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An Improved Splitting Function for Small-x Evolution

Matching together GLAP and BFKL

G. Altarelli
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Based on G.A., R. Ball, S.Forte

hep-ph/9911273 (NPB 575,313)
hep-ph/0001157 (lectures)
hep-ph/0011270 (NPB 599,383)
hep-ph/0104246

More specifically on

hep-ph/0109178 (NPB 621,359)
and on hep-ph/0306156 (NPB 674,459),
hep-ph/0310016, hep-ph/0407153

Related work (same physics, similar conclusion, different techniques): Ciafaloni, Colferai, Salam, Stasto [see also Thorne]

Our goal is to construct a relatively simple, closed form, improved anomalous dimension $\gamma_I(\alpha,N)$
(or splitting function $P_I(\alpha,x)$)

$P_I(\alpha,x)$ should

- reduce to perturbative results at large x
- contain BFKL corrections at small x
- include running coupling effects
- be sufficiently simple to be included in fitting codes

and of course

- closely reflect the trend of the data

Moments

$$\xi = \log\frac{1}{x}; \qquad t = \log\frac{Q^2}{\mu^2}$$

$$G(x, Q^2) \equiv G(\xi, t) = x[g(x, Q^2) + k\Sigma(x, Q^2)]$$

Singlet quark

For each moment: singlet eigenvector with largest anomalous dimension eigenvalue

$$G(N, t) = \int_0^1 x^{N-1} G(x, Q^2) dx = \int_0^\infty e^{-N\xi} G(\xi, t) d\xi$$

Mellin transf. (MT)

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{N\xi} G(N, t) \frac{dN}{2\pi i}$$

Inverse MT (ξ>0)

t-evolution eq.n

$$\frac{d}{dt} G(N, t) = \gamma(N, \alpha(t)) G(N, t)$$

γ: anom. dim

$$\gamma(N, \alpha) = \alpha \cdot \gamma_{1l}(N) + \alpha^2 \cdot \gamma_{2l}(N) + \ldots$$

Pert. Th.: LO NLO

known

Recall: $$\gamma(N) = \int_0^1 x^N P(x)dx$$

$P(x) = 1/x(\ln 1/x)^n$ ⟹ $\gamma(N) = n!/N^{n+1}$

At 1-loop:

$$\alpha \cdot \gamma_{1l}(N) = \alpha \cdot \left[\frac{1}{N} - A(N)\right]$$

This corresponds to the "double scaling" behavior at small x:

$$G(\xi, t) \sim \exp\left[\sqrt{\frac{4n_C}{\pi\beta_0} \cdot \xi \cdot \frac{\log Q^2/\Lambda^2}{\log \mu^2/\Lambda^2}}\right]$$

$$\beta(\alpha) = -\beta_0\alpha^2 + \ldots$$

A. De Rujula et al '74/Ball, Forte

Amazingly supported by the data

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In principle the BFKL approach provides a tool to control $(\alpha/N)^n$ corrections to $\gamma(N, \alpha)$,
that is $1/x(\alpha \log 1/x)^n$ to splitting functions.

Define t- Mellin transf.:

$$G(\xi, M) = \int_{-\infty}^{+\infty} e^{-Mt} G(\xi, t) dt$$

with inverse:

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{Mt} G(\xi, M) \frac{dM}{2\pi i}$$

ξ-evolution eq.n (BFKL) [at fixed α]:

$$\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha) G(\xi, M)$$

with $\chi(M, \alpha) = \alpha \cdot \chi_0(M) + \alpha^2 \cdot \chi_1(M) + \ldots$

known

Bad behaviour, bad convergence

At 1-loop:

$\psi(M) = \Gamma'(M)/\Gamma(M)$

$$\alpha\chi_0(M) = \frac{\alpha n_C}{\pi}\int_0^1 [z^{M-1} + z^{-M} - 2]\frac{dz}{1-z} = \frac{\alpha n_C}{\pi}\cdot[2\psi(1) - \psi(M) - \psi(1-M)]$$

Near M=0:

$$\alpha\chi_0(M) \sim \frac{\alpha n_C}{\pi}[\frac{1}{M} + 2\zeta(3)M^2 + 2\zeta(5)M^4 + \ldots.]$$

At M=1/2

$$\lambda_0 = \alpha\chi_0\left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi}4\ln 2 = \alpha c_0 \sim 2.65\alpha \sim 0.5$$

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The minimum value of $\alpha\chi_0$ at M=1/2 is the Lipatov intercept:

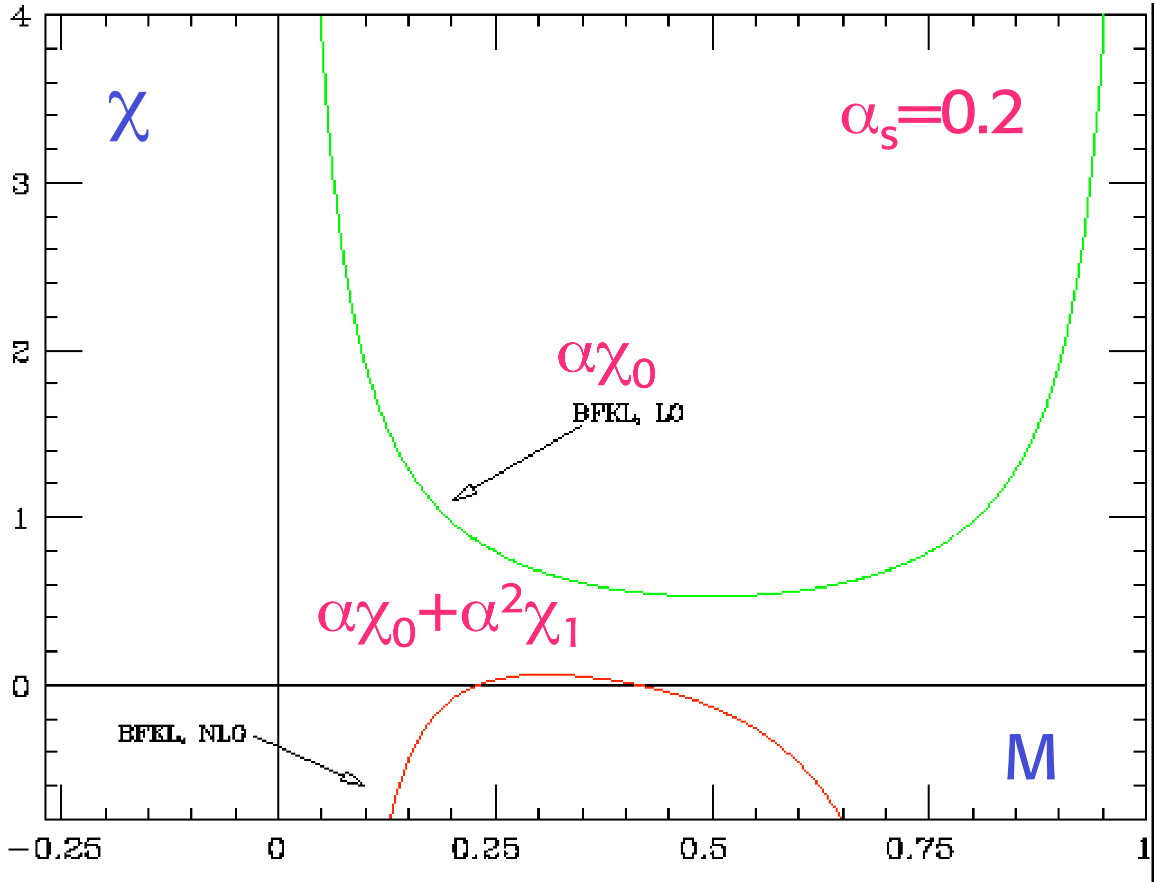
$$\lambda_0 = \alpha\chi_0\left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi} 4\ln 2 = \alpha c_0 \sim 2.65\alpha \sim 0.5$$

It corresponds to (for x->0):

$xP(x) \sim x^{-\lambda_0}$ Too hard, not supported by data

But the NLO terms are very large

χ_1 totally overwhelms χ_0!!



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In the region of t and x where both

$$\frac{d}{dt}G(N,t) = \gamma(N,\alpha)G(N,t)$$

$$\frac{d}{d\xi}G(\xi,M) = \chi(M,\alpha)G(\xi,M)$$

are approximately valid, the "duality" relation holds:

$$\chi(\gamma(\alpha,N),\alpha) = N$$

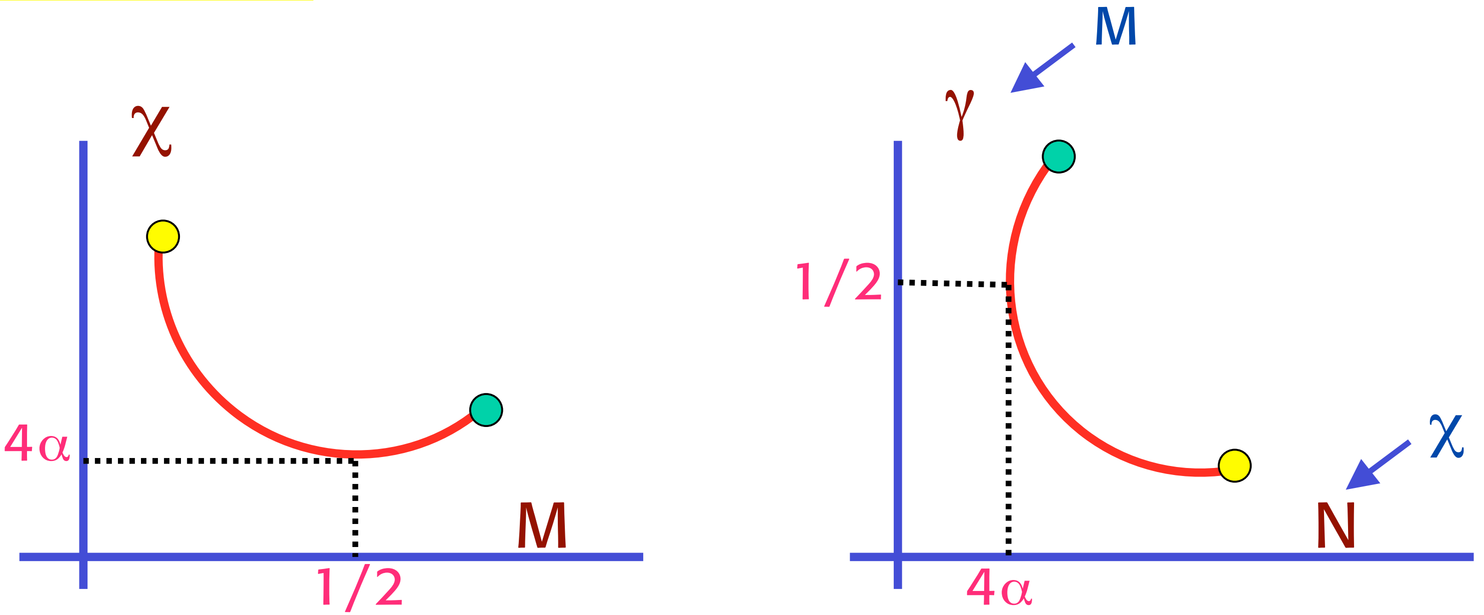
We skip the proof

Note: γ is leading twist while χ is all twist.

Still the two perturbative exp.ns are related and improve each other.

