New Possibilities in the Study of NLL BFKL

How to solve the BFKL Equation at NLLA in 4 slides

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Outline of the talk

- Introduction to the BFKL framework
- The solution of the BFKL equation at Leading Logarithmic Accuracy:
  - Analytic (and its intrinsic problems)
  - Iterative (plus new results)
- Formalism at Next to Leading Logarithmic Accuracy
  - Problems
  - ...and the iterative solution

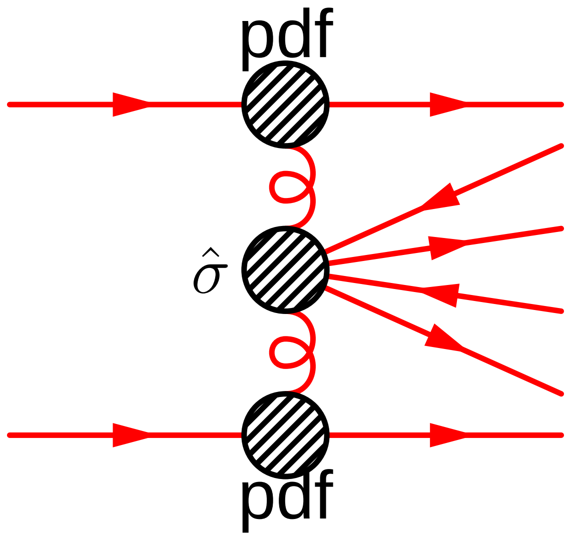
BFKL formalism

- BFKL (**B**alitskii, **F**adin, **K**uraev, **L**ipatov): resummation of large logarithms in the perturbation series for processes with two large (perturbative) and disparate energy scales ($\hat{s} \gg |\hat{t}|$) (forward scattering, small x DIS...)
- The cross section for the process $A + B \rightarrow A' + B'$ factorises as

$$\sigma(s) = \int \frac{d^2\mathbf{k}_a}{2\pi\mathbf{k}_a^2} \int \frac{d^2\mathbf{k}_b}{2\pi\mathbf{k}_b^2} \, \Phi_A(\mathbf{k}_a) \, f\left(\mathbf{k}_a, \mathbf{k}_b, \Delta = \ln\frac{s}{s_0}\right) \Phi_B(\mathbf{k}_b)$$

- $\Phi_A(\mathbf{k}_a), \Phi_B(\mathbf{k}_b)$ process dependent *impact factors* (calculated for many process at LL and for e.g. gg and $\gamma^*\gamma^*$ scattering at NLL)
- $f\left(\mathbf{k}_a, \mathbf{k}_b, \Delta\right)$ process independent *Gluon Green's function*

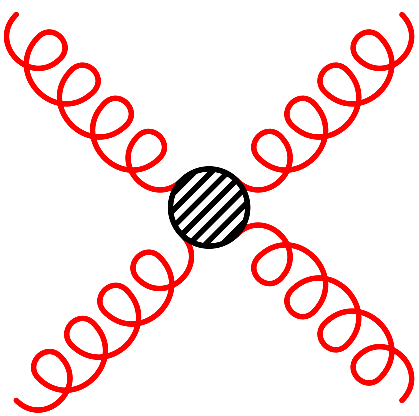
High Energy Limit



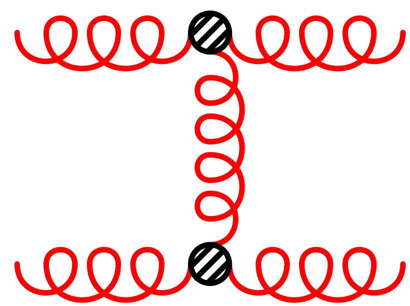
High energy limit:

$\frac{\hat{s}}{|\hat{t}|} \longrightarrow \infty$

$|\mathcal{M}|^2$ factorises.

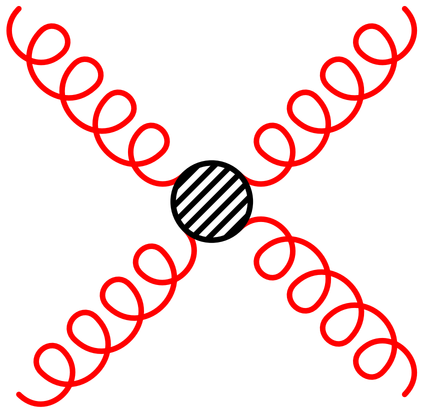


High Energy Limit $\longrightarrow$

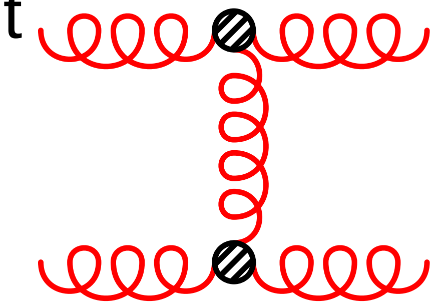


Diagrams with a t-channel gluon exchange dominate the cross section.

Dijet Production

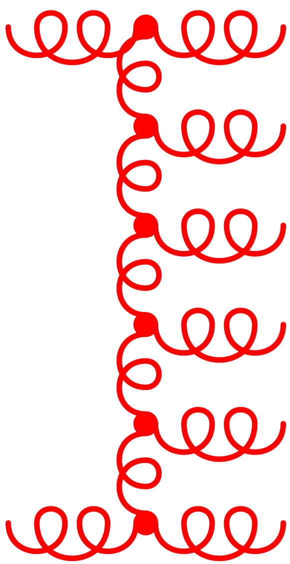


High Energy Limit $\xrightarrow{\hspace{1cm}}$ $\hat{s}/|\hat{t}| \to \infty$



BFKL evolution of the t-channel gluon

$P_{Ta}, \Delta y$



$P_{Tb}, 0$

$$\hat{s} \sim p_T^2 e^{\Delta y}$$

$$|\hat{t}| \sim p_T^2$$

$$\ln \frac{\hat{s}}{|\hat{t}|} \sim \Delta y$$

BFKL resums to all orders terms in the perturbative expansion of the form

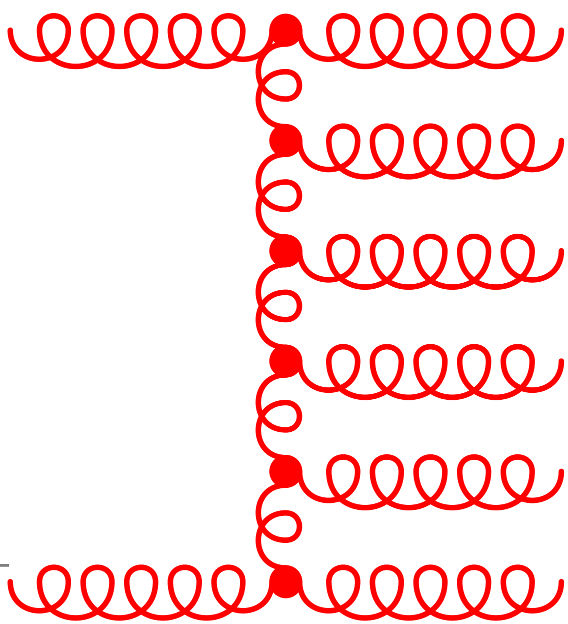
$$\left(\alpha_s \ln \frac{\hat{s}}{|\hat{t}|}\right)^n \sim (\alpha_s \Delta y)^n$$

BFKL at LLA

gluon-gluon scattering:

$$\frac{\mathrm{d}\hat{\sigma}_{gg}}{\mathrm{d}^2\vec{p}_{T_a}\mathrm{d}^2\vec{p}_{T_b}} = \underbrace{\left[\frac{C_A\alpha_s}{p_{T_a}^2}\right]}_{\text{QCD IF}} \underbrace{f(\vec{q}_a, \vec{q}_b, \Delta y)}_{\text{BFKL effects}} \underbrace{\left[\frac{C_A\alpha_s}{p_{T_b}^2}\right]}_{\text{QCD IF}}$$

$$\vec{q}_a = \vec{p}_{T_a}, \vec{q}_b = -\vec{p}_{T_b}$$



Resum leading logarithms contributing to $f(\vec{q}_a, \vec{q}_b, \Delta y)$.

