\infty}^{\infty} \int \frac{d\omega}{2\pi i} e^{\omega \mathrm{Y}} \int \frac{d\gamma}{2\pi i} \left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}} \frac{e^{in\theta_{ab}}}{\omega - \omega_n(a,\gamma)}$$

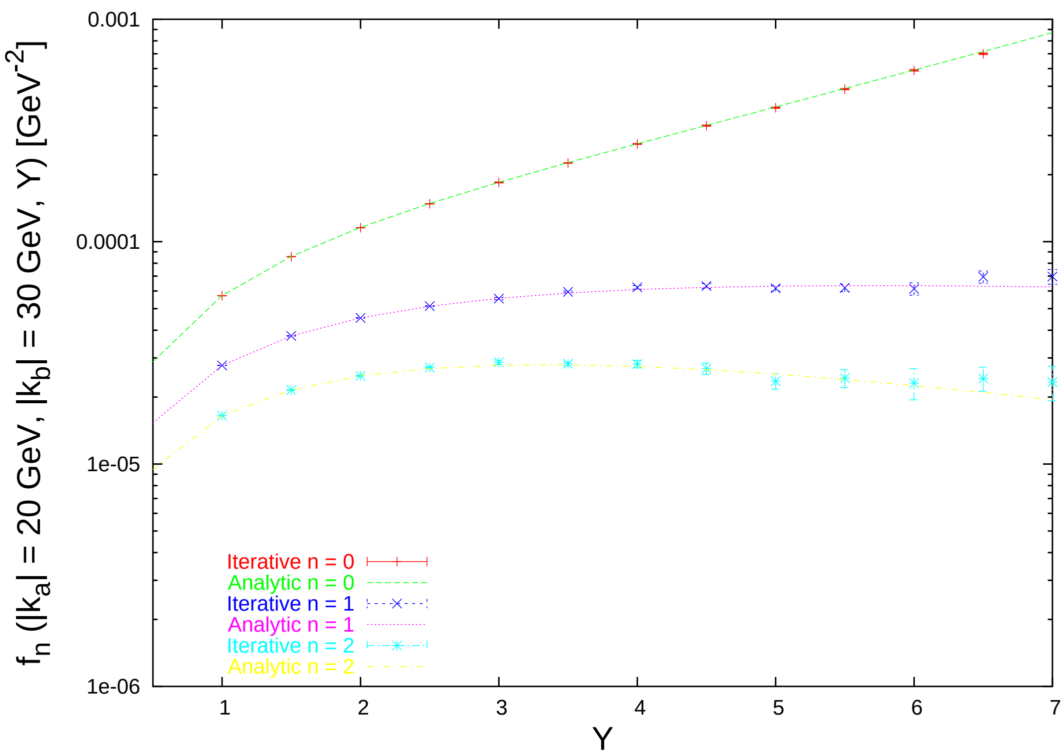
Test both solutions, use: (Andersen–Sabio Vera)

$$\begin{aligned} f(\vec{k}_a, \vec{k}_b, \mathrm{Y}) &= \exp\left(\omega_0\left(\vec{k}_a^2, \lambda\right)\mathrm{Y}\right)\left\{\delta^{(2)}(\vec{k}_a - \vec{k}_b)\right. \\ &+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2\vec{k}_i \left[\frac{\theta\left(\vec{k}_i^2 - \lambda^2\right)}{\pi \vec{k}_i^2} \xi^{\mathrm{MS}} + \widetilde{\mathcal{K}}_r(\cdots)\right] \\ &\times \int_0^{y_{i-1}} dy_i \exp\left[\left(\omega_0(\cdots) - \omega_0(\cdots)\right) y_i\right] \delta^{(2)}\left.\left(\sum_{l=1}^{n} \vec{k}_l + \vec{k}_a - \vec{k}_b\right)\right\} \end{aligned}$$

and project on conformal spins:

$$f\left(\vec{k}_a, \vec{k}_b, \mathrm{Y}\right) = \sum_{n=-\infty}^{\infty} f_n\left(|\vec{k}_a|, |\vec{k}_b|, \mathrm{Y}\right) e^{in\theta_{ab}}$$

$$
\begin{aligned}
f_n\left(|\vec{k}_a|, |\vec{k}_b|, \mathrm{Y}\right) &= \int_0^{2\pi} \frac{d\theta_{ab}}{2\pi} f\left(\vec{k}_a, \vec{k}_b, \mathrm{Y}\right) \cos\left(n\theta_{ab}\right) \quad (\mathrm{JRA-ASV}) \\
&= \frac{8\pi^2}{2\pi|\vec{k}_a||\vec{k}_b|} \int \frac{d\gamma}{2\pi i} \left(\frac{|\vec{k}_a|^2}{|\vec{k}_b|^2}\right)^{\gamma-\frac{1}{2}} e^{\omega_n(a,\gamma)\mathrm{Y}} \; (\mathrm{AK-LL})
\end{aligned}
$$



Perfect agreement. Very strong test.

