A complete formalism for one-loop multi-leg amplitudes

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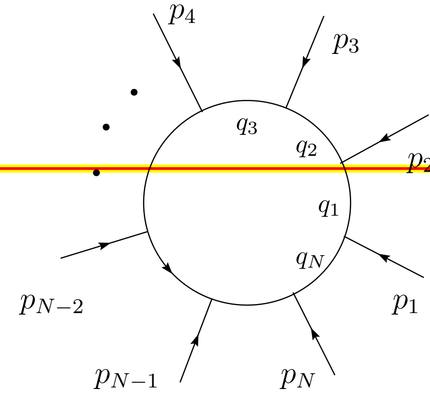
- evaluation of the one-loop amplitudes is the bottleneck for efficient calculations of multi-particle cross sections at NLO
- main difficulties are enormous sizes of expressions produced by conventional methods and numerical instabilities
- sizes becomes less important as computer memory/speed increases, as long as numerical behaviour can be controlled
- $\Rightarrow$ suitable methods should be numerically robust and easy to automate

hep-ph/0504267 [Binoth, Guillet, GH, Pilon, Schubert]

- valid for an arbitrary number N of external legs
- valid for massless and massive particles (uses dimensional regularisation for IR poles)
- designed to be numerically robust and fast due to a combination of semi-numerical and algebraic approaches

Tensor integrals



$$I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r) = \int \frac{d^n k}{i\,\pi^{n/2}}\,\frac{q_{a_1}^{\mu_1}\;\ldots\,q_{a_r}^{\mu_r}}{(q_1^2-m_1^2+i\delta)\ldots(q_N^2-m_N^2+i\delta)}$$

$q_i = k + r_i\,,\;\; r_j - r_{j-1} = p_j\,,\;\; \sum_{j=1}^{N} p_j = 0$

advantages of this representation:

- combinations $q_i = k + r_i$ appear naturally (e.g. fermion propagators)
- allows for a manifestly shift invariant formulation Lorentz structure carried by difference vectors $\Delta_{i\,j}^{\mu} = r_i^{\mu} - r_j^{\mu}$ and $g^{\mu\nu}$

Lorentz structure and form factors

$$I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;\,S) =$$

$$\sum_{l_1\cdots l_r\in S} \left[\Delta_{l_1\cdot}^{\cdot}\cdots\Delta_{l_r\cdot}^{\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}} A^{N,r}_{l_1\ldots,l_r}(S)$$

$$+ \sum_{l_1\cdots l_{r-2}\in S} \left[g^{\cdot\cdot}\Delta_{l_1\cdot}^{\cdot}\cdots\Delta_{l_{r-2}\cdot}^{\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}} B^{N,r}_{l_1\ldots,l_{r-2}}(S)$$

$$+ \sum_{l_1\cdots l_{r-4}\in S} \left[g^{\cdot\cdot}g^{\cdot\cdot}\Delta_{l_1\cdot}^{\cdot}\cdots\Delta_{l_{r-4}\cdot}^{\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}} C^{N,r}_{l_1\ldots,l_{r-4}}(S)$$

example

$$I_N^{n,\mu_1\mu_2}(a_1,a_2;S) = \sum_{l_1,l_2\in S} \Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\,A^{N,2}_{l_1\,l_2}(S) + g^{\mu_1\,\mu_2}\,B^{N,2}(S)$$

important: kinematic matrix $\mathcal{S}$: $\mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2$

$$I_N^n(S) \;=\; (-1)^N \Gamma(N - \frac{n}{2}) \int \prod_{i=1}^{N} dz_i \, \delta(1 - \sum_{l=1}^{N} z_l) \, \left(R^2\right)^{\frac{n}{2} - N}$$

$$R^2 = -\frac{1}{2} \sum_{i,j=1}^{N} z_i \, \mathcal{S}_{ij} \, z_j \; - i\delta$$

associate matrix $\mathcal{S}$ with set S of propagator labels

matrix corresponding to integral with propagator $q_j^2 - m_j^2$ missing is denoted by $\mathcal{S}^{\{j\}}$, corresponding to set $S \setminus \{j\}$