Take a good look at f!

The BFKL Equation

The Gluon Green's function fulfil (to LLA and NLLA) the BFKL equation (in dim. regularisation $(D = 4 + 2\epsilon)$):

$$\omega f_\omega \left(\mathbf{k}_a, \mathbf{k}_b\right) = \delta^{(2+2\epsilon)} \left(\mathbf{k}_a - \mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}' \mathcal{K} \left(\mathbf{k}_a, \mathbf{k}'\right) f_\omega \left(\mathbf{k}', \mathbf{k}_b\right)$$

where the BFKL kernel $\mathcal{K} \left(\mathbf{k}_a, \mathbf{k}'\right)$ is calculated to LLA or NLLA respectively. At LL the kernel is conformal invariant (no running coupling) with eigenfunctions $\mathbf{k}^{2(\gamma-1)}$. Use (transverse) Mellin transform!

$$e(\gamma) = \langle\gamma|\mathcal{K}(k,k)|\gamma\rangle$$

$$f_\omega \sim \sum_\gamma e(\gamma)|\gamma\rangle$$

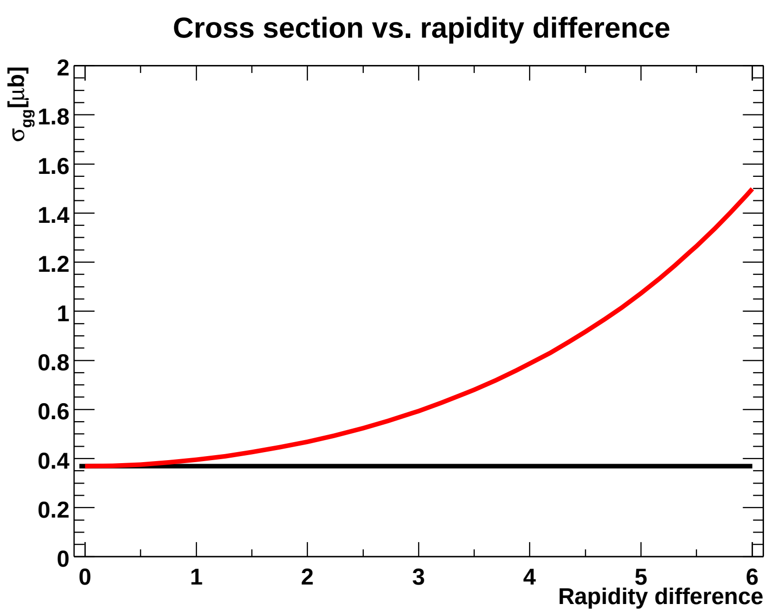
The BFKL Equation at LLA

Analytic solution for angular averaged gluon Green's function

$$\bar{f}\left(k_a, k_b, \Delta\right) = \frac{4}{k_a k_b} \int_0^\infty d\nu \left(\frac{k_a^2}{k_b^2}\right)^{i\nu} e^{\bar{\alpha}_s \Delta \chi_0(\nu)}$$

with the LL eigenvalue

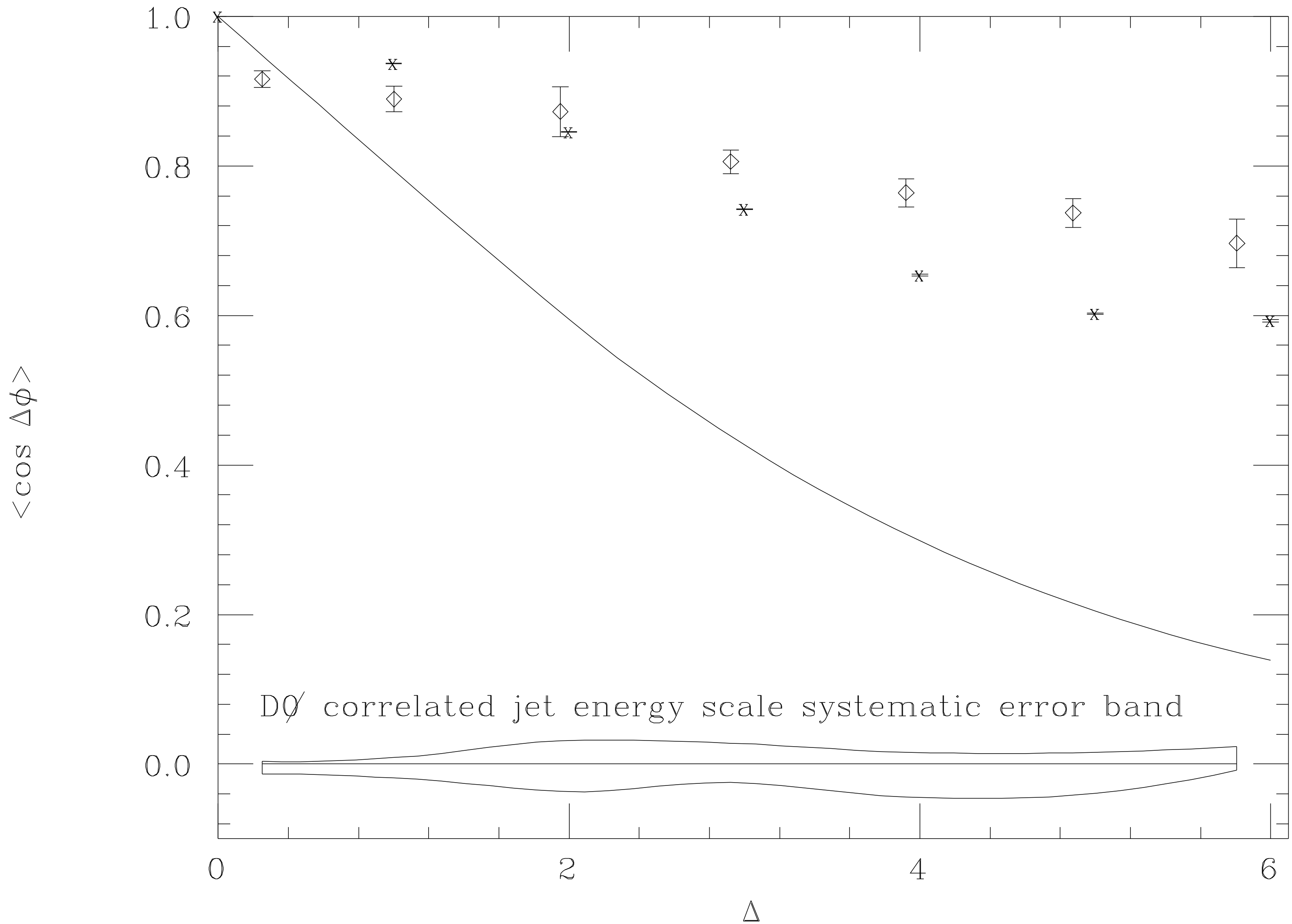
$$\chi_0(\nu) = -2\,\mathrm{Re}\left\{\psi\left(\frac{1}{2} + i\nu\right) - \psi(1)\right\}.$$



BFKL rise in cross section! Integrated over the full **k** phase space for gluon emission and allowing any number of gluons to radiate!!!

$$\hat{\sigma}_{gg} \to \frac{\pi C_A^2 \alpha_s^2}{2 P_{T,\min}^2} \frac{e^{\lambda \Delta y}}{\sqrt{\pi B \Delta y}}, B = 14\zeta(3)\bar{\alpha}_s, \quad \lambda = \frac{\alpha_s C_A}{\pi} 4 \ln 2 \approx 0.45$$

Effects of E&M Conservation



The QCD BFKL Equation at NLLA

- Both the *trajectory* $\omega(-\mathbf{k}_a^2)$ and the *real emission kernel* $\mathcal{K}_r$ are significantly more complicated than at LL
- Takes into account fermions and running coupling effects
- Breaks conformal invariance – Invalidates the Mellin transform approach. Much recent research from many groups has concentrated on the treatment of the running coupling effects in the solution of the NLL BFKL equation.