Non perturbative terms in χ correspond to power or exp. suppressed terms in γ.

$\chi(\gamma(N)) = N$



Example: if $\chi(M,\alpha) = \alpha\left[\frac{1}{M} + \frac{1}{1-M}\right]$ ⟹

⟹ $\alpha\left[\frac{1}{\gamma} + \frac{1}{1-\gamma}\right] = N$ ⟹ $\gamma = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4\alpha}{N}}\right]$

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For example at 1-loop: $\chi_0(\gamma_s(\alpha,N))=N/\alpha$

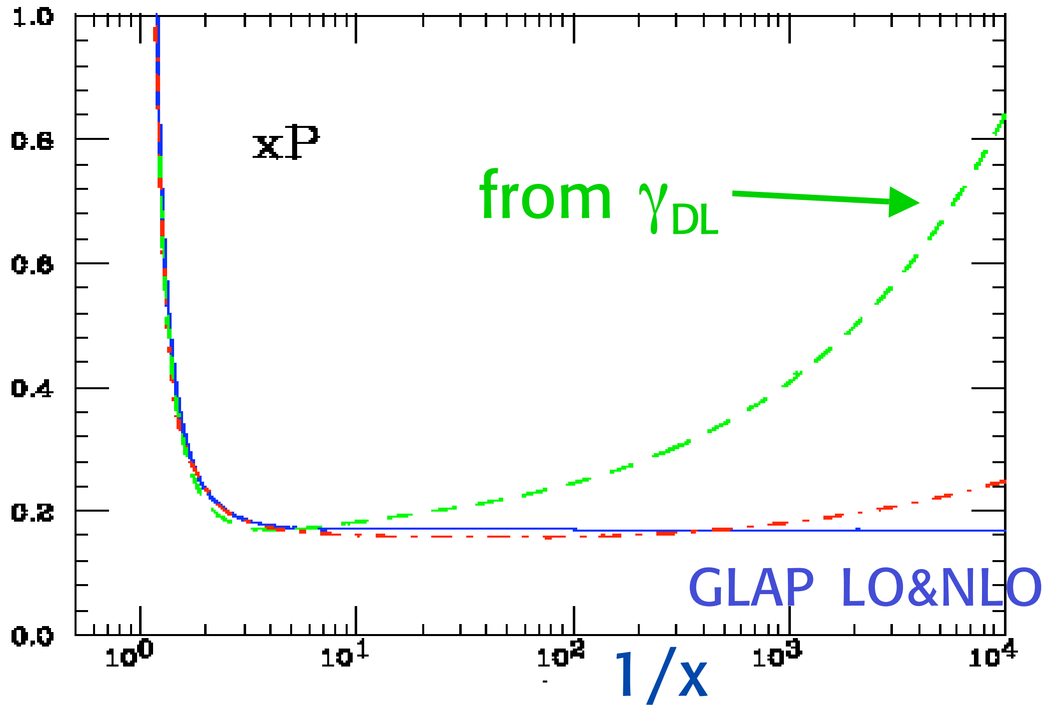
χ_0 improves γ by adding a series of terms in $(\alpha/N)^n$:

$\chi_0 \to \gamma_s\left(\frac{\alpha}{N}\right)$ $\qquad\qquad$ $\gamma_s\left(\frac{\alpha}{N}\right) = \sum_k c_k\left(\frac{\alpha}{N}\right)^k$

$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \ldots \text{ -double count.}$$

This is the naive result from GLAP+(LO)BFKL

The data discard such a large raise at small x



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Similarly it is very important to improve χ by using γ_{1l}

Near M=0, $\chi_0 \sim 1/M$, $\chi_1 \sim -1/M^2$

Duality + momentum cons. ($\gamma(\alpha, N=1)=0$)

$$\chi(\gamma(\alpha, N), \alpha) = N \longrightarrow \chi(0, \alpha) = 1$$

$$\lim_{M\to 0} \chi(M, \alpha) \approx \frac{\alpha}{M+\alpha}$$

$$\begin{cases} \gamma(\chi(M)) = M \rightarrow \gamma_{1l} \Rightarrow \chi_s\left(\frac{\alpha}{M}\right) \\ \chi_s\left(\frac{\alpha}{M}\right) = \sum_k d_k \left(\frac{\alpha}{M}\right)^k \end{cases}$$

$$\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$

G. Altarelli → Double Leading Expansion

$$\gamma(N,\alpha) = \alpha\cdot\gamma_{1l}(N) + \ldots \sim \alpha\cdot\left[\frac{1}{N} - A(N)\right]$$

Momentum conservation: $\gamma(1,\alpha)=0$ → $A(1)=1$

Duality: $\gamma(\chi(M)) = M$ → $\alpha\cdot\left[\frac{1}{\chi} - A(\chi)\right] = M$ →

→ $\chi = \frac{\alpha}{M+\alpha A(\chi)}$ → $\chi(M\sim 0) \sim \frac{\alpha}{M+\alpha A(1)} \sim \frac{\alpha}{M+\alpha}$

$$\chi_{DL}(M,\alpha) = \alpha\cdot\chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right)$$ +... -double count.

$$\chi_0(M) = \alpha\cdot\left[\frac{1}{M} + 0(M^2)\right]$$

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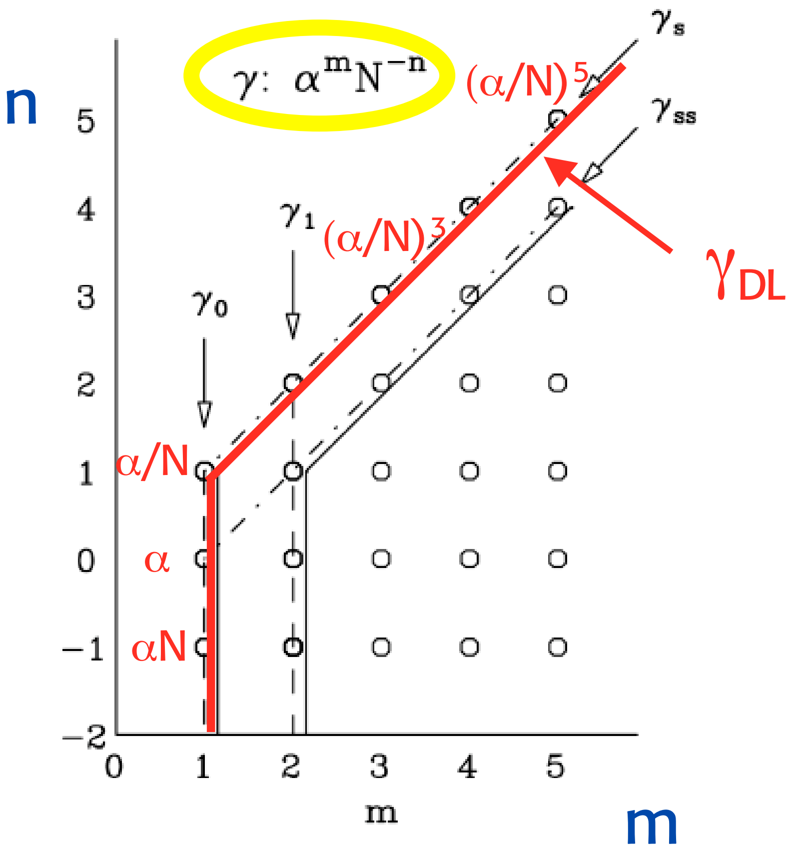
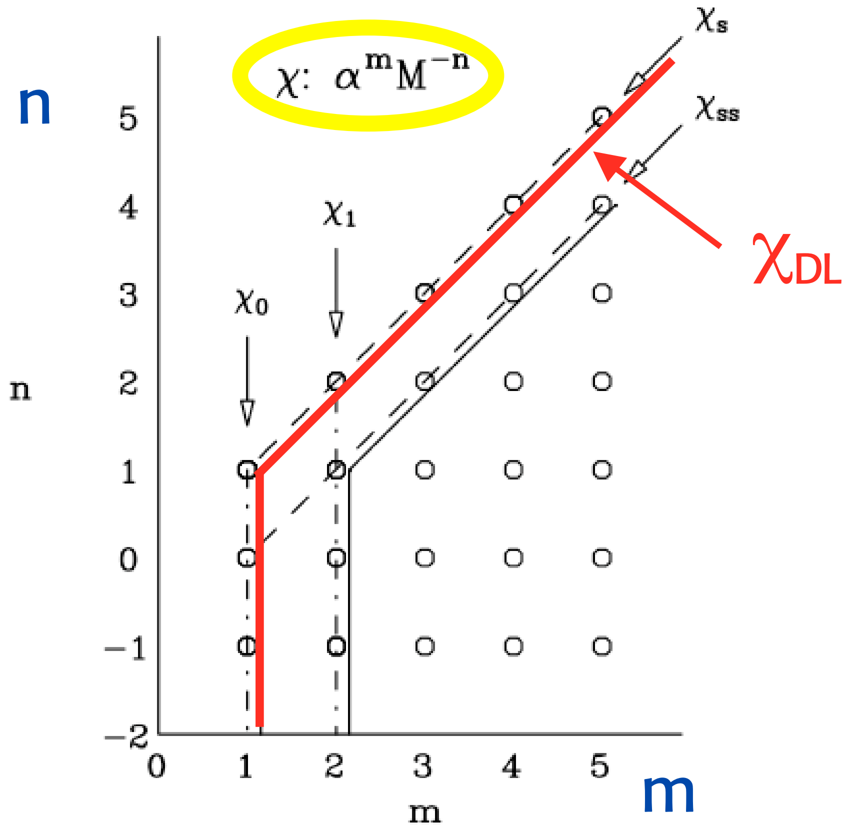
$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \ldots \text{ -double count.}$$