Reduction: I. scalar integrals

$$
\begin{aligned}
I_N^n(S) &= \sum_{i\in S} b_i(S) \int d^n\bar{k} \, \frac{(q_i^2 - m_i^2)}{\prod_{j\in S}(q_j^2 - m_j^2 + i\,\delta)} \\
&\quad + \int d^n\bar{k} \, \frac{1 - \sum_{i\in S} b_i(S)\,(q_i^2 - m_i^2)}{\prod_{j\in S}(q_j^2 - m_j^2 + i\,\delta)} \\
&\stackrel{!}{=} I_{div}(S) + I_{fin}(S)
\end{aligned}
$$

If $\sum_{i\in S} b_i(S)\,\mathcal{S}_{ij} = 1$ then

$$
\begin{aligned}
I_{fin}(S) &= -B(S)\,(N - n - 1)\, I_N^{n+2}(S) \\
B(S) &= \sum_{i\in S} b_i(S)\,, \qquad B\ \det\mathcal{S} = (-1)^{N+1}\det G
\end{aligned}
$$

Reduction: II. tensor integrals

$$I_N^{n,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;S) =$$

$$-\sum_{j\in S}\mathcal{C}^{\mu_1}_{ja_1}\int d\bar{k}\,\frac{(q_j^2-m_j^2)\;q_{a_2}^{\mu_2}\ldots q_{a_r}^{\mu_{a_r}}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}$$

$$+\int d\bar{k}\,\frac{\left[q_{a_1}^{\mu_1}+\sum_{j\in S}\mathcal{C}^{\mu_1}_{ja_1}\,(q_j^2-m_j^2)\right]\;q_{a_2}^{\mu_2}\ldots q_{a_r}^{\mu_r}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}$$

$$\stackrel{!}{=}\quad I_{div}+I_{fin}$$

If $\sum_{i\in S}\mathcal{C}^{\mu}_{i\,a}(S)\,\mathcal{S}_{ij}=\Delta^{\mu}_{j\,a}$ then

Reduction: II. tensor integrals

$$I_{fin}(S) = \int d\vec{Z} \int d^n\bar{l}\, \frac{\left[l_\nu \left(\mathcal{T}^{\mu_r \nu}_{a_r d} + 2\, \mathcal{V}^{\mu_r}_{a_r} \sum_{i\in S} z_i \Delta^\nu_{d\, i} \right)\right.}{(l^2 - R^2)^N} + \frac{\left.\mathcal{V}^{\mu_r}_{a_r} \, (l^2 + R^2) \right] \; \tilde{q}^{\mu_1}_{a_1} \ldots \tilde{q}^{\mu_{r-1}}_{a_{r-1}}}{(l^2 - R^2)^N}$$

where

$$l = k + \sum_{i\in S} z_i\, r_i$$

$$\mathcal{V}^\mu_a = \sum_{j\in S} \mathcal{C}^\mu_{j a} = \sum_{k\in S} b_k\, \Delta^\mu_{k\, a}$$

$$\mathcal{T}^{\mu\nu}_{a_1\, a_2} = g^{\mu\nu} + 2 \sum_{j\in S} \mathcal{C}^\mu_{j\, a_1}\, \Delta^\nu_{j\, a_2}$$

solving the defining equation for arbitrary N

if $\mathcal{S}$ is invertible, i.e. for $N < 7$ (non-exceptional kinematics):

$$\begin{aligned} \sum_{i \in S} \mathcal{C}^{\mu}_{i\,a}(S)\,\mathcal{S}_{ij} &= \Delta^{\mu}_{j\,a} \\ \Leftrightarrow\ \mathcal{C}^{\mu}_{i\,a}(S) &= \sum_{j \in S} \left(\mathcal{S}^{-1}\right)_{ij} \Delta^{\mu}_{ja} \end{aligned}$$