G. Altarelli, R. D. Ball, S. Forte
M. Ciafaloni, D. Colferai, G. P. Salam, A.M. Stasto
R.S. Thorne

Iterative Solution at NLLA

We propose an iterative approach to the BFKL equation at NLLA that solves the equation with *no approximations*

- Directly in the physical rapidity and transverse momentum space
  (avoids the use of the troublesome Mellin transform completely)
- The right language for use of impact factors (physics predictions!)
- Hopeful in extending the approach to final state studies like at LL
- Expresses the solution in terms of effective vertices and no-emission probabilities (physical insight into the BFKL solution at NLLA!)

Iteration at NLL

Start from the BFKL equation

$$\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}' \, \mathcal{K}(\mathbf{k}_a, \mathbf{k}') \, f_\omega(\mathbf{k}', \mathbf{k}_b)$$

$$\mathcal{K}(\mathbf{k}_a, \mathbf{k}) = 2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\,\delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}) + \mathcal{K}_r(\mathbf{k}_a, \mathbf{k})$$

Need all terms (IR) finite to be able to iterate: split the kernel $\mathcal{K}_r$ into two parts: a ϵ–dependent, $\mathcal{K}_r^{(\epsilon)}$, and a ϵ–independent, $\widetilde{\mathcal{K}}_r$

$$\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}\, 2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k})\, f_\omega(\mathbf{k}, \mathbf{k}_b)$$

$$+ \int d^{2+2\epsilon}\mathbf{k}\, \mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k})\, f_\omega(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}\, \widetilde{\mathcal{K}}_r(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k})\, f_\omega(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b)\,.$$

Iteration at NLL, 2

Introduce a slice in the phase space (no approximation)

$$\begin{aligned}\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b) &= \delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b)+\int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k})\,f_\omega(\mathbf{k},\mathbf{k}_b)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\left(\theta\left(\mathbf{k}^2-\lambda^2\right)+\theta\left(\lambda^2-\mathbf{k}^2\right)\right)f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\end{aligned}$$

approximate $f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\simeq f_\omega(\mathbf{k}_a,\mathbf{k}_b)$ for $|\mathbf{k}|<\lambda$

$$\begin{aligned}\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b) &= \delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b)\\ &+\left\{2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,\theta\left(\lambda^2-\mathbf{k}^2\right)\right\}f_\omega(\mathbf{k}_a,\mathbf{k}_b)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\left\{\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\,\theta\left(\mathbf{k}^2-\lambda^2\right)+\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\right\}f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\,.\end{aligned}$$

($\lambda\to 0$ limit can be obtained)

Iteration at NLL, 3

$$\left(\omega - \omega_0\left(\mathbf{k}_a^2, \lambda^2\right)\right) f_\omega\left(\mathbf{k}_a, \mathbf{k}_b\right) = \delta^{(2)}\left(\mathbf{k}_a - \mathbf{k}_b\right) + \int d^2\mathbf{k} \left(\frac{1}{\pi \mathbf{k}^2}\xi\left(\mathbf{k}^2\right)\theta\left(\mathbf{k}^2 - \lambda^2\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}\right)\right) f_\omega\left(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b\right)$$

$$\omega_0\left(\mathbf{q}^2, \lambda^2\right) \equiv -\xi\left(|\mathbf{q}|\,\lambda\right)\ln\frac{\mathbf{q}^2}{\lambda^2} + \eta$$

$$\xi\left(\mathrm{X}\right) \equiv \bar{\alpha}_s + \frac{\bar{\alpha}_s^2}{4}\left[\frac{4}{3} - \frac{\pi^2}{3} + \frac{5}{3}\frac{\beta_0}{N_c} - \frac{\beta_0}{N_c}\ln\frac{\mathrm{X}}{\mu^2}\right]$$

$$\eta \equiv \bar{\alpha}_s^2\frac{3}{2}\zeta(3).$$

$$\widetilde{\mathcal{K}}_r\left(\mathbf{q}, \mathbf{q}'\right) = \frac{\bar{\alpha}_s^2}{4\pi}\left\{\text{6 lines of equations}\ldots\right\}.$$

Iteration at NLL, 4

Iterate and take the inverse Mellin transform to find

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) &= \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
&+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2\mathbf{k}_i \left[\frac{\theta\left(\mathbf{k}_i^2 - \lambda^2\right)}{\pi \mathbf{k}_i^2}\xi\left(\mathbf{k}_i^2, \mu\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mathbf{k}_a + \sum_{l=1}^{i}\mathbf{k}_l, \mu\right)\right] \\
&\times \int_0^{y_{i-1}} dy_i \, \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right]\delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$

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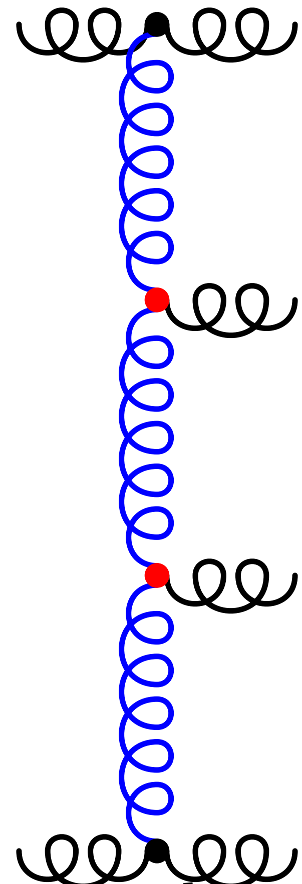
Solution at NLLA

$$
\begin{aligned}
f(\mathbf{k}_a,\mathbf{k}_b,\Delta) &= \exp\left(\omega_0\left(\mathbf{k}_a^2,\lambda^2,\mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a-\mathbf{k}_b) \\
&+ \sum_{n=1}^{\infty}\prod_{i=1}^{n}\int d^2\mathbf{k}_i\int_0^{y_{i-1}} dy_i \left[V\left(\mathbf{k}_i,\mathbf{k}_a+\sum_{l=0}^{i-1}\mathbf{k}_l,\mu\right)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2,\lambda^2,\mu\right)(y_{i-1}-y_i)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a+\sum_{l=1}^{n}\mathbf{k}_l\right)^2,\lambda^2,\mu\right)(y_n-0)\right] \\
&\times \delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l+\mathbf{k}_a-\mathbf{k}_b\right)
\end{aligned}
$$