$$\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$





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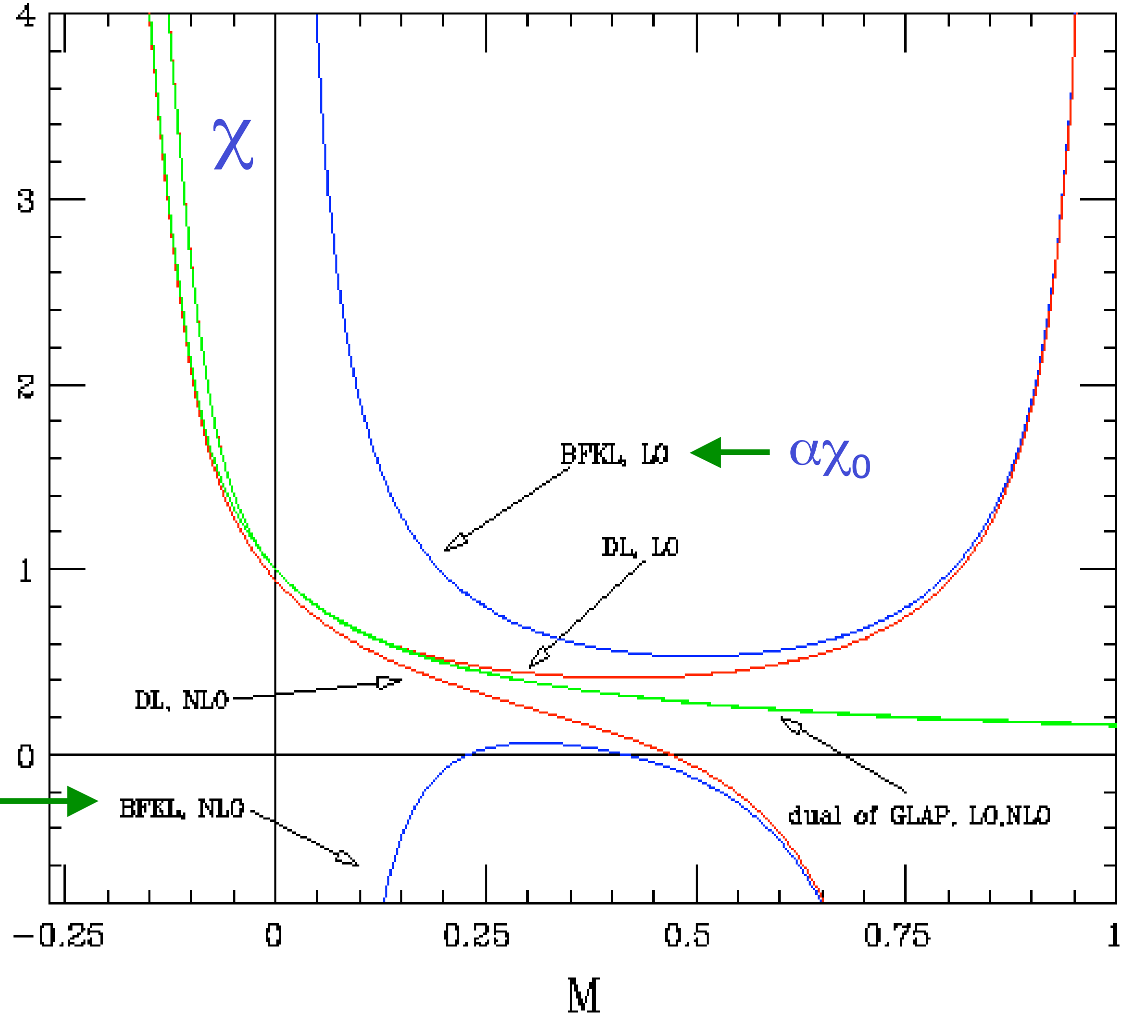
DL, LO: $\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots$ -double count.

BFKL, LO



$\alpha\chi_0$

$\alpha\chi_0 + \alpha^2\chi_1$

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A considerable improvement is obtained by including running coupling effects

Recall that the x-evolution equation was at fixed α

$$\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha) G(\xi, M)$$

In the following:

- Summary of general results
- Airy approximation
- Application to our problem

The implementation of running coupling in BFKL is not simple.
In M-space α becomes an operator

$$\alpha(t) = \frac{\alpha}{1+\beta_0 \alpha t} \Rightarrow \frac{\alpha}{1-\beta_0 \alpha \frac{d}{dM}}$$

In leading approximation:

$$\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha) G(\xi, M)$$

$$\frac{d}{d\xi} G(\xi, M) = \frac{\alpha}{1-\beta_0 \alpha \frac{d}{dM}} \chi_0(M) G(\xi, M)$$

A perturbative expansion in β_0 leads to validity of duality with modified χ and γ:

$$\Delta\chi_1(M) = \beta_0 \frac{\chi_0''(M)\chi_0(M)}{2\chi_0'(M)} \qquad \Delta\gamma_{ss}(N) = -\beta_0 \frac{\chi_0''(\gamma_s)\chi_0(\gamma_s)}{2\chi_0'^2(\gamma_s)}$$

But this expansion fails near M=1/2: $\chi_0'(1/2)=0$

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By taking a second MT the equation can be written as [F(M) is a boundary condition]

$$\left(1-\beta_0\alpha\frac{d}{dM}\right)NG(N,M)+F(M) = \alpha\chi_0(M)G(N,M)$$

It can be solved iteratively

$$G(N,M) = \frac{F(M)}{N-\alpha\chi_0(M)}+\frac{\alpha\beta_0}{N-\alpha\chi_0(M)}\frac{d}{dM}\frac{F(M)}{N-\alpha\chi_0(M)}+\ldots$$

or in closed form:

$$G(N,M) = H(N,M)+$$

$$+\int_{M_0}^{M}dM'\exp\left[\frac{M-M'}{\beta_0\alpha}-\frac{1}{\beta_0 N}\int_{M'}^{M}\chi_0(M'')dM''\right]\frac{F(M')}{\beta_0\alpha N}$$

H(N,M) is a homogeneous eq. sol. that vanishes faster than all pert. terms and can be dropped.

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The following properties can be proven:

• From G(N,M) we can obtain G(N,t) and evaluate it by saddle point expansion. The perturbative G(N,t) is reproduced and satisfies duality (in terms of modified χ and γ according to the perturbative results singular at $\chi'(1/2)=0$)
and factorisation (no t-dep. from the boundary condition).

• From G(N,M) we can get G(ξ,M). This presents unphysical oscillations when $\chi>0$ for all M.

These problems can be studied by using the Airy expansion:
The asymptotics is fixed by the behaviour of χ near the minimum, where a quadratic form is taken:

$$\chi_{eff}(M) = c + \frac{1}{2}k\left(M - \frac{1}{2}\right)^2$$

Lipatov; Collins,Kwiecinski;
Thorne; Ciafaloni, Taiuti,Mueller

G.A., R. Ball, S.Forte, hep-ph/0109178 (NPB 621,359)

For a quadratic kernel the explicit solution is

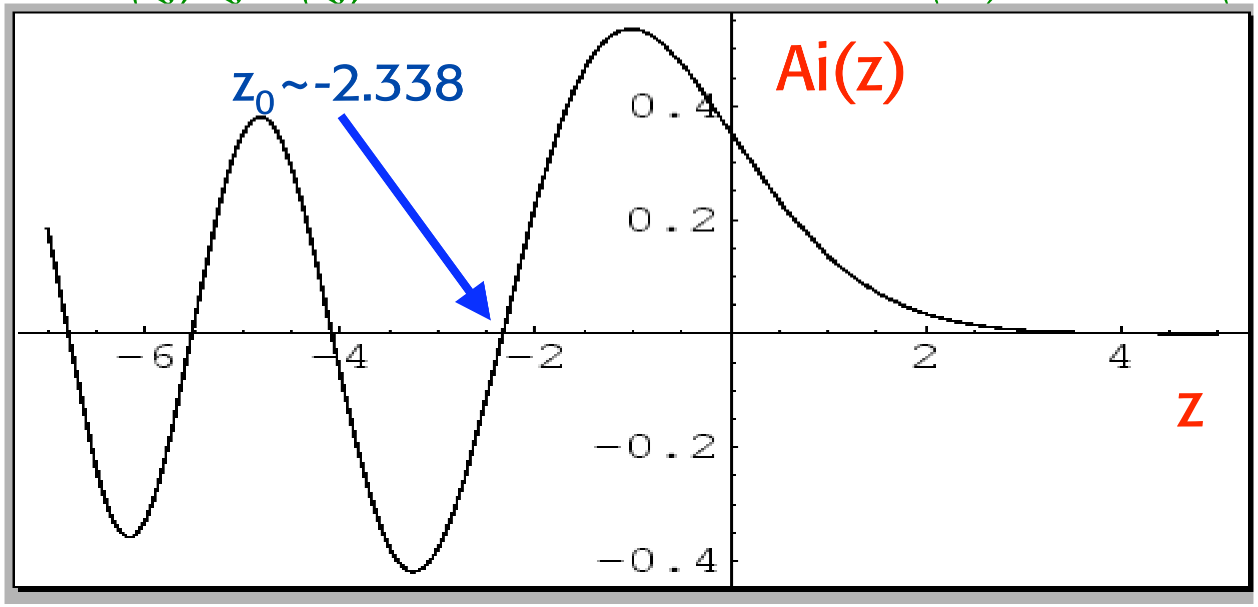
$$G(N,t) = K(N)\exp\frac{1}{2\beta_0\alpha(t)}\cdot Ai[z(\alpha(t),N)]$$

where

$$z(\alpha(t),N) = \left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}}\cdot\frac{1}{\beta_0}\cdot\left[\frac{1}{\alpha(t)}-\frac{c}{N}\right]$$

$$K(N) = \exp\frac{-1}{2\beta_0\alpha}\cdot\left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}}\cdot\frac{1}{\pi N}$$

$Ai''(z)-zAi(z) = 0$ $Ai(0) = 3^{-2/3}\Gamma(2/3)$



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From $$G(N,t) = K(N)\exp\frac{1}{2\beta_0\alpha(t)} \cdot Ai[z(\alpha(t),N)]$$

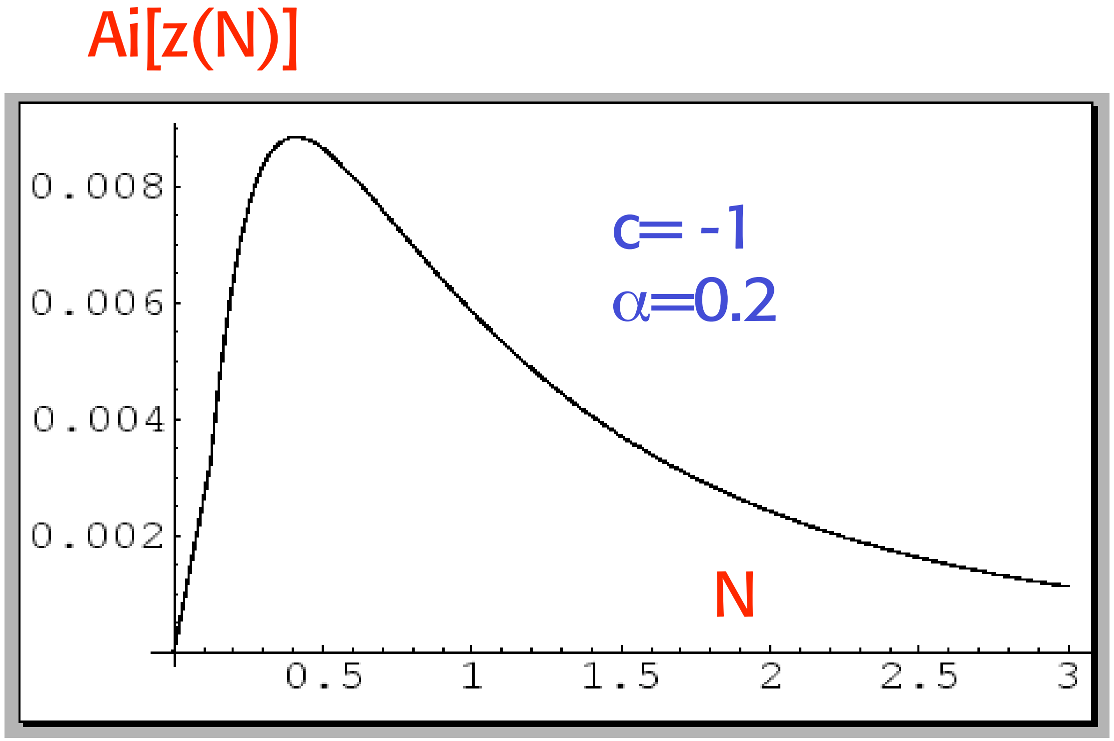
one obtains G(x,t) by inv. MT

$$G(\xi,t) = \int_{-i\infty}^{+i\infty} e^{N\xi} G(N,t)\frac{dN}{2\pi i}$$

The asymptotics is dominated by the saddle condition:

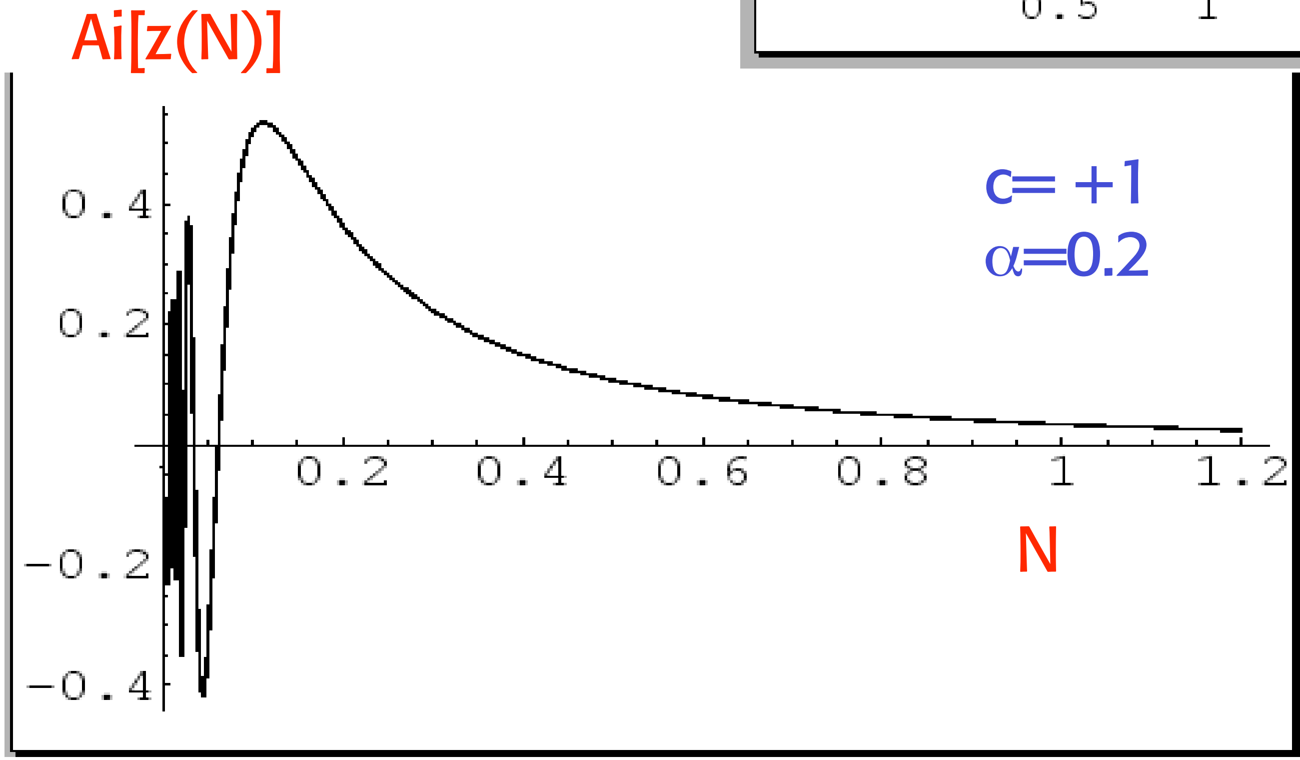
$$\xi = -\frac{1}{Ai[z(\alpha(t),N)]} \cdot \frac{d}{dN} Ai[z(\alpha(t),N)]$$

For c>0 at not too large ξ this is satisfied at large N. When ξ increases N gets smaller. Then oscillations start, d/dN changes sign and the real saddle is lost.
G(ξ,t) starts oscillating, in agreement with the general analysis.



$$\chi_{eff}(M) = c + \frac{1}{2}k\left(M-\frac{1}{2}\right)^2$$

$$z(\alpha(t), N) = \left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}} \cdot \frac{1}{\beta_0} \cdot \left[\frac{1}{\alpha(t)} - \frac{c}{N}\right]$$



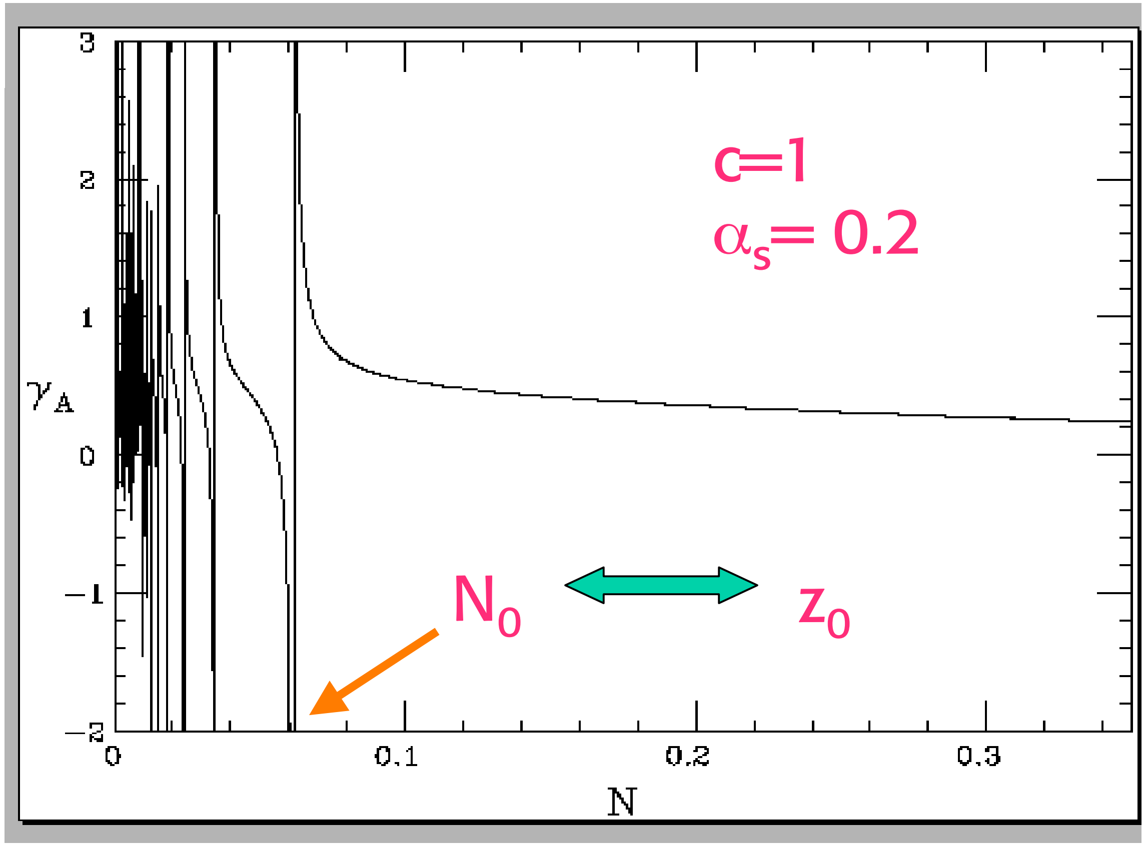
The dual anom. dim. γ_A is given by

$$\gamma_A(\alpha(t), N) = \frac{d}{dt}\log G(N,t) = \frac{1}{2} + \left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}} \frac{Ai'(z)}{Ai(z)}$$

z large $\longrightarrow$

$$\frac{1}{2} - \sqrt{\frac{2}{k}\left(\frac{N}{\alpha(t)} - c\right)} - \frac{1}{4} \cdot \frac{\beta_0 \alpha}{1 - \frac{\alpha}{N}c} + \ldots$$

$$z(\alpha(t), N) = \left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}} \cdot \frac{1}{\beta_0} \cdot \left[\frac{1}{\alpha(t)} - \frac{c}{N}\right]$$



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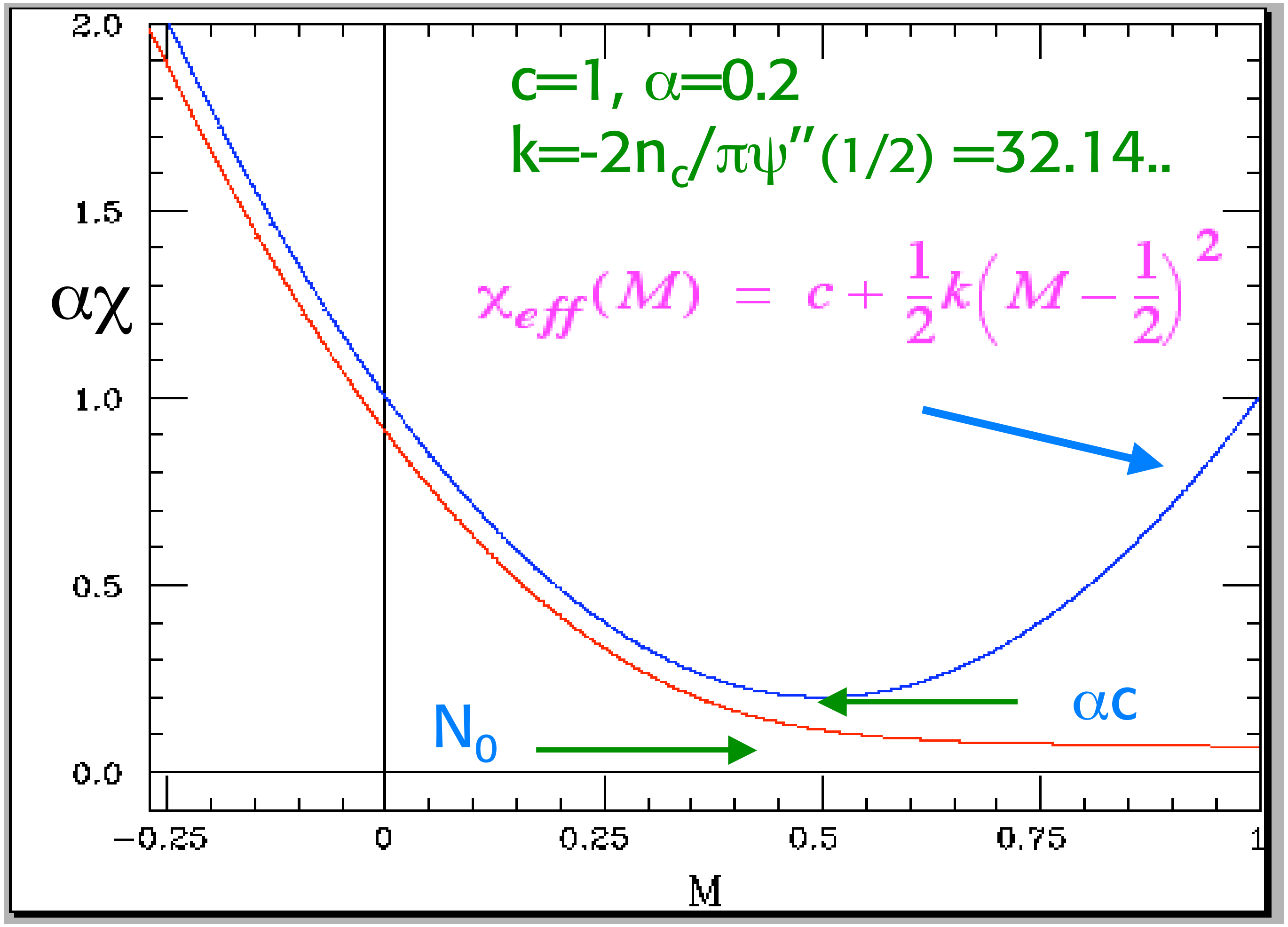
The effect of running on χ is a softer small-x behaviour

