1 Reweighted cross sections

In this note I shall discuss the problems related to the reweighting of differential distributions. The way in which such distributions are computed is irrelevant, and therefore what follows applies to perturbative fixed-order computations and Monte Carlo simulations, although I shall refer mainly to the latter. In order to simplify the discussion, and to perform all of the necessary calculations analytically, I consider the case of two kinematic variables with the following ranges

$$p\,,\quad 0\le p\le 1\,;\qquad x\,,\quad 0\le x\le 1\,. \tag{1}$$

I assume that the doubly-differential cross section is

$$\begin{aligned}\frac{d\sigma^T}{dxdp} &= \frac{2}{\Gamma(a-1,1)} \qquad\qquad (2)\\ &\times\ \frac{e^{-1/p}}{p^a}\left[\Theta\left(\frac{1}{2}-x\right)\left(\frac{1}{2}+\alpha\frac{1-p}{2}\right)+\Theta\left(x-\frac{1}{2}\right)\left(\frac{1}{2}-\alpha\frac{1-p}{2}\right)\right],\end{aligned}$$

where the superscript T means "true". In eq. (2), a and α are free parameters; I assume that

$$a\ge 1\,;\qquad -1\le\alpha\le 1\,, \tag{3}$$

where the latter condition implies that the cross section is positive definite. It is a matter of simple algebra to compute the single-inclusive and total cross sections

$$\frac{d\sigma^T}{dp} = \frac{1}{\Gamma(a-1,1)}\frac{e^{-1/p}}{p^a}\,, \tag{4}$$

$$\begin{aligned}\frac{d\sigma^T}{dx} &= \Theta\left(\frac{1}{2}-x\right)\left(1+\alpha-\alpha\frac{\Gamma(a-2,1)}{\Gamma(a-1,1)}\right)\\ &+\ \Theta\left(x-\frac{1}{2}\right)\left(1-\alpha+\alpha\frac{\Gamma(a-2,1)}{\Gamma(a-1,1)}\right),\end{aligned} \tag{5}$$

$$\sigma^T = 1\,. \tag{6}$$

The total rate is equal to one thanks to the prefactor that contains the Γ function in eq. (2), which has actually been chosen for this purpose. Eqs. (2)–(5) suggests that p is a rescaled (transverse) momentum, such as the transverse momentum of the WW pair; the larger the parameter a, the more steeply falling the distribution. The nature of the x doesn't need to be specified here, since what follows actually applies to *any* observable; in order to simplify the discussion, I assumed that the cross section is a constant in the ranges $x < 1/2$ and $x > 1/2$ (see eq. (5)); the difference between its values in those ranges (i.e. the steepness of $d\sigma^T/dx$) is proportional to α. Notice that x and p are correlated, and that the slope in x is flatter the larger p.

I now want to apply the reweighting procedure of ref. [1] to compute $d\sigma/dx$. The correct answer is that of eq. (5), which in a MC simulation is obtained by filling x bins with the weights computed with eq. (2) for all of the phase-space points (x, p) sampled during the run. However, ref. [1] assumes that the true doubly-differential cross section (i.e., the correct MC simulation) is not available. What is available is a MC simulation which is known to necessitate corrections. In the present simplified approach, this corresponds to a doubly-differential cross sections that I write as follows:

$$\begin{aligned} \frac{d\sigma^W}{dxdp} &= \frac{2}{\Gamma(b-1,1)} \\ &\times \frac{e^{-1/p}}{p^b}\left[\Theta\left(\frac{1}{2}-x\right)\left(\frac{1}{2}+\beta\frac{1-p}{2}\right)+\Theta\left(x-\frac{1}{2}\right)\left(\frac{1}{2}-\beta\frac{1-p}{2}\right)\right], \end{aligned} \tag{7}$$

where the superscript W stands for "wrong". The functional form of eq. (7) is identical to that of eq. (2); this obviously doesn't need to be so, but it simplifies the computations. The two cross sections are different, however, since in general $a \neq b$ and $\alpha \neq \beta$. The single-inclusive "wrong" cross sections can be obtained from eqs. (4) and (5) by the formal replacements $a \to b$ and $\alpha \to \beta$. In particular, we have