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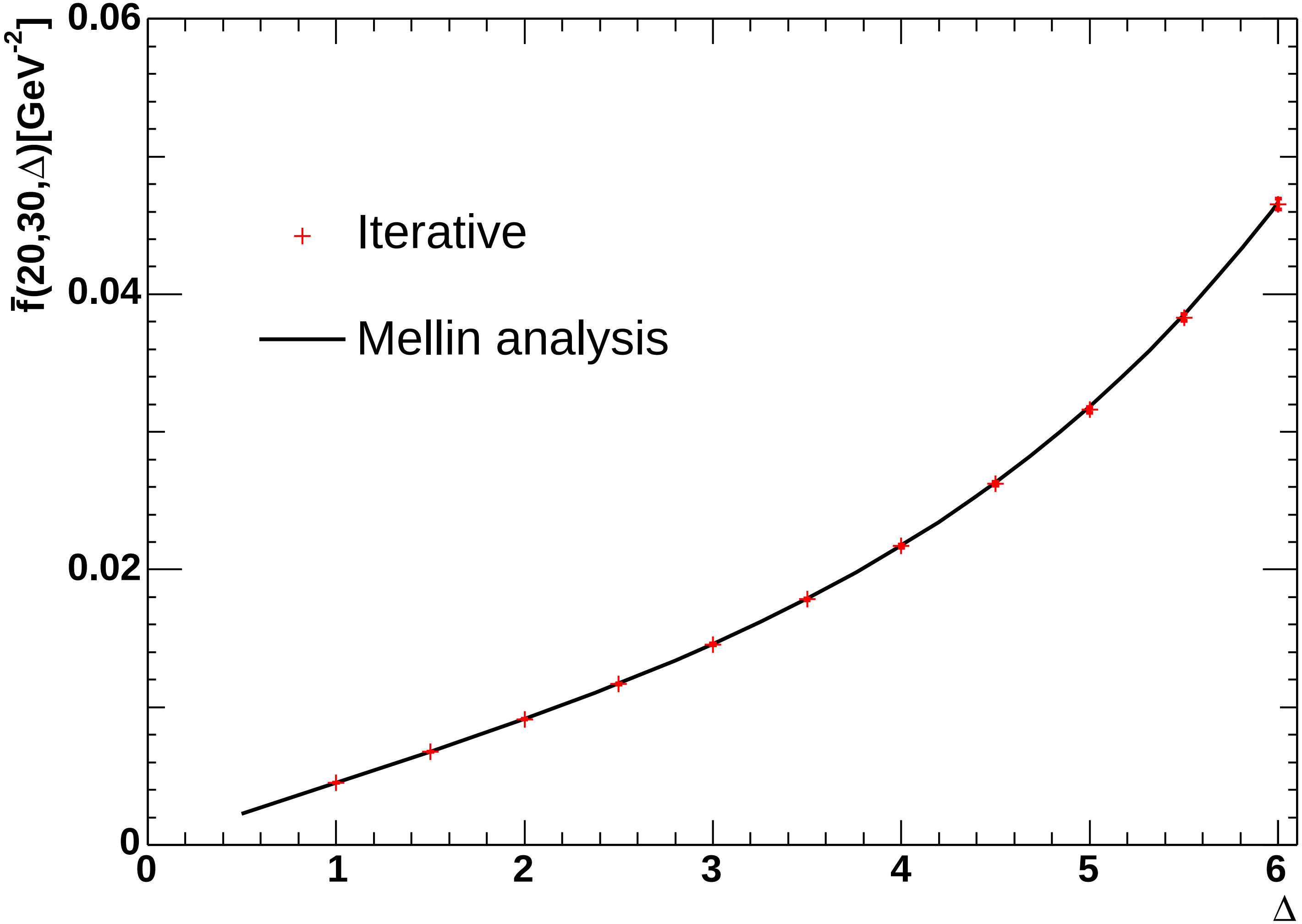
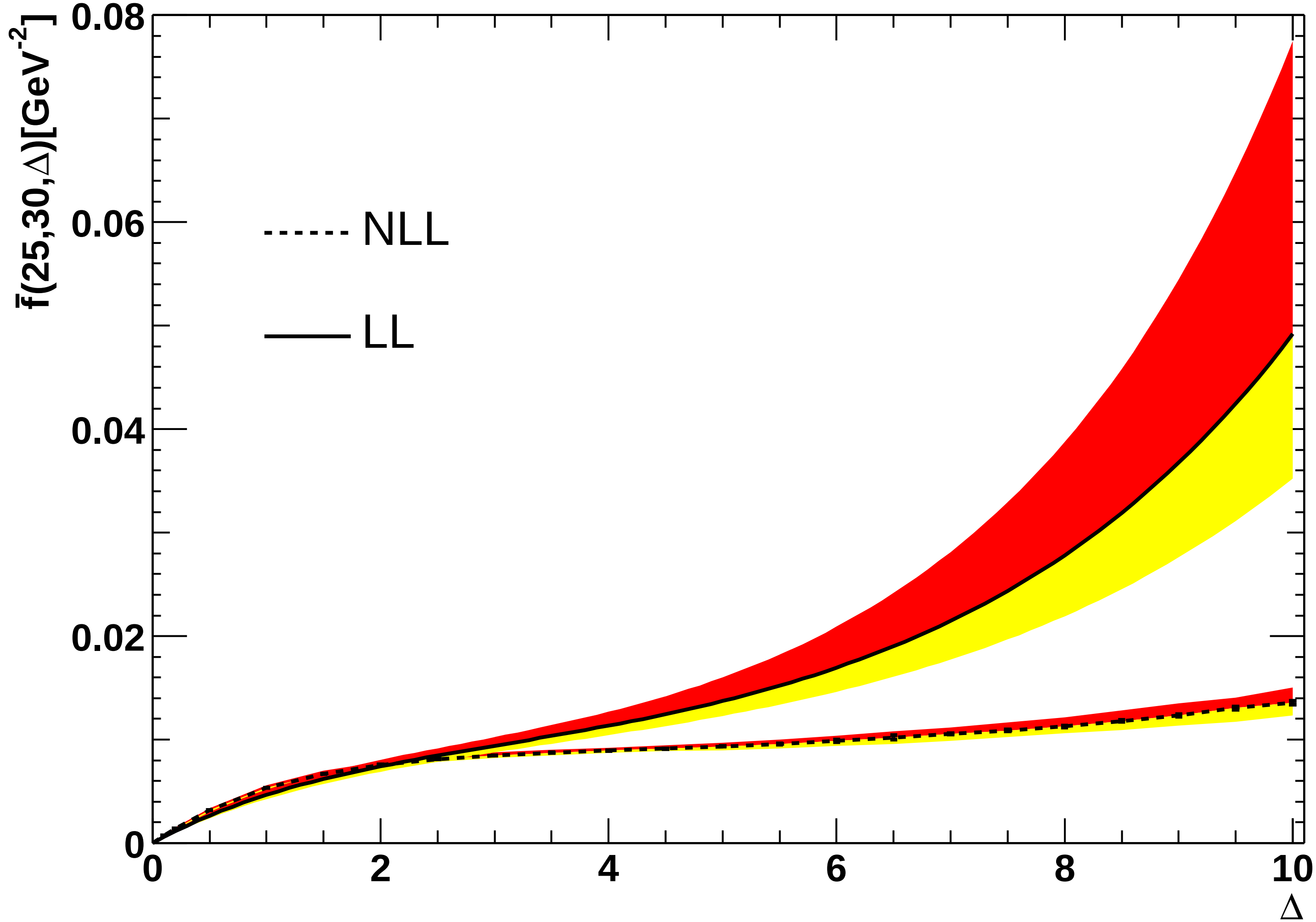
$N = 4$ **SYM**