$xP \sim x^{-\lambda}$ → $xP \sim x^{-N_0}$



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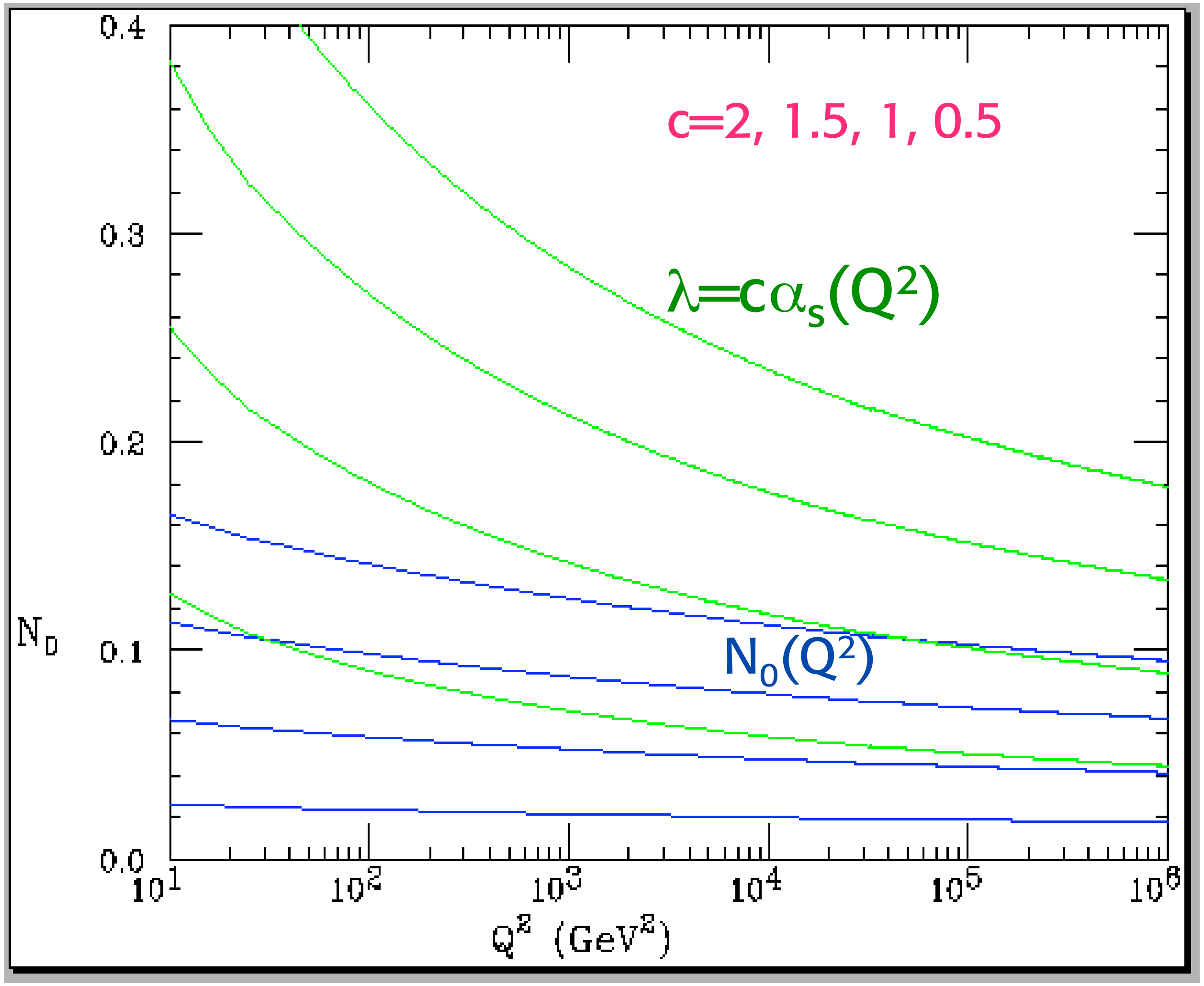
As an effect of running, the small-x asymptotics is much softened:

$xP \sim x^{-\lambda}$ → $xP \sim x^{-N_0}$



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The Airy result is free of the perturbative β_0 singularities.

At NLL order we can add the full γ_A and subtract its large N limit:

$\chi_0 \to \gamma_s$ $\chi_1 \to \gamma_{ss}$

$$\gamma(\alpha, N) \approx \gamma_s\left(\frac{\alpha}{N}\right) + \alpha\gamma_{ss}\left(\frac{\alpha}{N}\right) + \alpha\Delta\gamma_{ss}\left(\frac{\alpha}{N}\right) +$$

$$+ \gamma_A(\alpha, N) - \frac{1}{2} + \sqrt{\frac{2}{k}\left(\frac{N}{\alpha} - c\right)} + \frac{1}{4} \cdot \frac{\beta_0\alpha}{1 - \frac{\alpha}{N}c}$$

The last term cancels the sing. of $\alpha\Delta\gamma_{ss}$
($N=\alpha c$ corresponds to $M=1/2$)

The goal of our recent work is to use these results to construct a relatively simple, closed form, improved anom. dim. $\gamma_I(\alpha,N)$ or splitting funct.n $P_I(\alpha,x)$

G.A., R. Ball, S.Forte, hep-ph/ 0306156 (NPB 674,459), 0310016

$P_I(\alpha,x)$ should

- reduce to pert. result at large x
- contain BFKL corr's at small x
- include running coupling effects (Airy)
- be sufficiently simple to be included in fitting codes

and of course

- closely follow the trend of the data

Improved anomalous dimension

1st iteration: optimal use of $\gamma_{1l}(N)$ and $\chi_0(M)$

$$\gamma_I(\alpha, N) = \alpha\gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) - \frac{\alpha n_c}{\pi N} + $$
$$+\gamma_A(\alpha, N) - \frac{1}{2} + \sqrt{\frac{2}{k_0}\left(\frac{N}{\alpha} - c_0\right)} + \frac{1}{4}\beta_0\alpha - \text{mom sub}$$

Properties:

- Pert. Limit α->o , N fixed

$$\gamma_I(\alpha, N) \longrightarrow \alpha\gamma_{1l}(N) + o(\alpha^2)$$

- Limit α->o , α /N fixed

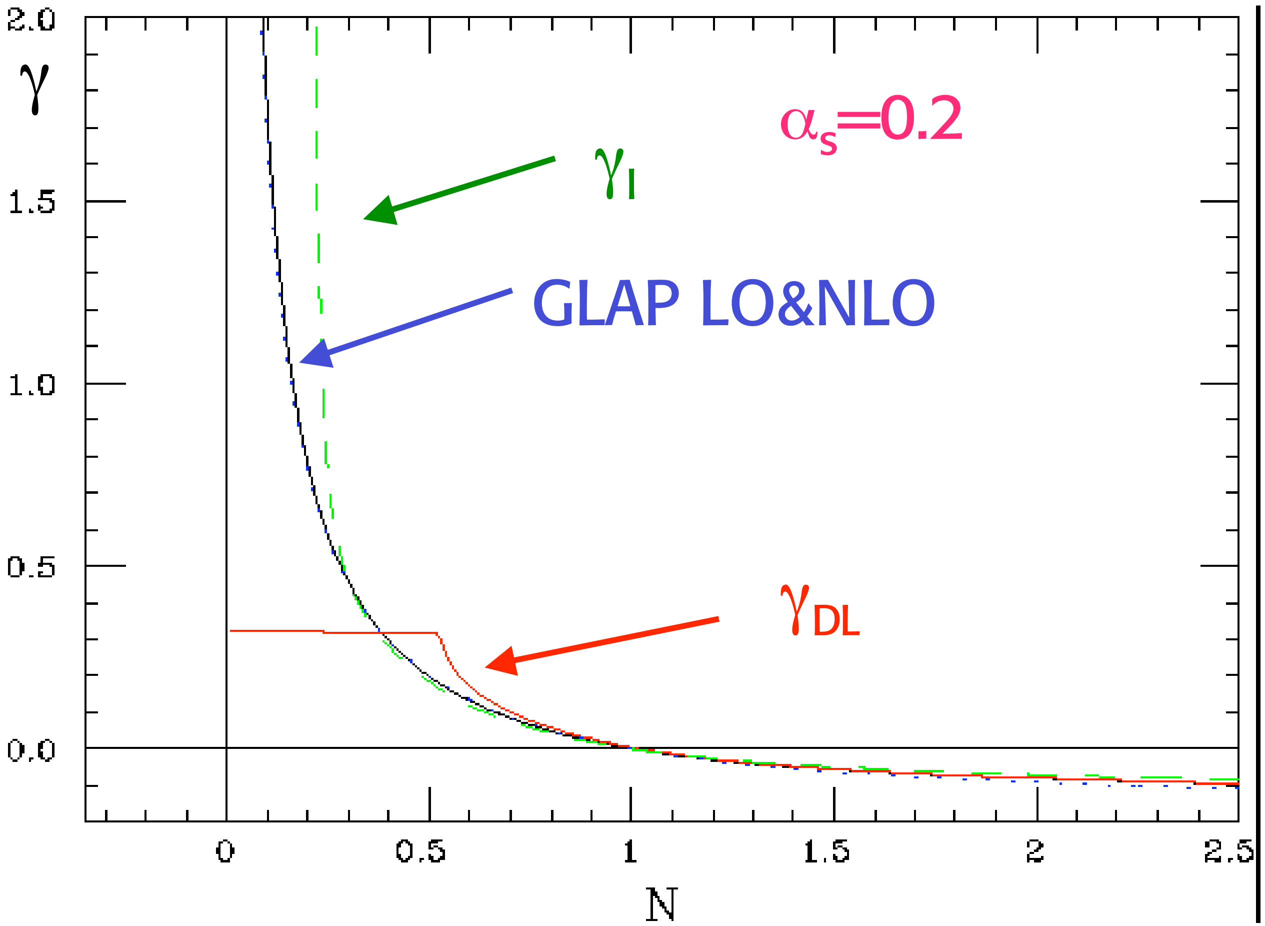
$$\gamma_I(\alpha, N) \longrightarrow \alpha\gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) - \frac{\alpha n_c}{\pi N} + o(\alpha\, \alpha/N)$$

$\alpha\gamma_{1l}(N)$ → Pole in 1/N

$\gamma_s\left(\frac{\alpha}{N}\right)$ → Cut with branch in $\alpha\, c_0$

the Airy term cancels the cut and introduces a pole at $N=N_0$

- $$\gamma_I(\alpha, N) = \alpha\gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) - \frac{\alpha n_c}{\pi N} +$$
$$+\gamma_A(\alpha, N) - \frac{1}{2} + \sqrt{\frac{2}{k_0}\left(\frac{N}{\alpha} - c_0\right)} + \frac{1}{4}\beta_0\alpha - \text{mom sub}$$
- $$\gamma_{DL}(\alpha,N) = \alpha\gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) - \frac{\alpha n_c}{\pi N} - \text{mom sub}$$