Otherwise use pseudo-inverse H_{ij} to Gram matrix

$$\begin{aligned} \mathcal{C}^{\mu}_{i\,b} &= -\sum_{j \in S \setminus \{a\}} H_{ij}\,\Delta^{\mu}_{j\,a} + W^{\mu}_{i} \quad , \quad i \in S \setminus \{a\} \\ \mathcal{C}^{\mu}_{a\,b} &= -\sum_{j \in S \setminus \{a\}} \mathcal{C}^{\mu}_{j\,b} \end{aligned}$$

The case $N \geq 5$

important:

$$\mathcal{V}_b^{\mu} \equiv 0 \text{ and } \mathcal{T}^{\mu\nu}_{a_1\,a_2} \equiv 0 \text{ for } N > 5$$

$\Rightarrow$ NO higher dimensional integrals for $N > 5$!

$$I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r\,;S) = -\mathcal{C}^{\mu_r}_{j\,a_r}\; I_{N-1}^{n,\,\mu_1\ldots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\backslash\{j\})$$

case $N = 5$ more tricky, proof relies on $\mathcal{T}^{\mu\,\nu}_{a\,b} = 2\,\mathcal{V}^{\mu}_a\,\mathcal{V}^{\nu}_b/B$

result:

- no higher dimensional integrals I_N^{n+2m} $(m > 0)$ for $N > 4$
- no inverse Gram determinants

N=5 rank 2 example

$$I_5^{n,\,\mu_1\mu_2}(a_1,a_2;S) \;=\; g^{\mu_1\,\mu_2}\,B^{5,2}(S) + \sum_{l_1,l_2\in S} \Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\,A^{5,2}_{l_1\,l_2}(S)$$

$$B^{5,2}(S) \;=\; -\frac{1}{2}\sum_{j\in S} b_j\, I_4^{n+2}(S\setminus\{j\})$$

$$\begin{aligned} A^{5,2}_{l_1 l_2}(S) \;=\; & \sum_{j\in S}\Big(\mathcal{S}^{-1}{}_{j\,l_1}\,b_{l_2} + \mathcal{S}^{-1}{}_{j\,l_2}\,b_{l_1} \\ & -2\,\mathcal{S}^{-1}{}_{l_1\,l_2}\,b_j + b_j\,\mathcal{S}^{\{j\}-1}{}_{l_1\,l_2}\Big)\; I_4^{n+2}(S\setminus\{j\}) \\ & +\frac{1}{2}\sum_{j\in S}\;\sum_{k\in S\setminus\{j\}}\Big[\mathcal{S}^{-1}{}_{j\,l_2}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_1} + \\ & \mathcal{S}^{-1}{}_{j\,l_1}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_2}\Big]\; I_3^{n}(S\setminus\{j,k\}) \end{aligned}$$

N=6 rank 1 example

$$I_6^{n,\,\mu}(a;S) = -\sum_{j\in S} \mathcal{C}^{\mu}_{j\,a}\, I_5^n(S\setminus\{j\})$$

$$= -\sum_{j,l\in S} \Delta^{\mu}_{la}\, \mathcal{S}^{-1}_{lj} \sum_{k\in S} b_k^{\{j\}}$$

$$\left[B^{\{j,k\}}\, I_4^{n+2}(S\setminus\{j,k\}) + \sum_{m\in S\setminus\{j,k\}} b_m^{\{j,k\}}\, I_3^n(S\setminus\{j,k,m\}) \right]$$

basis integrals

$$I_3^n(j_1,\ldots,j_r) = -\Gamma\left(3-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\,\frac{z_{j_1}\ldots z_{j_r}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{3-n/2}}$$

$$I_3^{n+2}(j_1) = -\Gamma\left(2-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\,\frac{z_{j_1}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{2-n/2}}$$