Conformal invariant theory



Dependence of f on Δ

QCD



Leading Log tools at NLL

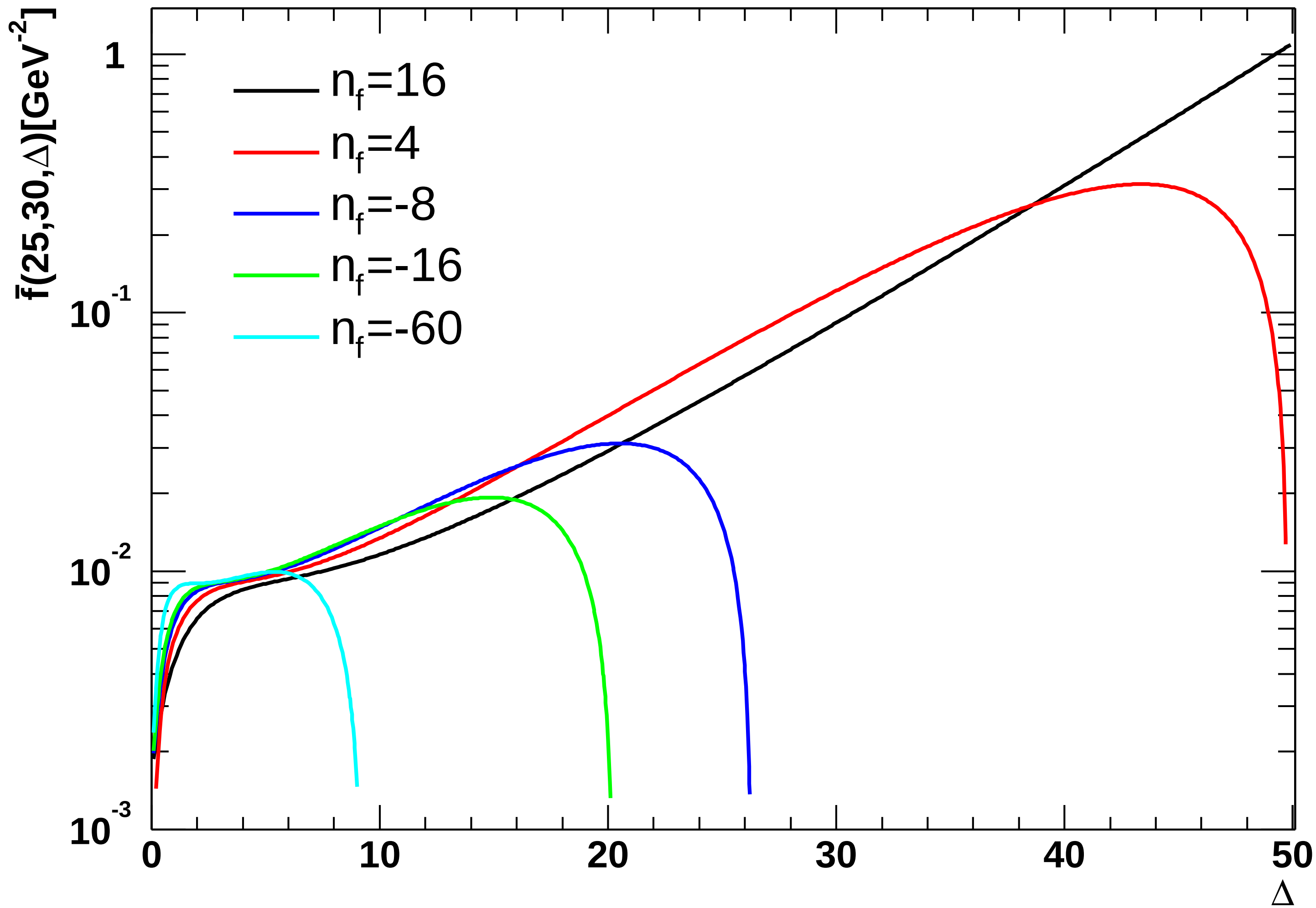
$$\omega^{\rm NLL}(\gamma) = \int \mathrm{d}^{D-2}\mathbf{k}\, \mathcal{K}^{\rm NLL}(\mathbf{k}_a,\mathbf{k}) \left(\frac{\mathbf{k}^2}{\mathbf{k}_a^2}\right)^{\gamma-1}$$

$$= \frac{\alpha_s(\mathbf{k}_a^2)N}{\pi} \left(\chi^{\rm LL}(\gamma) + \chi^{\rm NLL}(\gamma)\frac{\alpha_s(\mathbf{k}_a^2)N}{\pi}\right)$$

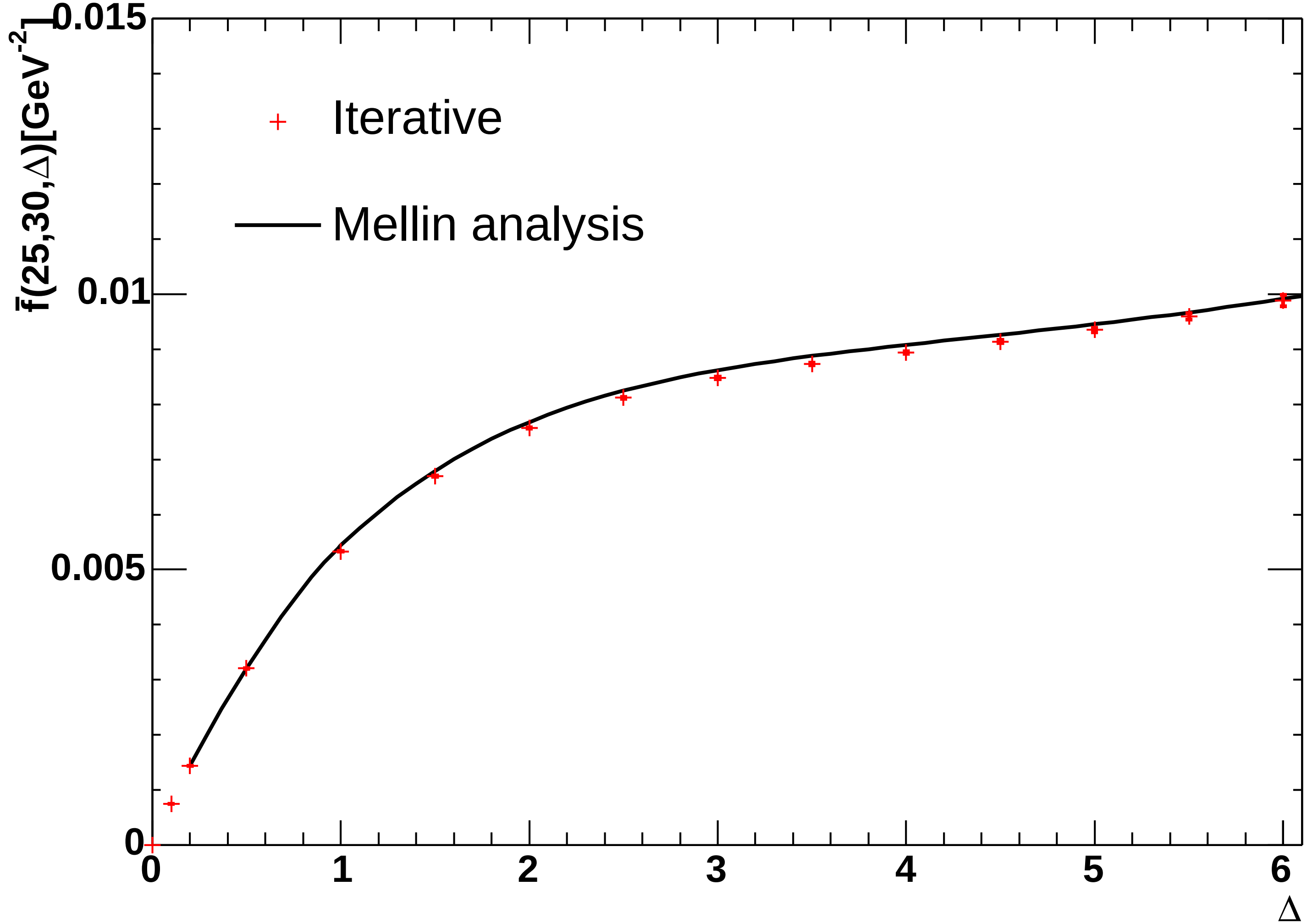
$$\chi^{\rm NLL}(\gamma) = -\frac{1}{4}\Big[\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N}\right)\frac{1}{2}\left(\chi^{\rm LL}(\gamma) {\color{red}-\psi'(\gamma)+\psi'(1-\gamma)}\right)$$
$$-6\zeta(3) + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3+\left(1+\frac{n_f}{N^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$
$$-\left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N}\right)\chi^{\rm LL}(\gamma) - \psi''(\gamma) - \psi''(1-\gamma) - \frac{\pi^3}{\sin(\pi\gamma)} + 4\phi(\gamma)\Big],$$

$$\bar{f}(k_a,k_b,\Delta) = \frac{1}{k_a^2}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathrm{d}\gamma}{2\pi i}\, e^{\Delta\omega^{\rm LL}(\gamma)} \left(\frac{k_b^2}{k_a^2}\right)^{\gamma}$$