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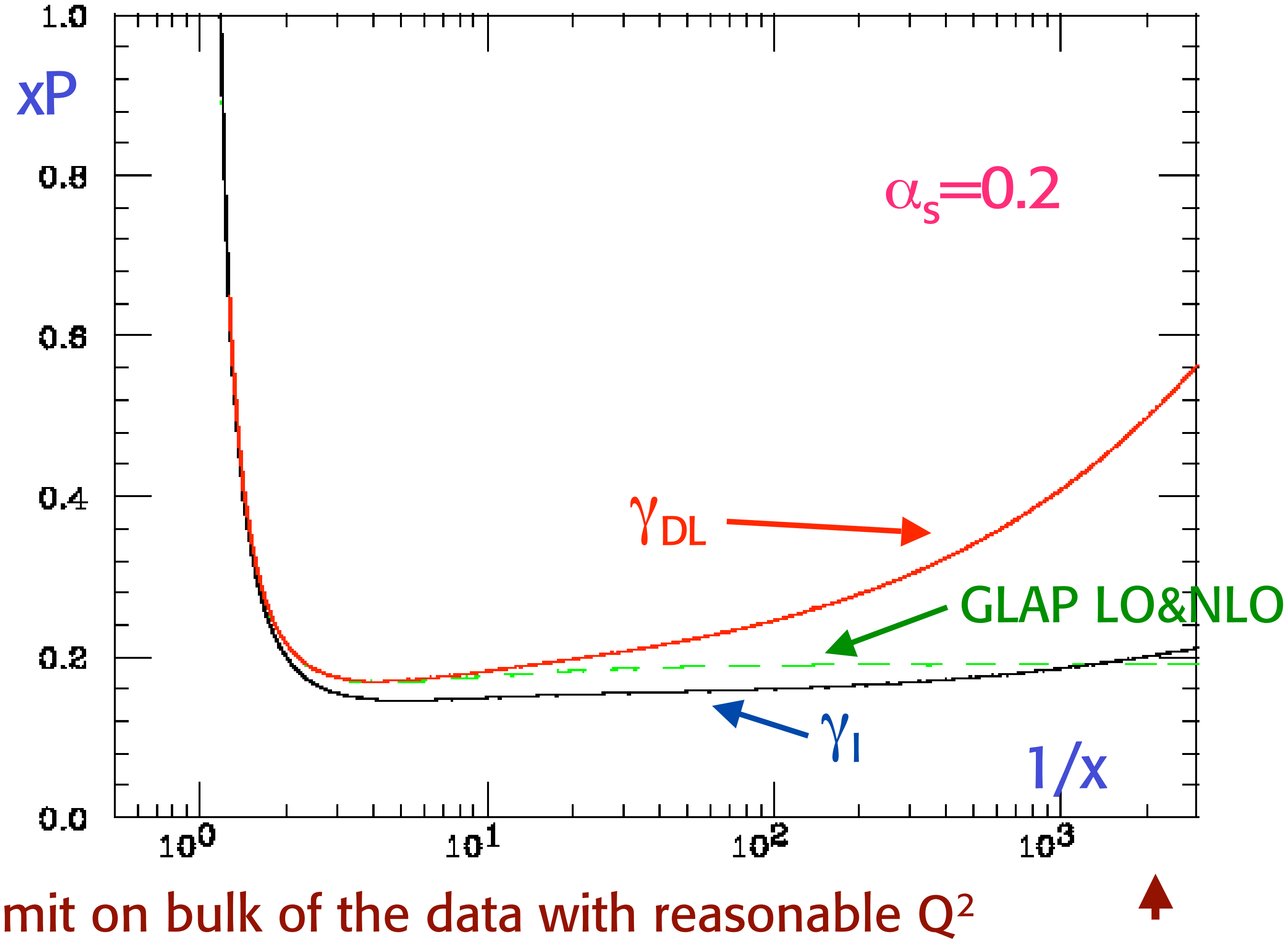
Here is the same plot for the corresponding splitting fncts.

Note: for α_s=0.2 the pole in GLAP is ~0.191/N
while the pole in γ_I is ~0.014/(N-N$_0$)
(only visible at very small x)



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Limit on bulk of the data with reasonable Q^2

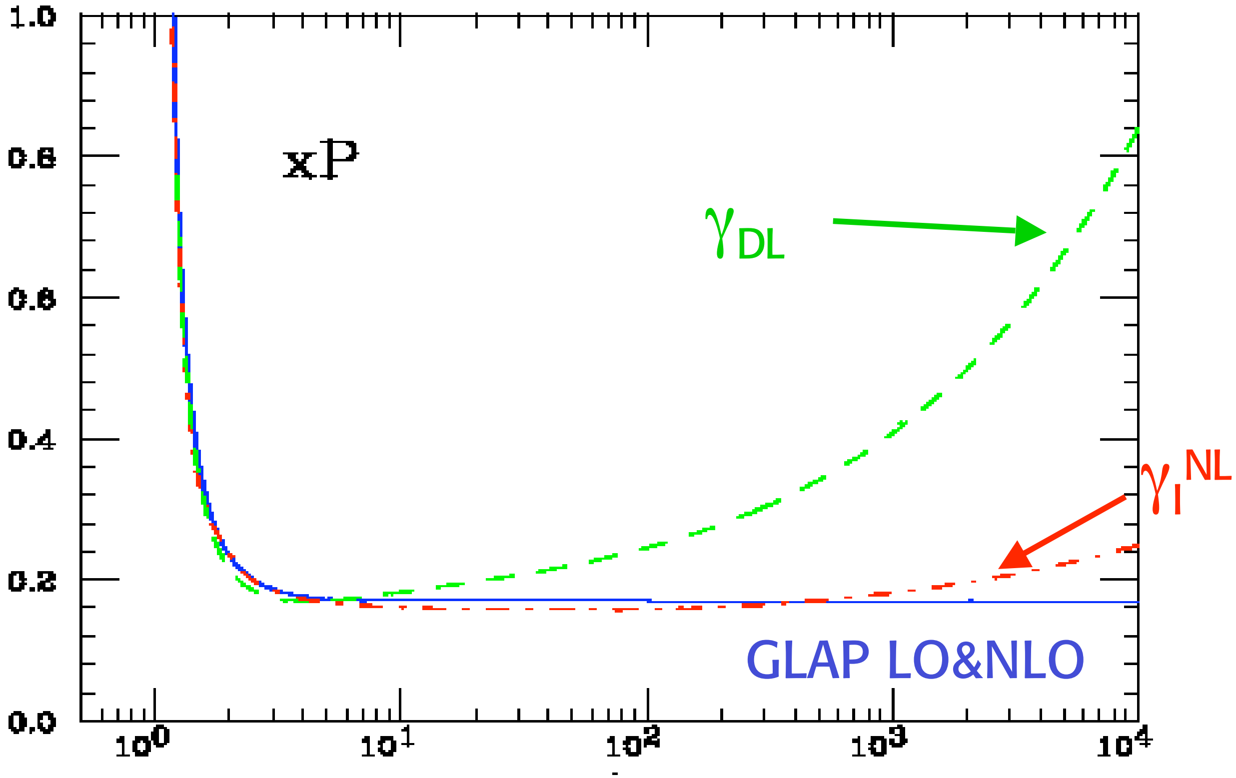
We can add the 2-loop perturbative result γ_{2l}:

$$\gamma_I^{NL}(\alpha, N) = \alpha\gamma_{1l}(N) + \alpha^2\gamma_{2l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) - \frac{\alpha n_c}{\pi N} +$$

$$+\gamma_A(\alpha, N) - \frac{1}{2} + \sqrt{\frac{2}{k_0}\left(\frac{N}{\alpha} - c_0\right)} +$$

$$+\frac{1}{4}\beta_0\alpha\left(1 + \frac{\alpha}{N}c_0\right) - \mathrm{mom.sub}$$

This is our main result



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Preview of our next paper (in advanced preparation)

BFKL kernel symmetric in k_1^2, k_2^2 (gluon virtualities)

Mellin variable $\left(\frac{s}{k_1 k_2}\right)$ in DIS $Q^2 \gg k^2$ $\left(\frac{s}{Q^2}\right)$

The symm. and the DIS kernels are related by:

$\chi \equiv \chi_{DIS}$

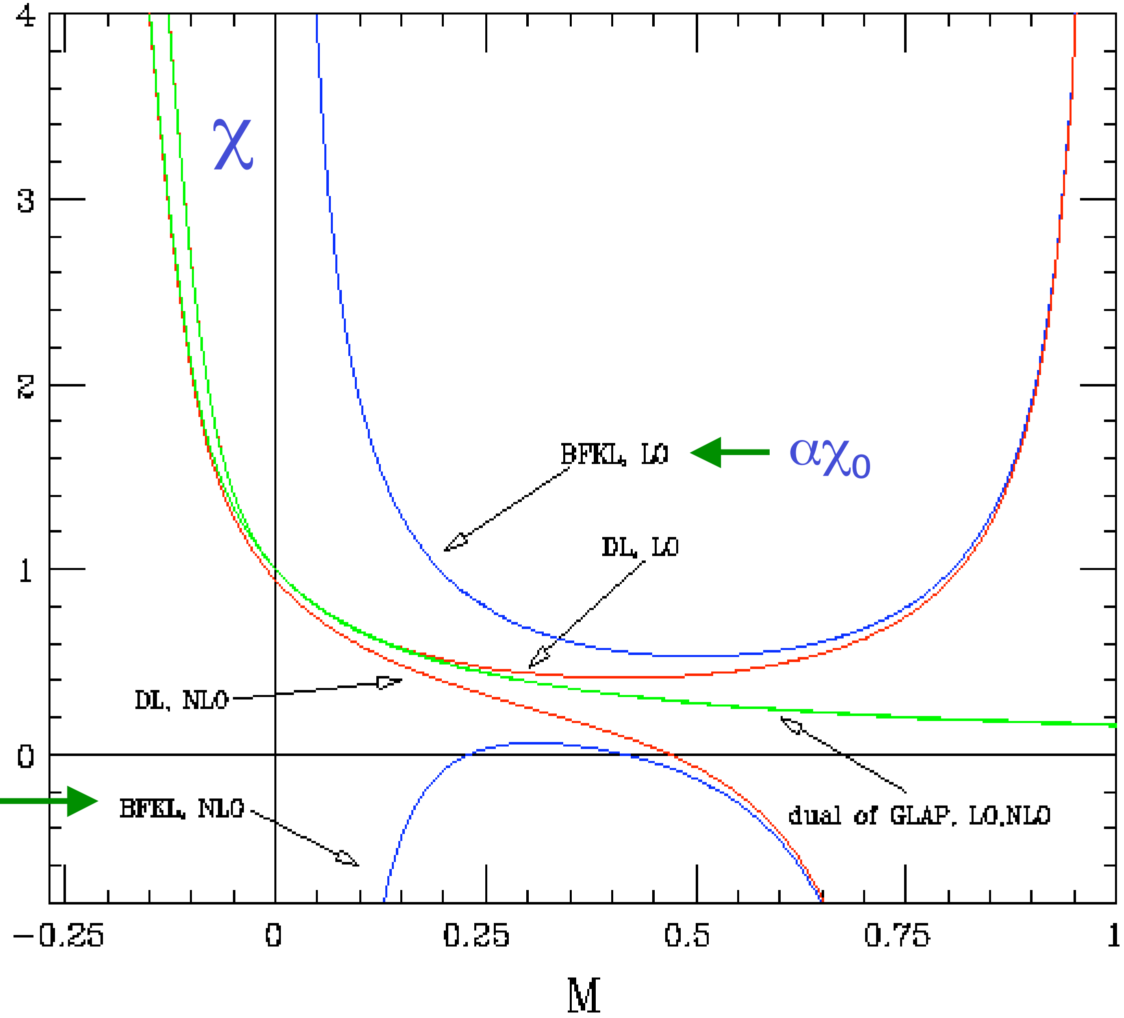
$$\chi_{DIS}\left(M + \frac{N}{2}\right) = \chi_{SYMM}(M)$$

Fadin, Lipatov

We use the underlying symmetry to improve the DL expansion (both 1/M and 1/(1-M) fixed) Ciafaloni, Salam

DL, LO: $\chi_{DL}(M,\alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots$ -double count.