$$\begin{aligned} \frac{d\sigma^W}{dx} &= \Theta\left(\frac{1}{2}-x\right)\left(1+\beta-\beta\frac{\Gamma(b-2,1)}{\Gamma(b-1,1)}\right) \\ &+ \Theta\left(x-\frac{1}{2}\right)\left(1-\beta+\beta\frac{\Gamma(b-2,1)}{\Gamma(b-1,1)}\right). \end{aligned} \tag{8}$$

Ref. [1] proceeds by computing the p-dependent correction factor

$$K(p) \equiv \frac{d\sigma^T}{dp}\bigg/\frac{d\sigma^W}{dp} = \frac{\Gamma(b-1,1)}{\Gamma(a-1,1)}\,p^{b-a}\,, \tag{9}$$

which is then applied event-by-event in the MC simulation. In the formalism of this note, this is equivalent to defining a "corrected" doubly-differential cross section

$$\begin{aligned}
\frac{d\sigma^C}{dxdp} &\equiv K(p)\,\frac{d\sigma^W}{dxdp} \qquad (10)\\
&= \frac{2}{\Gamma(a-1,1)}\\
&\times \frac{e^{-1/p}}{p^a}\left[\Theta\left(\frac{1}{2}-x\right)\left(\frac{1}{2}+\beta\frac{1-p}{2}\right)+\Theta\left(x-\frac{1}{2}\right)\left(\frac{1}{2}-\beta\frac{1-p}{2}\right)\right],
\end{aligned}$$

from which we obtain the "corrected" differential distribution

$$\begin{aligned}
\frac{d\sigma^C}{dx} &= \Theta\left(\frac{1}{2}-x\right)\left(1+\beta-\beta\,\frac{\Gamma(a-2,1)}{\Gamma(a-1,1)}\right)\\
&+ \Theta\left(x-\frac{1}{2}\right)\left(1-\beta+\beta\,\frac{\Gamma(a-2,1)}{\Gamma(a-1,1)}\right). \qquad (11)
\end{aligned}$$

According to ref. [1], eq. (11) is a good approximation of eq. (5). In order to study this, I introduce the shorthand notation

$$\frac{d\sigma^A_<}{dx} = \frac{d\sigma^A}{dx}\Theta\left(\frac{1}{2}-x\right), \qquad (12)$$

$$\frac{d\sigma^A_>}{dx} = \frac{d\sigma^A}{dx}\Theta\left(x-\frac{1}{2}\right), \qquad (13)$$

where A=T,W,C, and define

$$R^{W,C}_< = \frac{d\sigma^{W,C}_<}{dx}\Bigg/\frac{d\sigma^T_<}{dx}, \qquad (14)$$

$$R^{W,C}_> = \frac{d\sigma^{W,C}_>}{dx}\Bigg/\frac{d\sigma^T_>}{dx}, \qquad (15)$$

$$S^{W,C} = \left(\frac{d\sigma^{W,C}_>}{dx}-\frac{d\sigma^{W,C}_<}{dx}\right)\Bigg/\left(\frac{d\sigma^T_>}{dx}-\frac{d\sigma^T_<}{dx}\right). \qquad (16)$$

If the "wrong" cross section were not wrong at all, the R's and S's defined in eqs. (14)–(16) would be all equal to one. On the other hand, the larger the values of $|R-1|$ and $|S-1|$, the worse the agreement between the "true" cross section and the "wrong" or "corrected" ones. By construction, $R_<$ and $R_>$ are relevant to the rate for $x < 1/2$ and $x > 1/2$ respectively, while S is relevant to the slope.