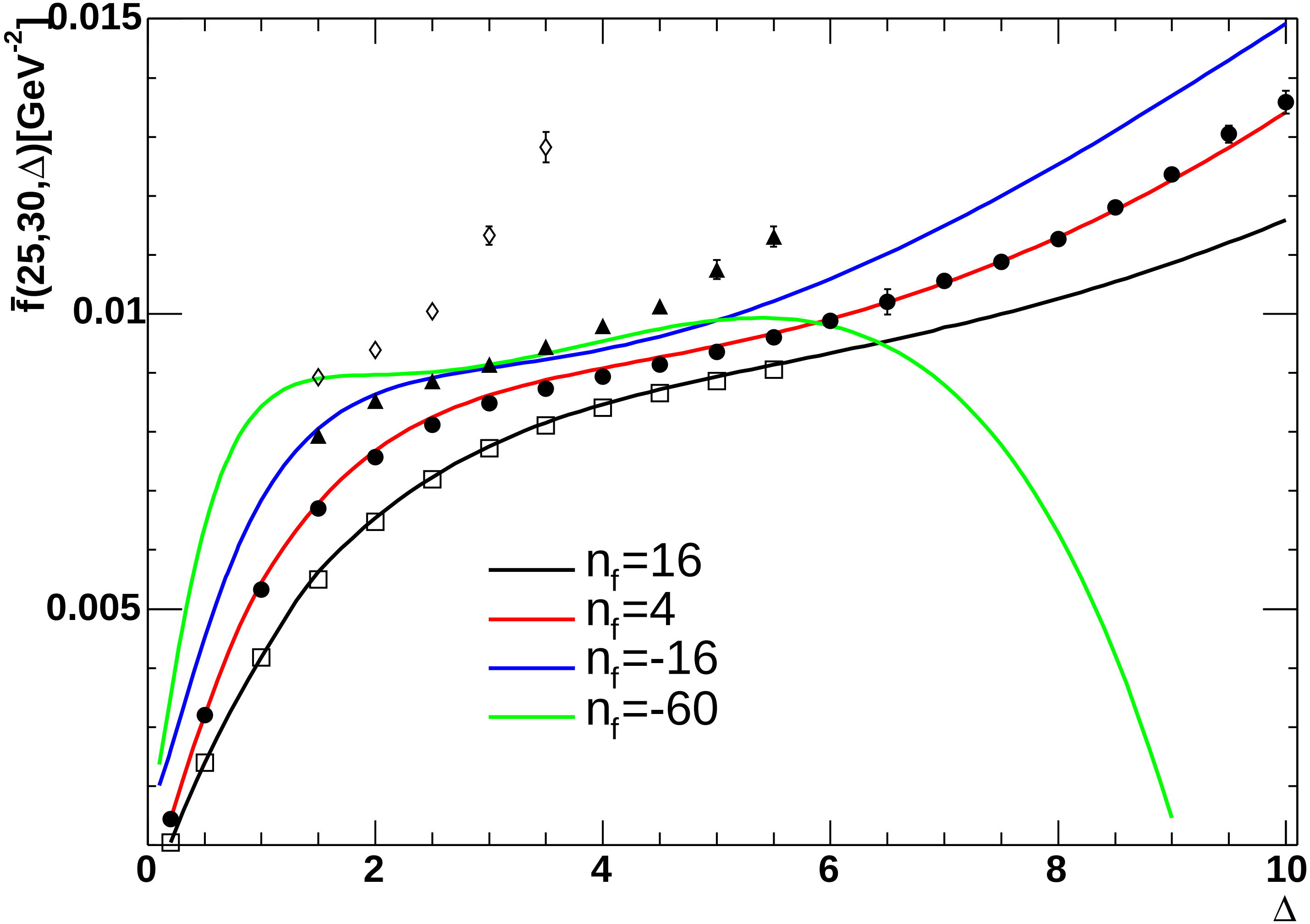
Leading Log tools at NLL



Comparison with Mellin Analysis



Comparison with Mellin Analysis



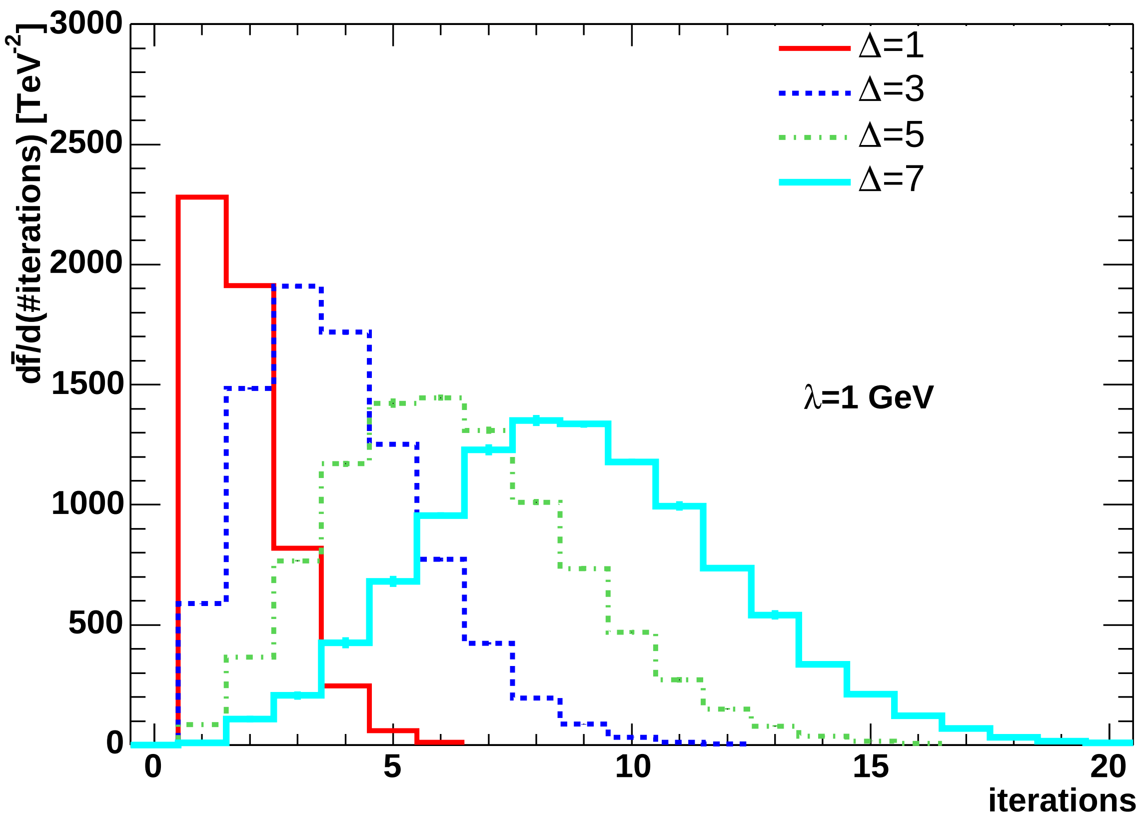
Conclusions

1. We have solved the BFKL equation at Next–to-leading logarithmic accuracy (No approximation: keeping all scale invariant and scale dependent terms, and full angular information.)
2. . . . in a form that is directly suitable for calculation of cross sections with already calculated impact factors
3. This method avoid the problems introduced by treating the running of the coupling as a perturbation that have to be dealt with in other approaches