$$I_4^{n+2}(j_1,\ldots,j_r) = \Gamma\left(3-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^4 dz_i\,\delta(1-\sum_{l=1}^4 z_l)\,\frac{z_{j_1}\ldots z_{j_r}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{3-n/2}}$$

$$I_4^{n+4}(j_1) = \Gamma\left(2-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^4 dz_i\,\delta(1-\sum_{l=1}^4 z_l)\,\frac{z_{j_1}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z - i\delta)^{2-n/2}}$$

and scalar $I_3^n, I_3^{n+2}, I_4^{n+2}, I_4^{n+4}$. important: $r_{\max} = 3$

two alternatives:

1. direct numerical evaluation (contour deformation)

2. further algebraic reduction to scalar integrals

algebraic reduction: example N=4

algebraic reduction re-introduces inverse Gram determinants $\sim 1/B$

$$I_4^{n+2}(l;S) \quad = \quad \frac{1}{B}\left\{ b_l\, I_4^{n+2}(S) + \frac{1}{2}\sum_{j\in S} \mathcal{S}^{-1}{}_{j\,l}\, I_3^n(S\setminus\{j\}) - \frac{1}{2}\sum_{j\in S\setminus\{l\}} b_j\, I_3^n(l;S\setminus\{j\}) \right\}$$

- does not pose a problem in most regions of phase space
- allows fast evaluation
- $\Rightarrow$ use numerical method only in regions where $1/B$ becomes small

numerical behaviour

test process $p_1 + p_2 \to p_3 + p_4$ with $p_1^2 = p_2^2 = M^2$

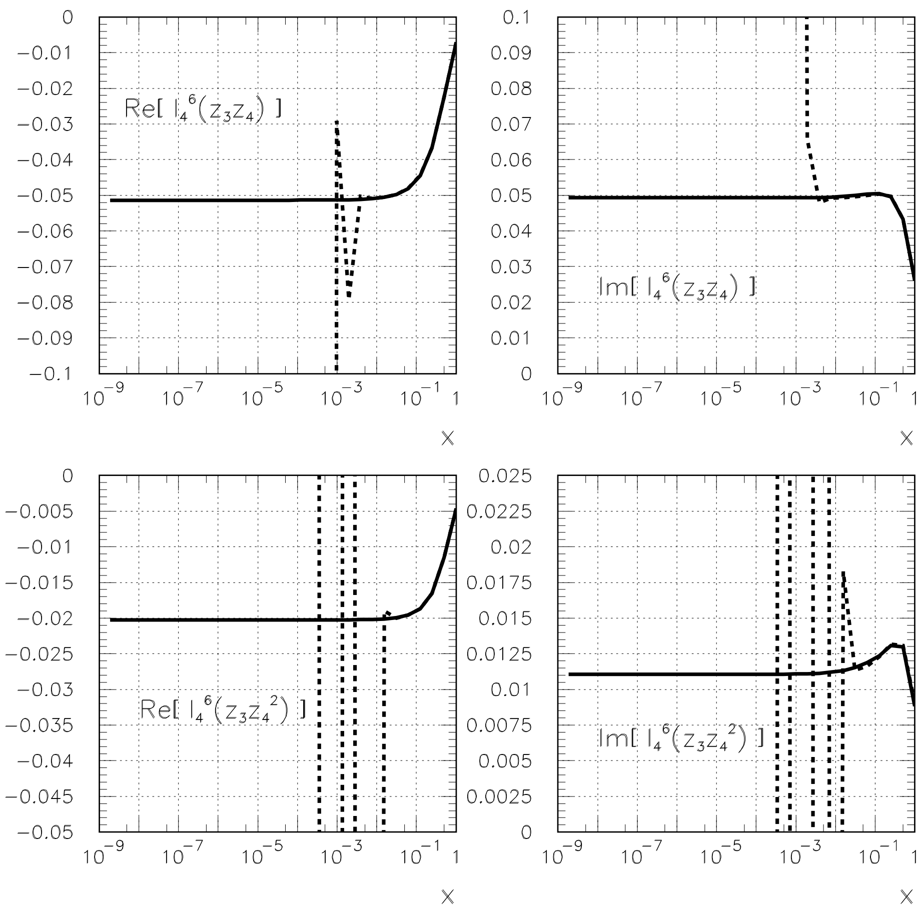
$$\begin{aligned} p_1 &= (E(x), 0, 0, M\,x) \\ p_2 &= (E(x), 0, 0, -M\,x) \\ p_3 &= E(x)\,(1, 0, \sin\theta, \cos\theta) \\ p_4 &= E(x)\,(1, 0, -\sin\theta, -\cos\theta) \\ E(x) &= M\,\sqrt{1+x^2} \end{aligned}$$