BFKL, LO



$\alpha\chi_0 + \alpha^2\chi_1$

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LO

Naive symmetrization:

$$\chi_0(M) = [2\psi(1) - \psi(M) - \psi(1-M)]$$

$$\chi_s\left(\frac{\alpha}{M}\right) + \chi_s\left(\frac{\alpha}{1-M}\right) + \alpha\left[\chi_0(M) - \frac{1}{M} - \frac{1}{1-M}\right]$$

Not relevant:
χ_{DIS} not symm.!

In "symmetric" variables:

$$\chi_s\left(\frac{\alpha}{M+\frac{N}{2}}\right) + \chi_s\left(\frac{\alpha}{1-M+\frac{N}{2}}\right) + \alpha[\psi(1) + \psi(1+N) - \psi\left(M+\frac{N}{2}\right) - \psi\left(1-M+\frac{N}{2}\right) - \frac{1}{M+\frac{N}{2}} - \frac{1}{1-M+\frac{N}{2}}]$$

Note: N determined self-consistently

The actual function for DIS:

$$\chi_s\left(\frac{\alpha}{M}\right) + \chi_s\left(\frac{\alpha}{1-M+N}\right) + \alpha[\psi(1) + \psi(1+N) - \psi(M) - \psi(1-M+N) - \frac{1}{M} - \frac{1}{1-M+N}]$$

to be called $\chi_{DLSymm,LO}$ (after mom. cons. subtraction)

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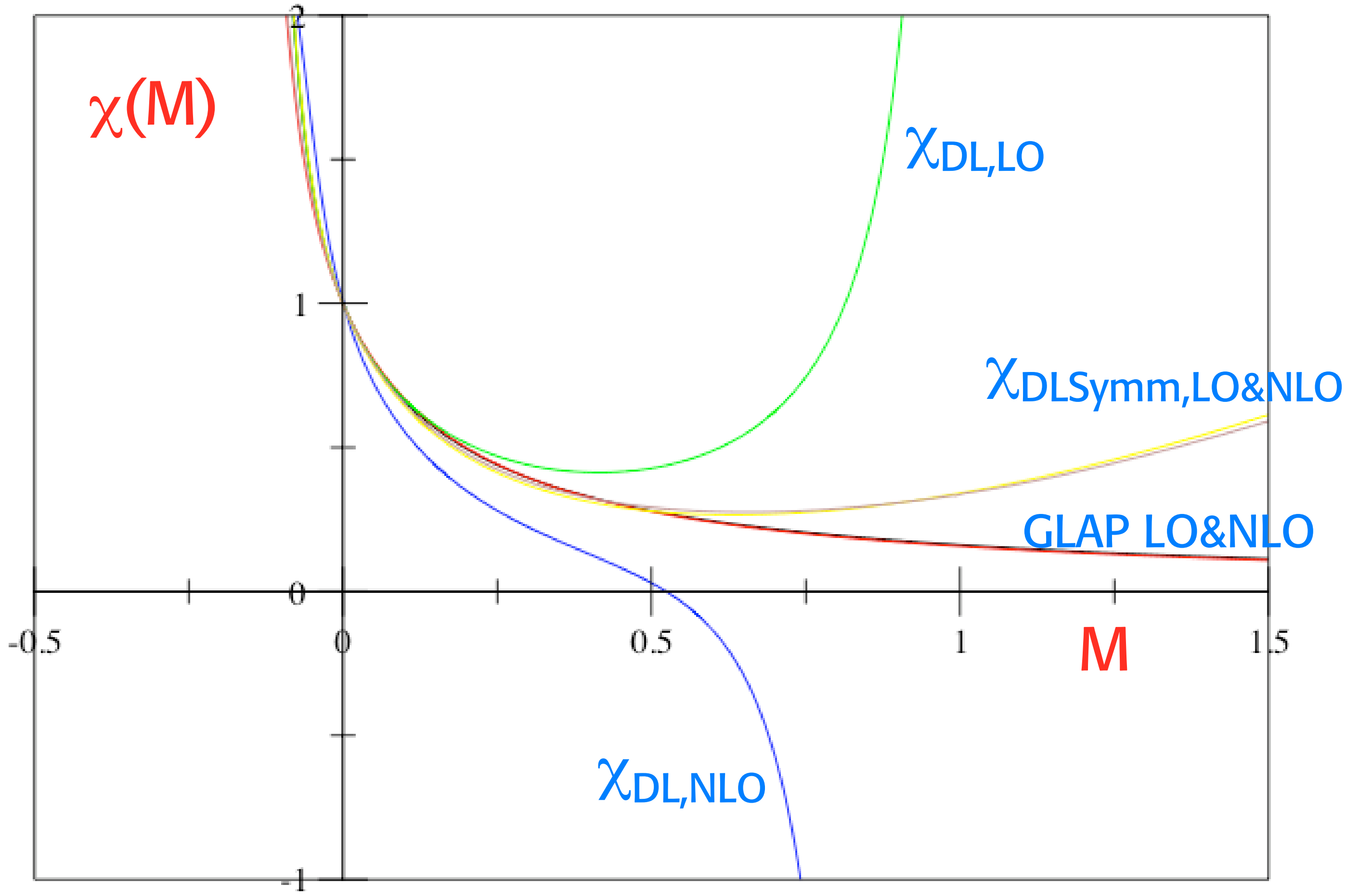
Similarly for NLO



We use $\chi_{DLSymm,LO}$ for implementing the Airy procedure:

Theorem: c, k are the same as for "symmetric" variables

$$\chi_{eff}(M) = c + \frac{1}{2}k\left(M - \frac{1}{2}\right)^2$$

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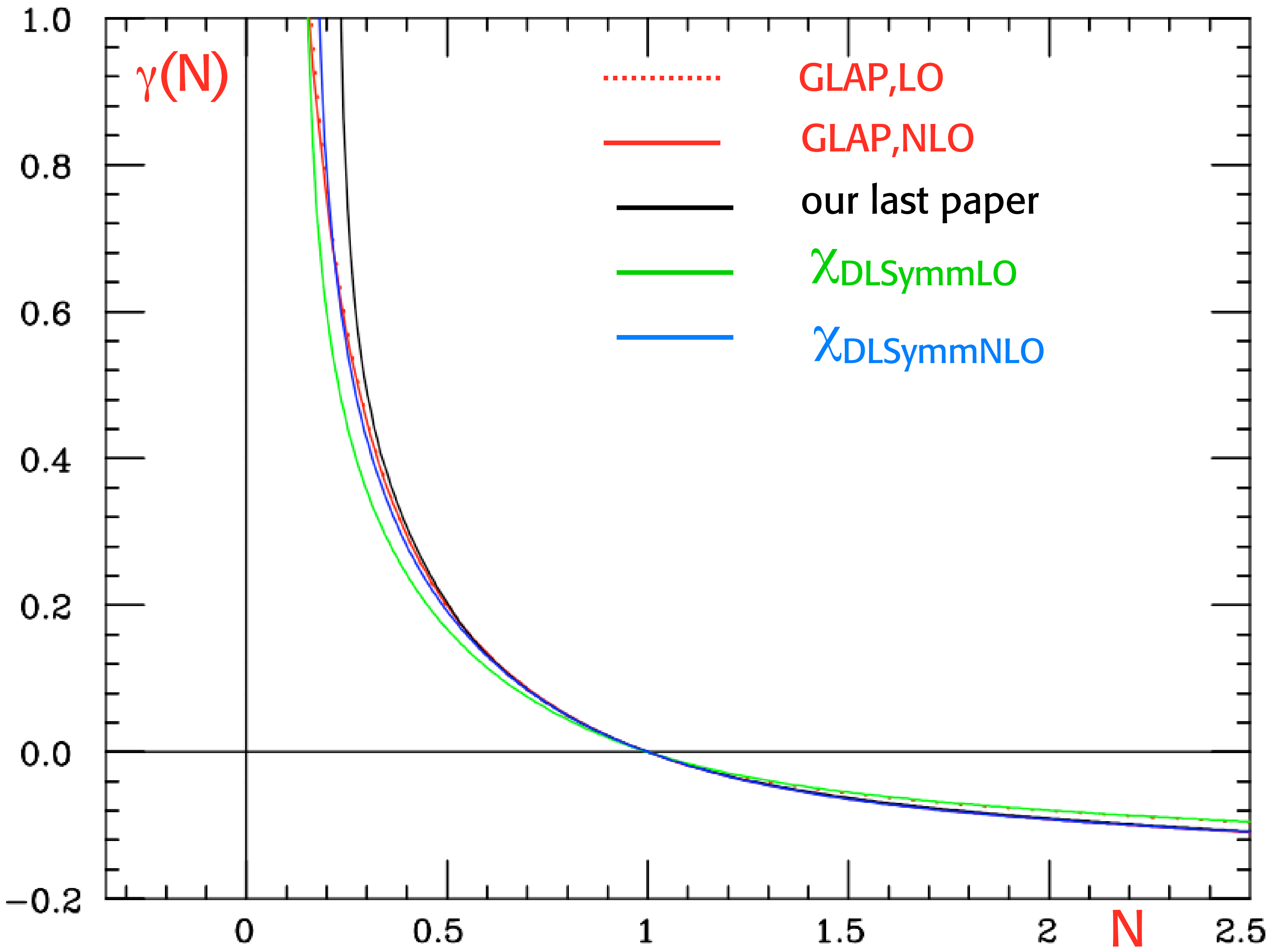

γ(N)
GLAP,LO
GLAP,NLO
our last paper
χDLSymmLO
χDLSymmNLO
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
0.5
1
1.5
2
2.5
N