Convergence, 1

$$\bar{f}\left(k_a, k_b, \Delta\right) \;=\; \int_0^{2\pi} d\theta\, f\left(k_a, k_b, \theta, \Delta\right),$$

$k_a = 25$ GeV, $k_b = 30$ GeV, $\lambda = 1$ GeV



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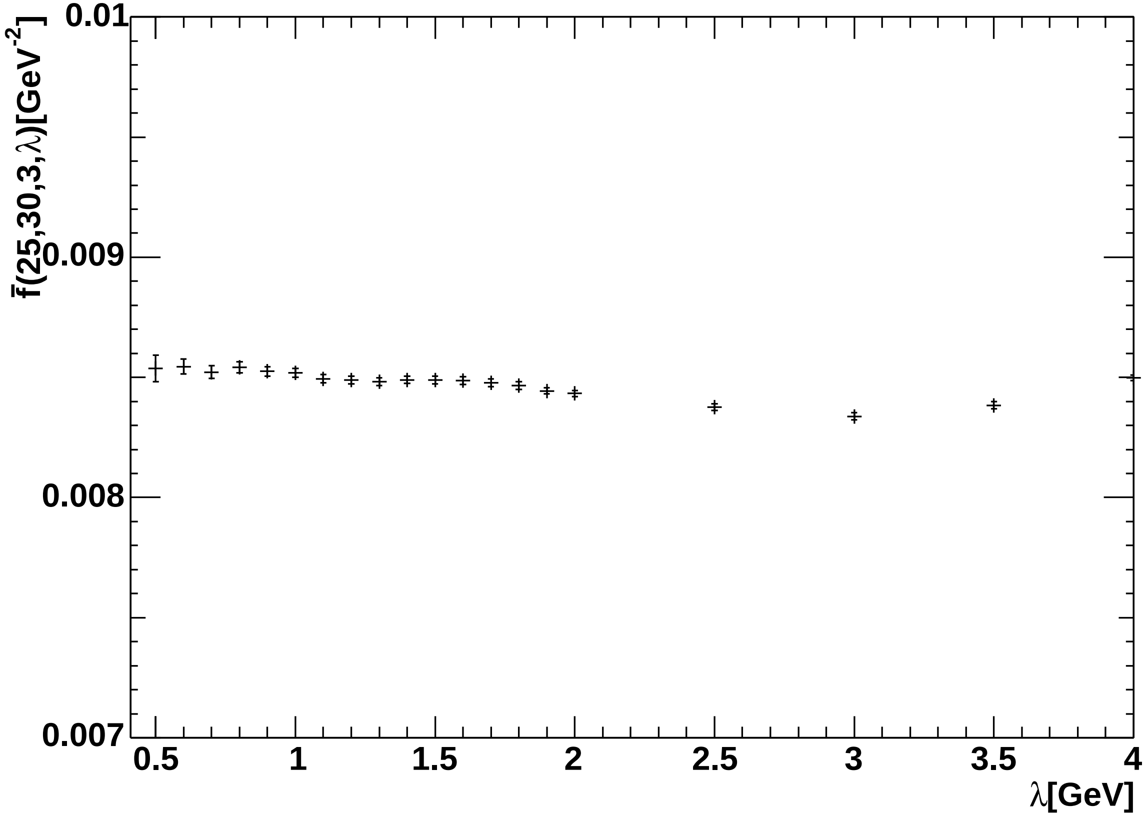
Convergence, 2

$$\bar{f}\left(k_a, k_b, \Delta\right) \;=\; \int_0^{2\pi} d\theta \, f\left(k_a, k_b, \theta, \Delta\right),$$



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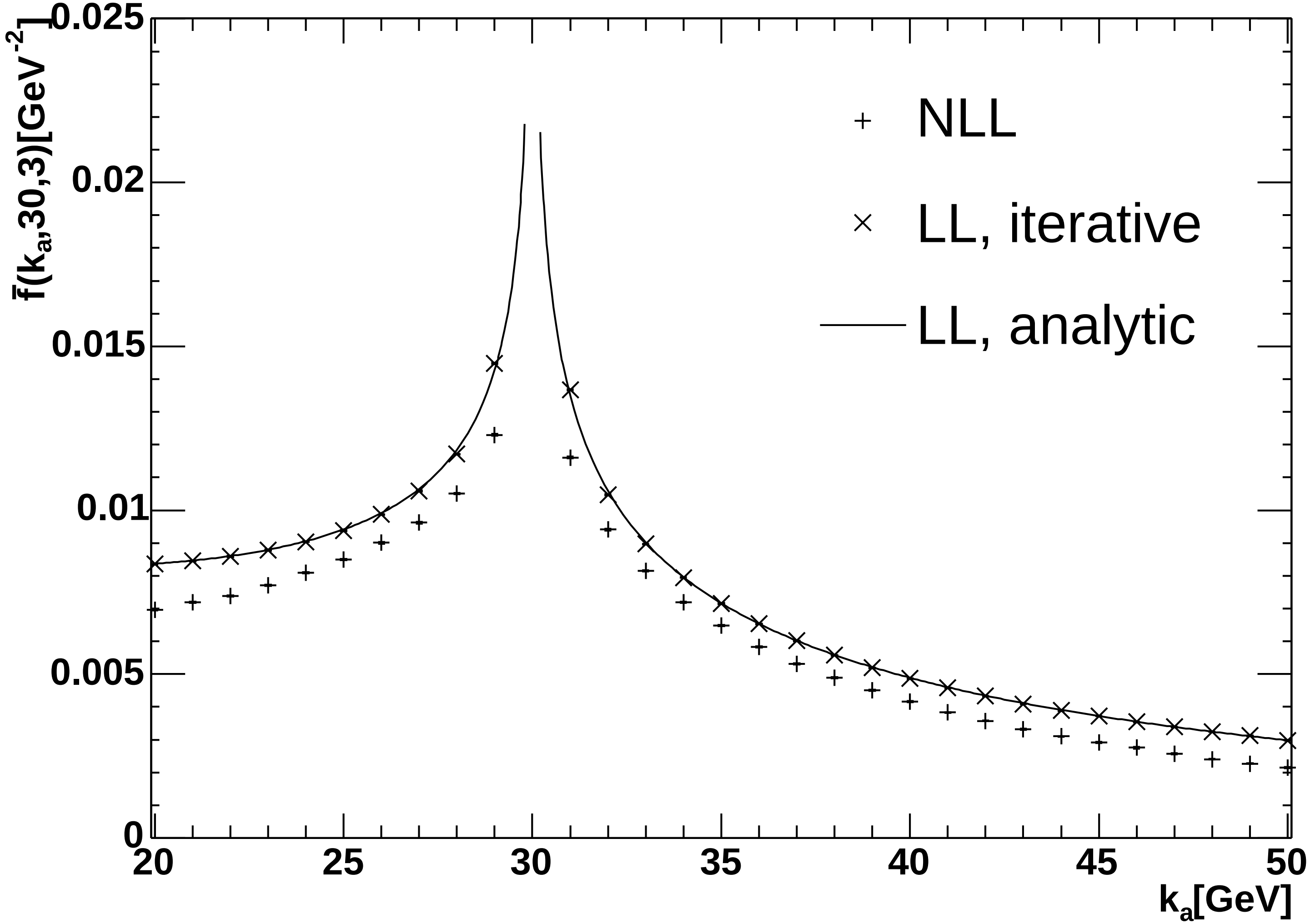
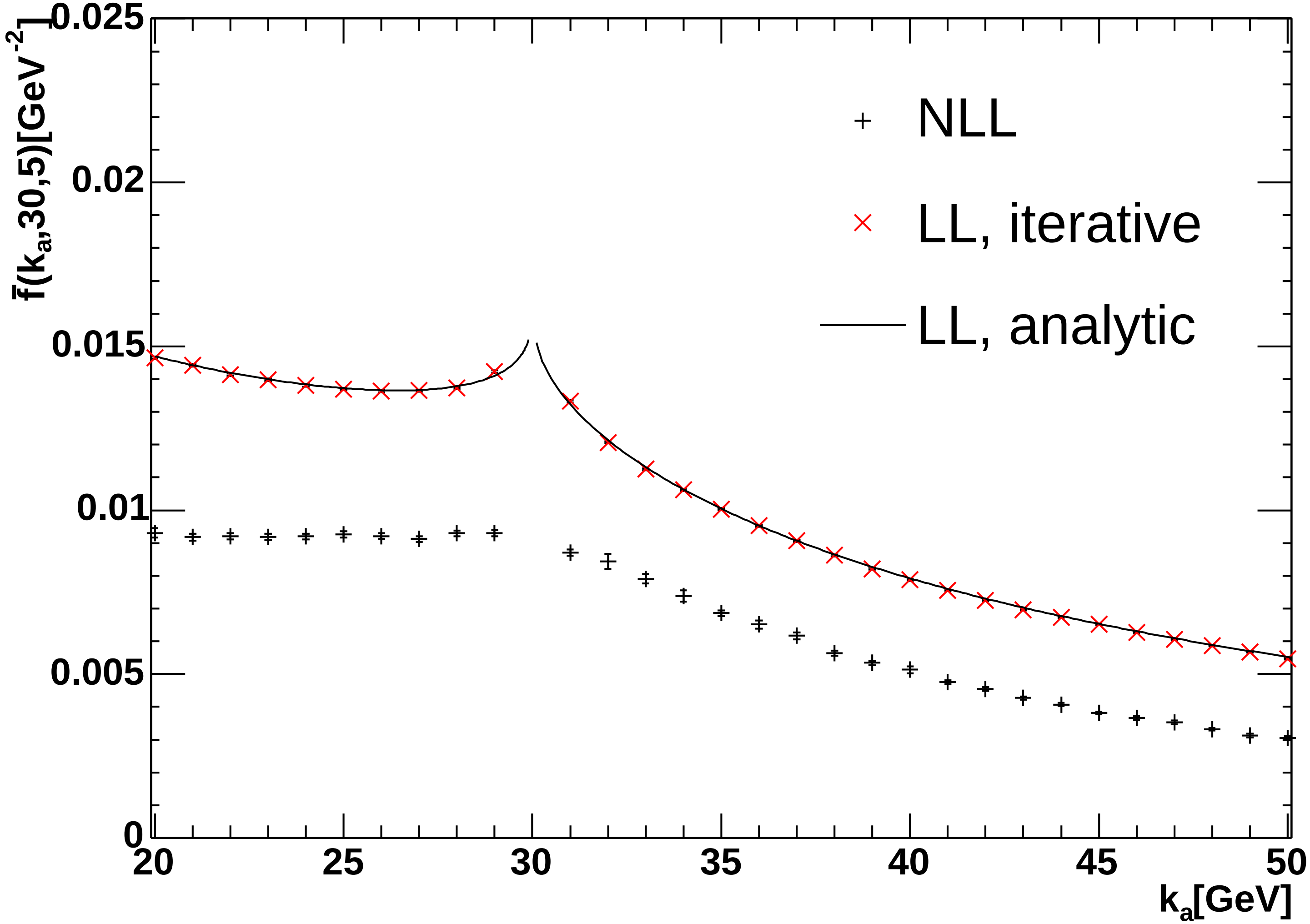
Dependence of f on Momenta