Using eqs. (5), (8), and (11) we readily get

$$R^{W}_{<} = \frac{\Gamma(a-1,1)}{\Gamma(b-1,1)} \frac{(1+\beta)\,\Gamma(b-1,1) - \beta\,\Gamma(b-2,1)}{(1+\alpha)\,\Gamma(a-1,1) - \alpha\,\Gamma(a-2,1)}, \tag{17}$$

$$R^{C}_{<} = \frac{(1+\beta)\,\Gamma(a-1,1) - \beta\,\Gamma(a-2,1)}{(1+\alpha)\,\Gamma(a-1,1) - \alpha\,\Gamma(a-2,1)}, \tag{18}$$

$$S^{W} = \frac{\beta\,\Gamma(a-1,1)}{\alpha\,\Gamma(b-1,1)} \frac{\Gamma(b-1,1) - \Gamma(b-2,1)}{\Gamma(a-1,1) - \Gamma(a-2,1)}, \tag{19}$$

$$S^{C} = \frac{\beta}{\alpha}, \tag{20}$$

and the results for $R^{W,C}_{>}$ can be obtained from eqs. (17) and (18) with the formal replacements $\alpha \to -\alpha$, $\beta \to -\beta$. With the equations above we can explicitly verify that when $b = a$ and $\beta = \alpha$ (i.e., the "wrong" cross section is identical to the "true" one), then $R = S = 1$.

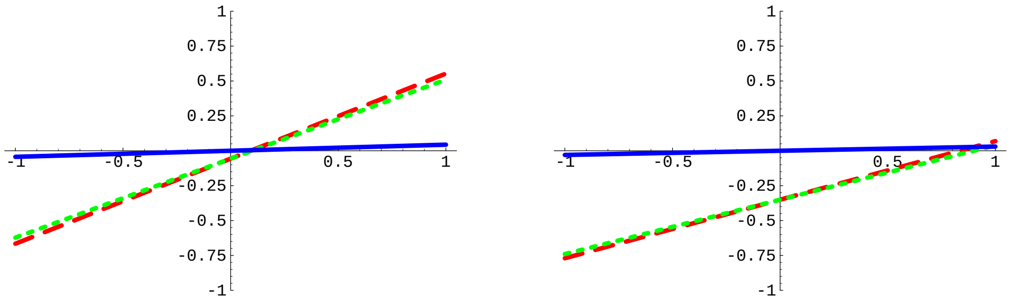

Fig. 1: I plot here $1 - R^{W}_{<}$ (red dashed line), $1 - R^{C}_{<}$ (green dotted line), and $R^{C}_{<} - R^{W}_{<}$ (blue line) as a function of β, for $a = 4$, $b = 4.5$, and $\alpha = 0.1$ (left panel) or $\alpha = 0.9$ (right panel).

I proceed by observing that, if the derivative of the "true" x distribution has the opposite sign of that of the "wrong" x distribution (say, the former is decreasing while the latter is increasing, i.e. $\alpha > 0$ and $\beta < 0$), then clearly the reweighting cannot correct this behaviour, as is most evident from eq. (20). In order to have an idea of what happens in general, I plot in figs. 1 and 2 the quantities $1 - R^{W}_{<}$, $1 - R^{C}_{<}$, and $R^{W}_{<} - Rc_{<}$, as functions of β for given values of α, a, and b. In figs. 3 and 4 I plot $1 - S^{W}$, $1 - S^{C}$, and $S^{W} - S^{C}$. By inspection of the figures, we can see that when the p "true" and "wrong" distributions are similar ($a = 4$, $b = 4.5$), the difference

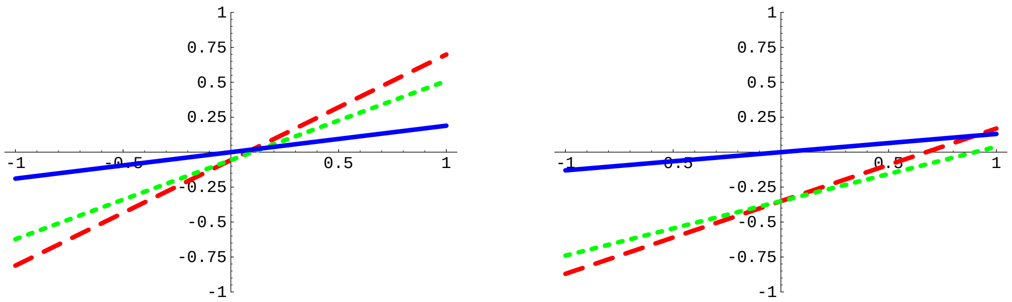

Fig. 2: As in fig. 1, except for the value of b; here $b = 7$.