$\det(G) = 32M^6\,(1+x^2)^2\,x^2\,\sin^2\theta$

Exceptional configurations:

$\theta = 0, \pi$ and $x = 0$

numerical behaviour



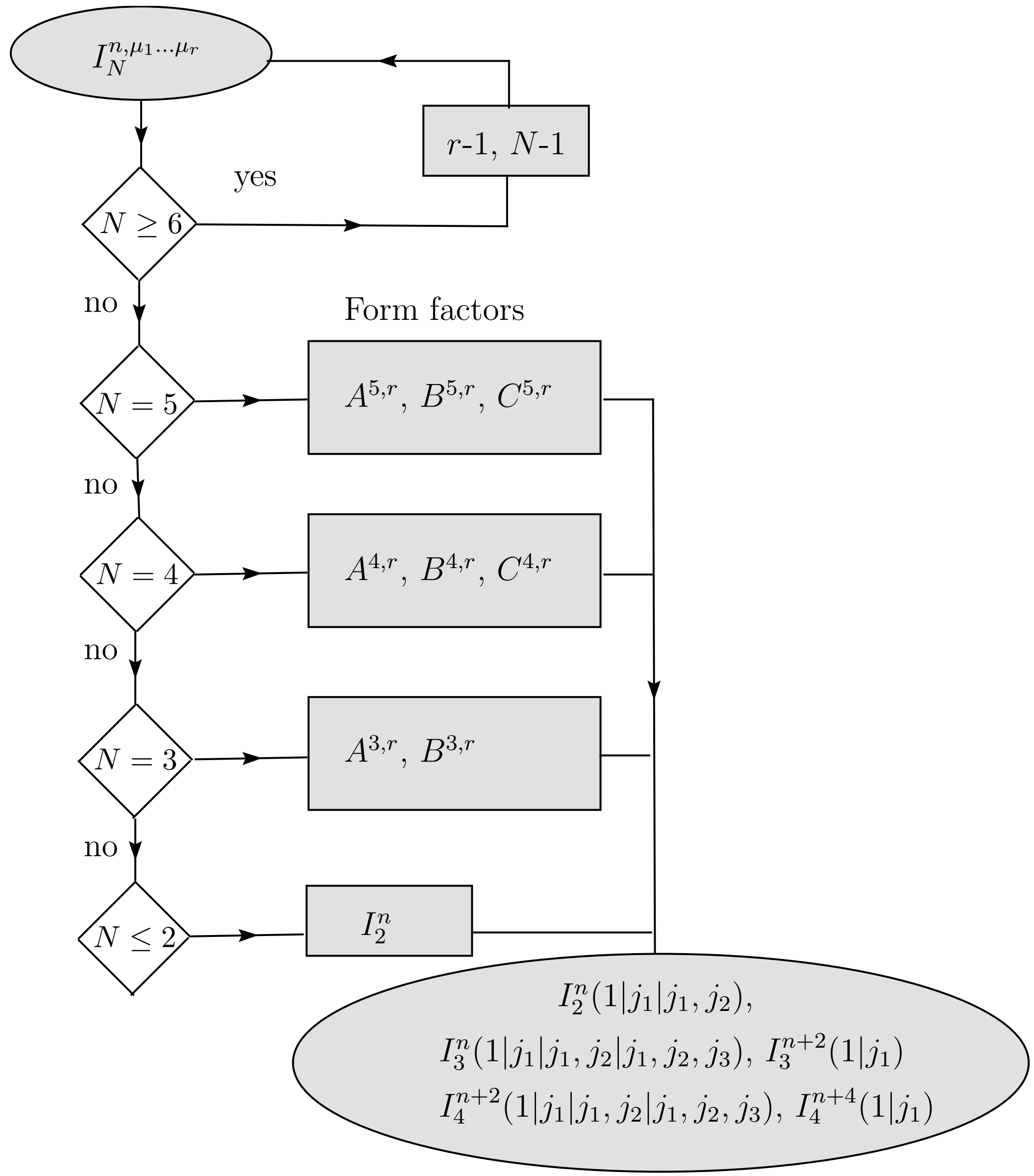
$I_4^6(z_3 z_4)$ and $I_4^6(z_3 z_4^2)$

exceptional kinematics for $x \longrightarrow 0$

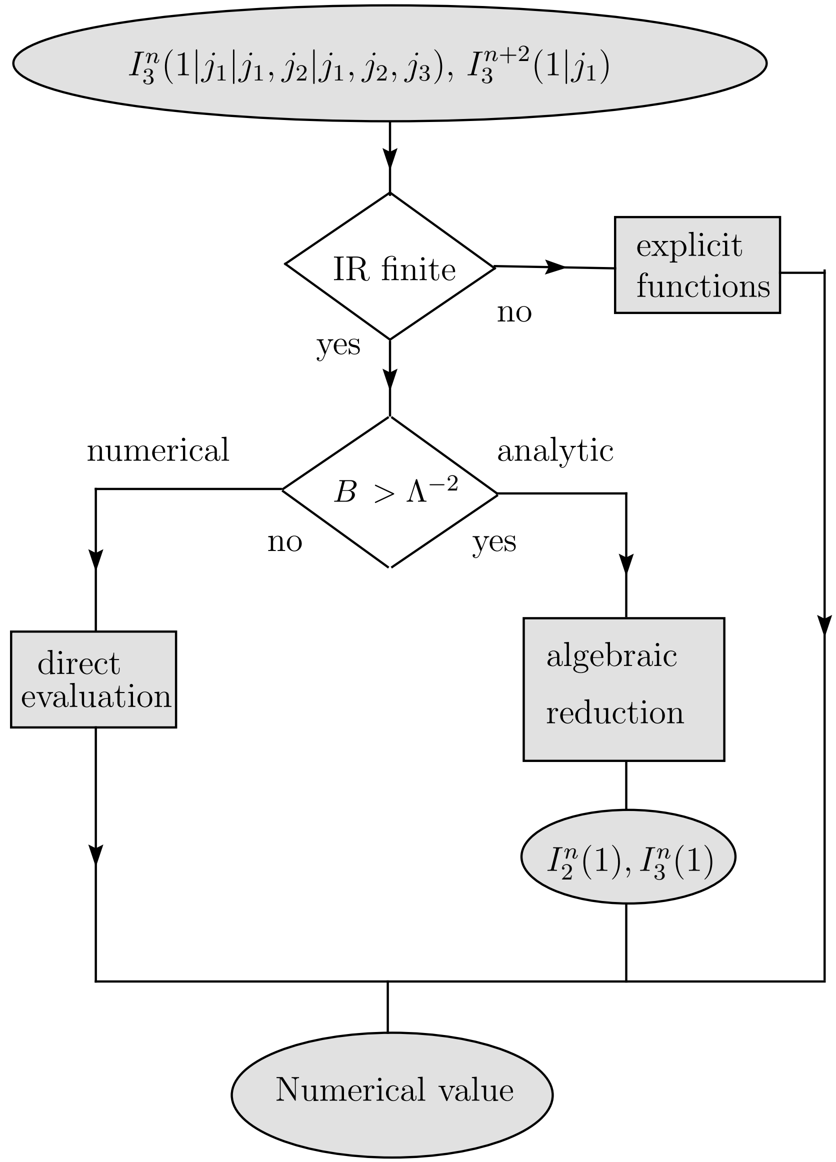