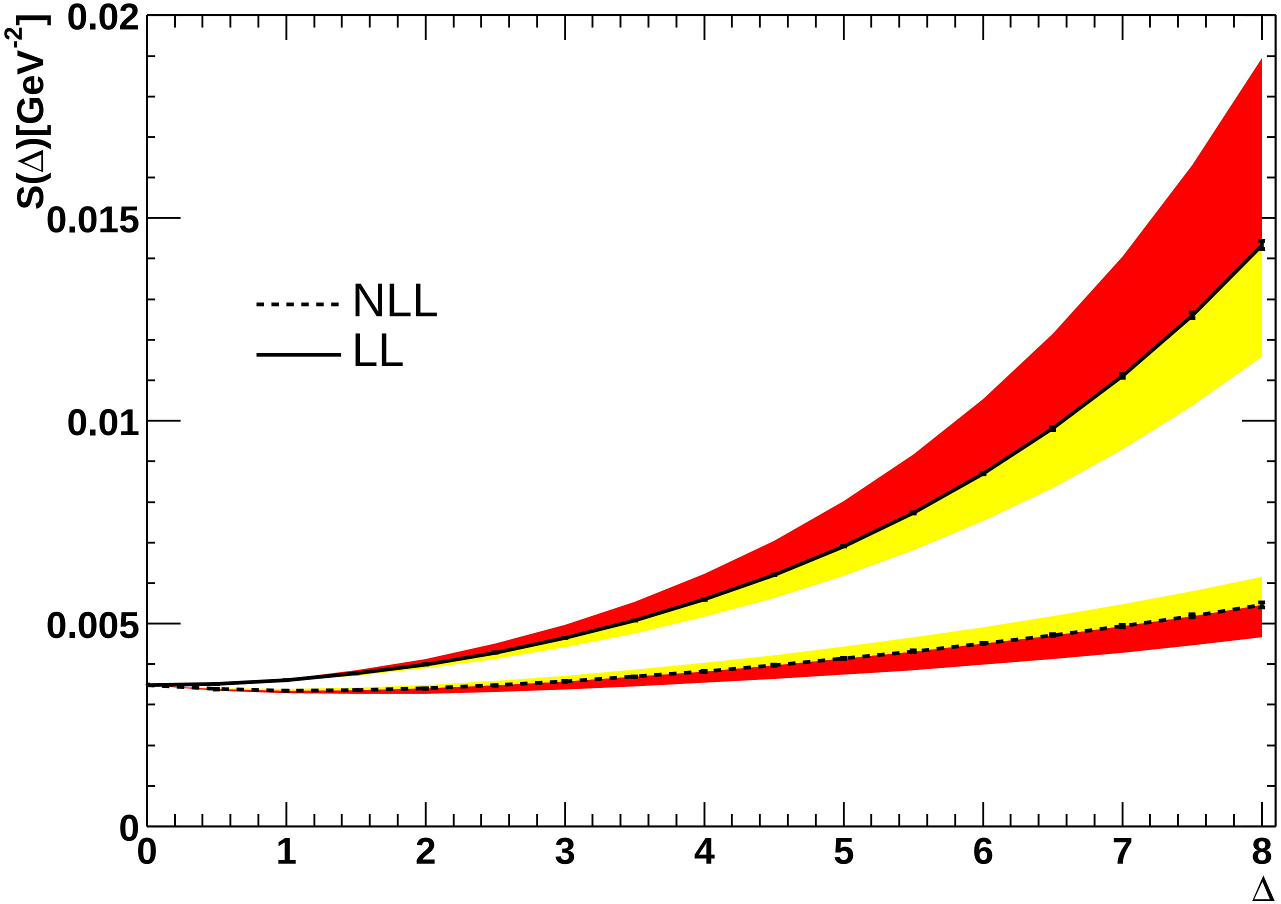


Dependence of f on Momenta



Toy Cross Section

$$S(\Delta) = \int_{k_a>30\mathrm{GeV}} \frac{d^2\mathbf{k}_a}{\mathbf{k}_a^2} \int_{k_b>30\mathrm{GeV}} \frac{d^2\mathbf{k}_b}{\mathbf{k}_b^2} f\left(\mathbf{k}_a, \mathbf{k}_b, \Delta\right)$$



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Angular Correlation