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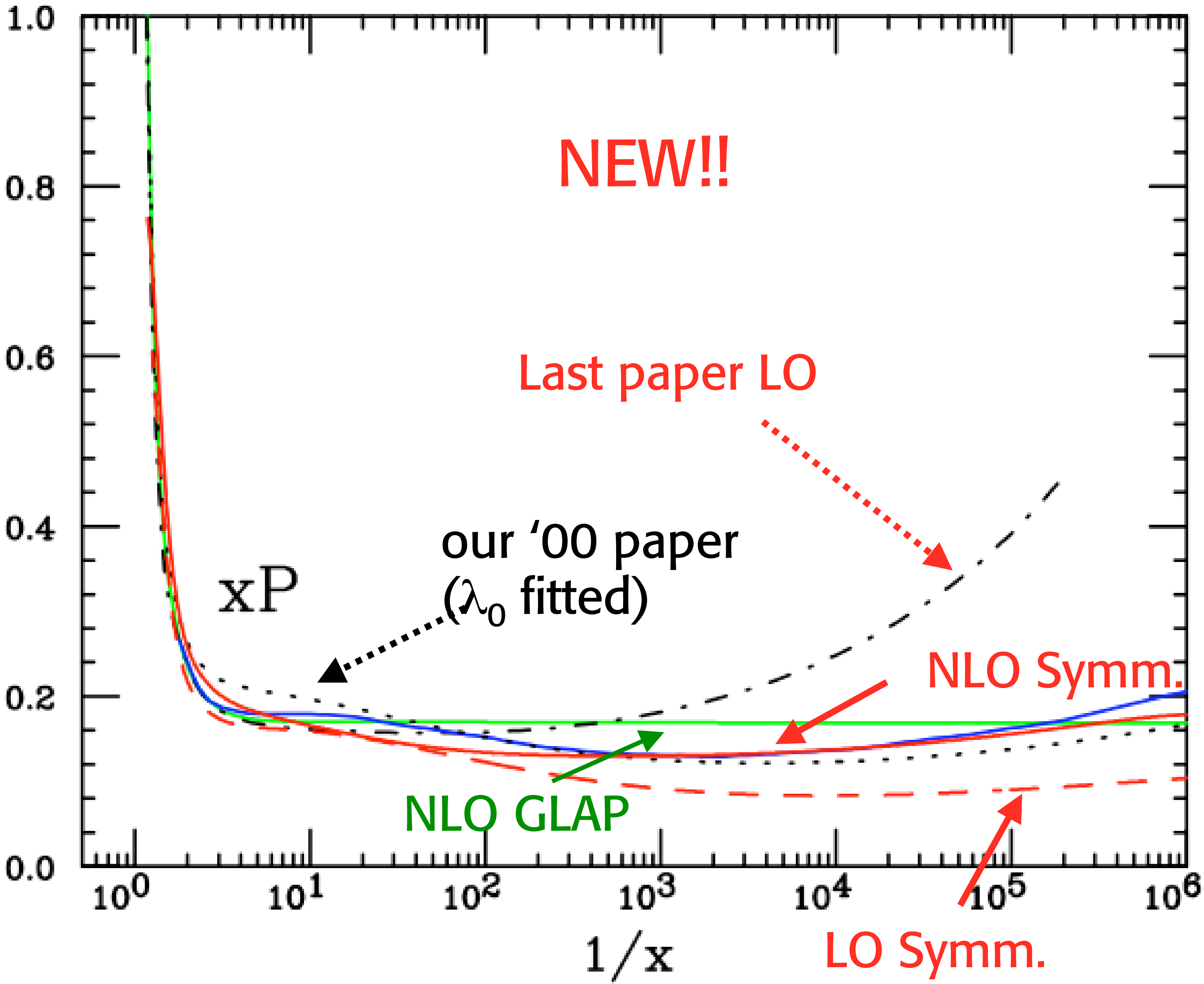
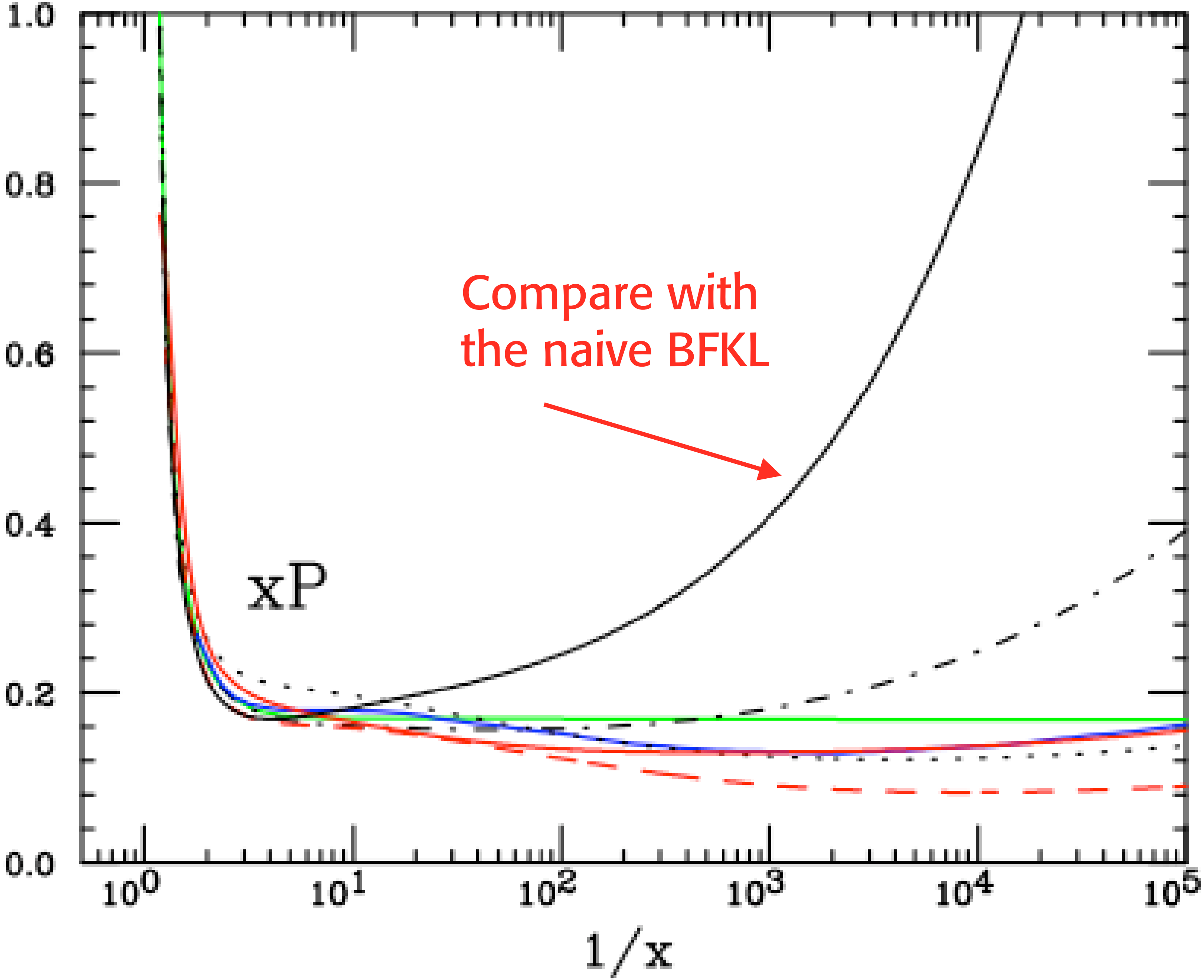
Here are the results for splitting functions (n_f=0)



G. /


Compare with
the naive BFKL
xP
1.0
0.8
0.6
0.4
0.2
0.0
10^0
10^1
10^2
10^3
10^4
10^5
1/x

G. Alt

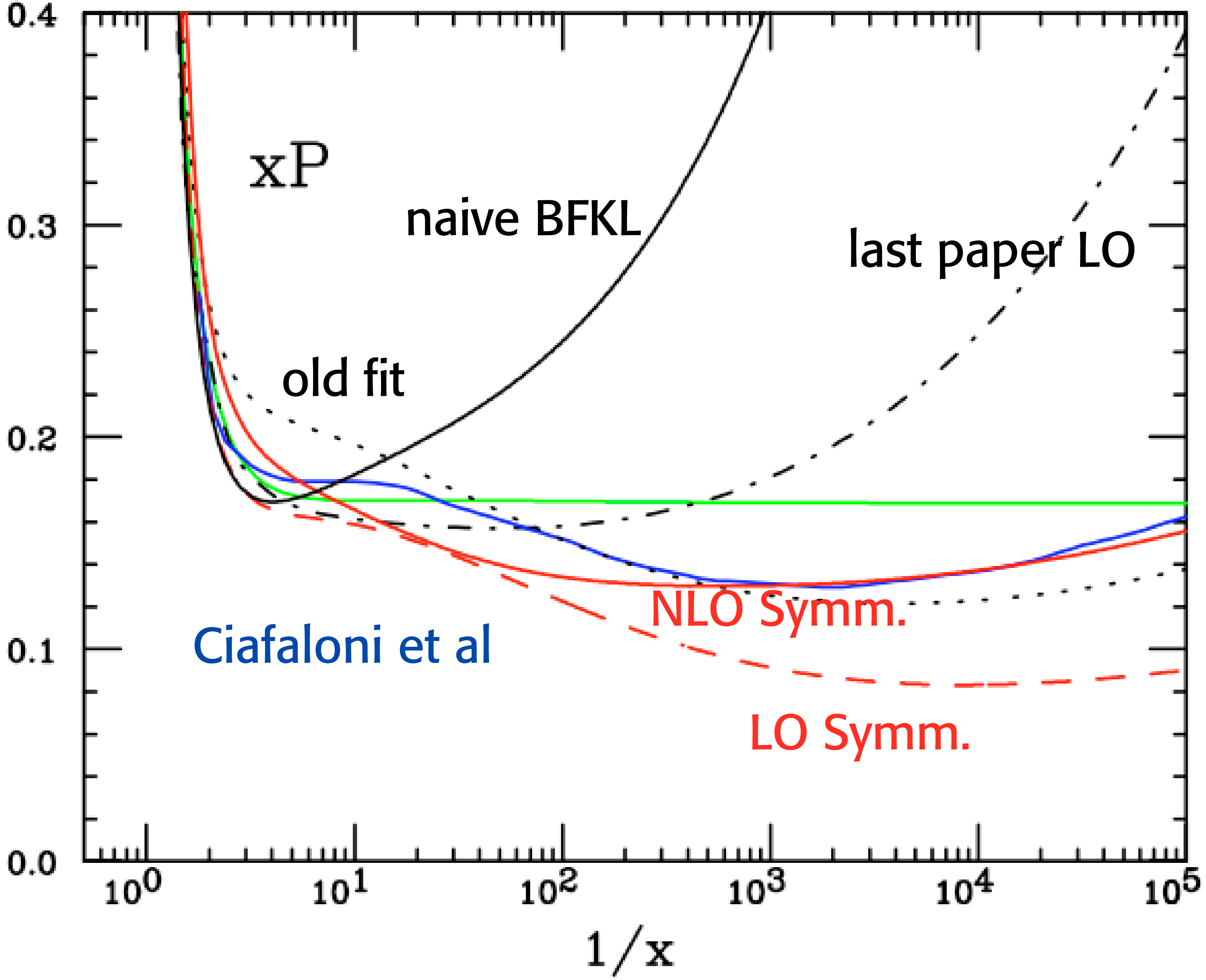
Our most important competitors:

Ciafaloni, Colferai, Salam, Stasto hep-ph/0307188. Also Thorne

Same physics: regularisation of M=0 pole in χ (and of M=1 pole using symmetrisation) and running coupling effects

Different resummation technique, no Airy expansion (num. sol of evol eqn.), and they include χ_1 but not γ_{2l}

Same curves on an expanded scale



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Summary and Conclusion

- We have constructed an improved an. dim. that reduces to the pert. result at large x and incorporates BFKL with running coupling effects at small x.
- We think we now know how to get the best use of the joint info from γ and χ.
- Properly introducing running coupling effects in the LO softens the asympt. small-x behaviour as shown by the data.

A clearer picture of the matching of GLAP and BFKL is finally emerging

G. Altarelli