between the "wrong" and "corrected" x distributions is fairly marginal; the "corrected" x distribution may display a disagreement wrt the "true" x distributions that can be as large as 50%. The agreement with the "true" result obviously improves when $\beta \simeq \alpha$, but in such a case one wouldn't advocate the necessity of a reweighting procedure at all. In the case in which the p "wrong" distribution is much steeper than the "true" one ($a = 4$, $b = 7$), the effect of the reweighting is more pronounced, but this doesn't imply that the "corrected" x distribution improves the "wrong" one, since this appears to depend on the value of α. In any case, the "corrected" x distribution agrees better with the "wrong" than with the "true" one.

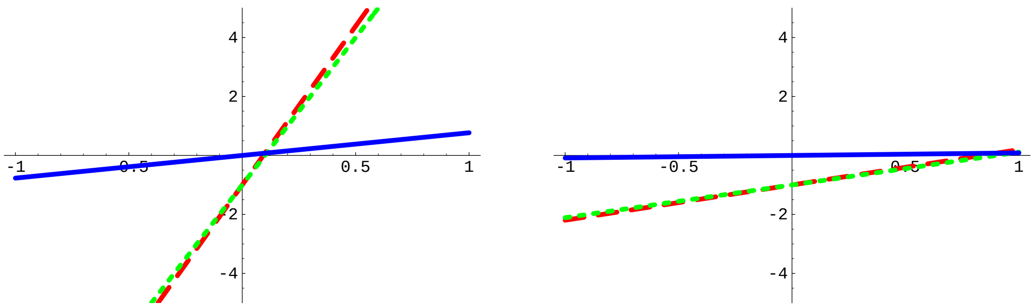

Fig. 3: As in fig. 1, for S^W and S^C.

Conclusions: it is pretty obvious that the functional form chosen here for the cross section is too simplistic to give a proper description of the complex final state which emerges from a hadronic collision. It does show, however, that the results of the reweighting are to a large extent unpredictable, the

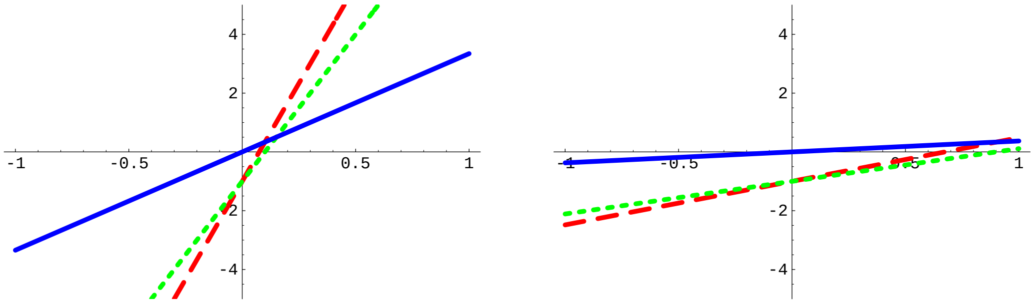

Fig. 4: As in fig. 2, for S^W and S^C.

procedure being guaranteed to do what it is supposed to do only on the very observable that is reweighted. For the other observables it appears that only the comparison with the "true" result allows one to assess the accuracy with which the reweighting does its job. If the "true" result is not available, it is impossible to give a sound estimate of the uncertainties involved in the procedure.

References

[1] G. Davatz, G. Dissertori, M. Dittmar, M. Grazzini and F. Pauss, JHEP **0405** (2004) 009 [arXiv:hep-ph/0402218].