“Gluon Green Function in N=4 SUSY Yang–Mills Theory”

Jeppe R Andersen, ASV - hep–th/0406009

Theoretical Excursions:

Resumming running coupling effects:

Structure of equation in this regularisation:

$\overline{MS}$–scheme

$$\left(\omega - \omega_0\left(|\vec{k}_a|^2, \lambda\right)\right) f_\omega\left(\vec{k}_a, \vec{k}_b\right) = \delta^{(2)}\left(\vec{k}_a - \vec{k}_b\right)$$
$$+ \int d^2\vec{k}\left(\frac{1}{\pi|\vec{k}|^2}\xi\left(|\vec{k}|^2\right)\theta\left(|\vec{k}|^2 - \lambda^2\right) + \widetilde{\mathcal{K}}_r\left(\vec{k}_a, \vec{k}_a + \vec{k}\right)\right) f_\omega\left(\vec{k}_a + \vec{k}, \vec{k}_b\right)$$

$$\xi\left(|\vec{k}|^2\right) \equiv \bar{\alpha}_s^{\overline{\rm MS}} + \bar{\alpha}_s^{\overline{\rm MS}^2}\left(\mathcal{S} - \frac{\beta_0}{4N_c}\ln\frac{|\vec{k}|^2}{\mu^2}\right)$$

$$\mathcal{S} = \frac{1}{3} - \frac{\pi^2}{12} + \frac{5}{12}\frac{\beta_0}{N_c}$$

$$\omega_0\left(|\vec{q}|^2, \lambda\right) = -\xi\left(|\vec{q}|\lambda\right)\ln\frac{|\vec{q}|^2}{\lambda^2} + \eta$$

$$\eta = \bar{\alpha}_s^{\overline{\rm MS}2}\frac{3}{2}\zeta(3)$$

It is λ independent because:

$$\frac{1}{2}\frac{\partial}{\partial\ln\lambda}\omega_0\left(|\vec{q}|^2, \lambda\right) = \xi\left(\lambda^2\right)$$

Relation to cusp anomalous dimension

(Korchemsky et al. / Ktorides et al.)

$$\Gamma_{\text{cusp}}(\overline{\alpha}_s) \;=\; \overline{\alpha}_s + \overline{\alpha}_s^2 \mathcal{S}$$

$$\omega_0\left(|\vec{q}|^2, \lambda\right) \;\equiv\; -\frac{1}{2}\int_{\lambda^2}^{|\vec{q}|^2} \frac{d|\vec{k}|^2}{|\vec{k}|^2}\Gamma_{\text{cusp}}\left(\overline{\alpha}_s^{\overline{\text{MS}}}\left(|\vec{k}|^2\right)\right) + \text{constant}$$

$$\overline{\alpha}_s^{\overline{\text{MS}}}\left(|\vec{k}|^2\right) \;\approx\; \overline{\alpha}_s^{\overline{\text{MS}}}\left(\mu^2\right) - \overline{\alpha}_s^{\overline{\text{MS}}^2}\left(\mu^2\right)\frac{\beta_0}{4N_c}\ln\frac{|\vec{k}|^2}{\mu^2}$$

$$\omega_0\left(|\vec{q}|^2, \lambda\right) \;=\; -\left[\overline{\alpha}_s^{\overline{\text{MS}}} + \overline{\alpha}_s^{\overline{\text{MS}}^2}\left(\mathcal{S} - \frac{\beta_0}{4N_c}\ln\frac{|\vec{q}|\lambda}{\mu^2}\right)\right]\ln\frac{|\vec{q}|^2}{\lambda^2} + \text{constant}$$

Use the correct one (Fadin–Lipatov):

$$\text{constant} \;=\; \eta \;=\; \overline{\alpha}_s^{\overline{\text{MS}}^2}\frac{3}{2}\zeta(3)$$

Resummation achieved using:

$$\overline{\alpha}_s^{\text{R}}\left(|\vec{k}|^2\right) \;=\; \frac{1}{\frac{\beta_0}{4N_c}\ln\frac{|\vec{k}|^2}{\Lambda_{\text{QCD}}^2}}.$$

$$\omega_0\left(|\vec{q}|^2, \lambda\right) \;=\; \frac{4N_c}{\beta_0}\ln\frac{\overline{\alpha}_s^{\text{R}}\left(|\vec{q}|^2\right)}{\overline{\alpha}_s^{\text{R}}\left(\lambda^2\right)} + \frac{4N_c}{\beta_0}\mathcal{S}\left[\overline{\alpha}_s^{\text{R}}\left(|\vec{q}|^2\right) - \overline{\alpha}_s^{\text{R}}\left(\lambda^2\right)\right] + \eta$$

$$\frac{1}{2}\frac{\partial}{\partial\ln\lambda}\omega_0\left(|\vec{q}|^2, \lambda\right) \;=\; \xi\left(\lambda^2\right)$$

Also, try the gluon–bremsstrahlung (GB) scheme ...