solid line: numerical implementation

dashed: numerical behaviour of algebraic representation

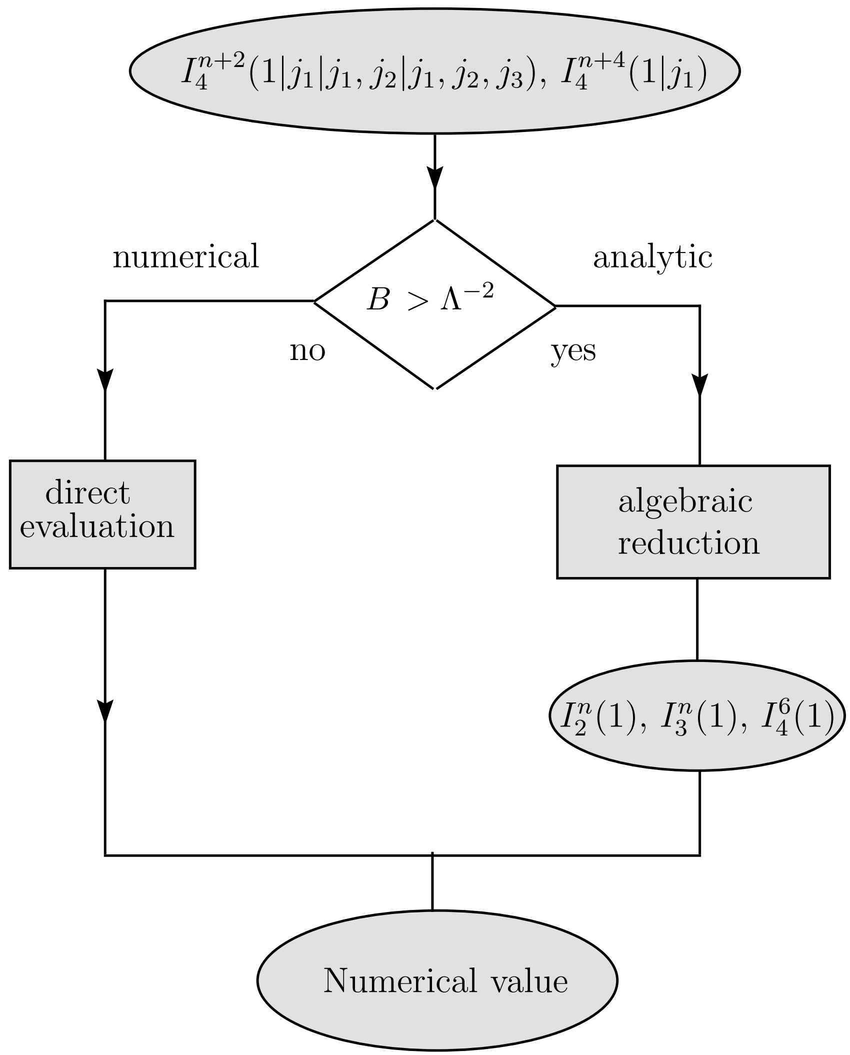
schematic overview



treatment of basis integrals: N=3



treatment of basis integrals: N=4



Summary

- formalism valid for arbitrary N
- massive and massless particles
- efficient isolation of IR divergences
- manifestly shift invariant
- no inverse Gram determinants
- basis integrals: representation very suitable for numerical integration
- further algebraic reduction optional
- $\Rightarrow$ balance between numerical robustness and speed