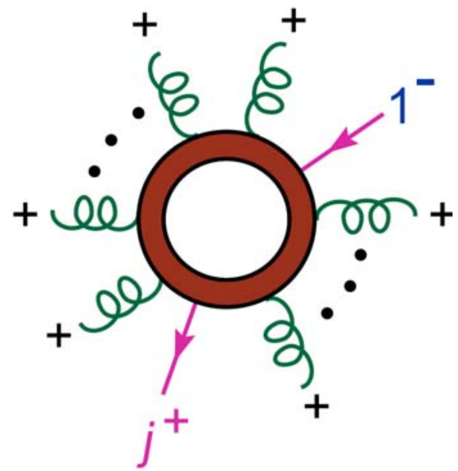
# Twistoresque Methods for Perturbative QCD

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# Introduction

- Need a flexible, efficient method to extend range of known tree, and particularly 1-loop QCD amplitudes with many external legs, for use in NLO corrections to many LHC processes, some ILC processes, etc.
- 1-loop not known beyond *n=5* legs, except for special helicity configurations
- Semi-numerical approaches to 1-loop amplitudes are one way to go, *e.g.*

Denner, Dittmaier, …, hep-ph/0212259,…; Nagy, Soper, hep-ph/0308127;
Giele, Glover, hep-ph/0402152; Andonov *et al.,* hep-ph/0411186;
van Hameren, Vollinga, Weinzierl, hep-ph/0502165; Binoth *et al.,* hep-ph/0504267;
Ellis, Giele, Zanderighi, hep-ph/0506196 [*Hgggg, Hqqgg, Hqqqq*]

# Introduction (cont.)

- Another approach is to pay attention to the analytic properties of amplitudes
  - poles (factorization) at tree level
  - poles and branch cuts (unitarity) at loop level
- In this approach one can incorporate hidden symmetries of tree level QCD, due to its relation to N=4 super-Yang-Mills theory:
  - supersymmetry Ward identities — Grisaru, Pendleton, van Nieuwenhuizen (1977)
  - connection to twistor space — Penrose (1967)
    and to twistor string theory — Witten, hep-th/0312171
- These symmetries have loop-level implications for QCD *via* unitarity

# Outline

- Motivation
- Role of N=4 super-Yang-Mills theory
- Color & helicity
- Supersymmetry Ward identities
- Twistor space, twistor strings, & MHV tree rules
- On-shell recursion relations at tree level
- (Generalized) unitarity and twistor structure of 1-loop amplitudes in N=4 super-Yang-Mills theory
- On-shell recursion relations at 1-loop, leading to new QCD amplitudes with 6 or more legs
- Conclusions

# Role of N=4 super-Yang-Mills theory

- Essentially unique, maximally supersymmetric, conformal field theory
- Topological string in twistor space  Witten, hep-th/0312171
  is most directly for N=4 SYM
- N=4 SYM ⇔ QCD at tree level;
  can be thought of as 1 component of QCD at 1 loop
- Loop-level scattering amplitudes share many properties with those of QCD, but are simpler

  ⇒ "theoretical playground"

Bern, LD, Kosower, hep-ph/9403226, 9409265

# N=4 super-Yang-Mills theory

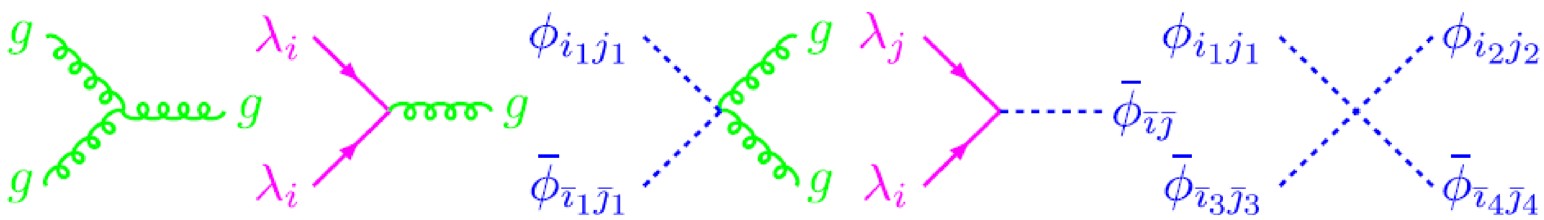
State multiplicities:

$$\begin{array}{lccccccccc}
 & & & & & 1 & & & & \\
 & & & & 1 & & 1 & & & \\
\mathcal{N}=2 & & & 1 & & 2 & & 1 & & \\
 & & 1 & & 3 & & 3 & & 1 & \\
\end{array}$$

| | | | | | |
|---|---|---|---|---|---|
| $\mathcal{N}=4$ | 1 | 4 | 6 | 4 | 1 |
| | $g^-$ | $\lambda_{\bar{\imath}}^-$ | $\bar{\phi}_{\bar{\imath}\bar{\jmath}}, \phi_{ij}$ | $\lambda_i^+$ | $g^+$ |
| helicity | $-1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ |

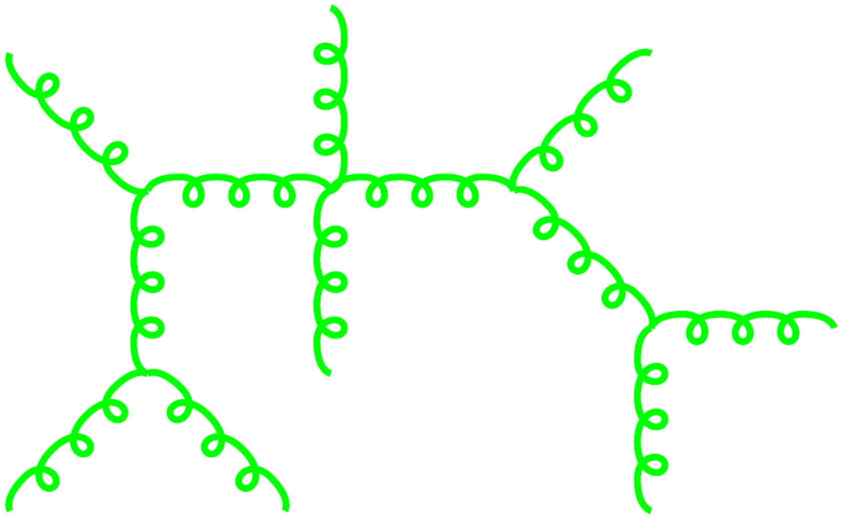
all in adjoint representation

**Feynman rules:** Usual gauge interactions, plus $W(\Phi)$



# Tree level

- N=4 SYM $\Leftrightarrow$ QCD at tree level for *n*-gluon amplitude because no fermions & scalars enter (as they must be pair-produced)

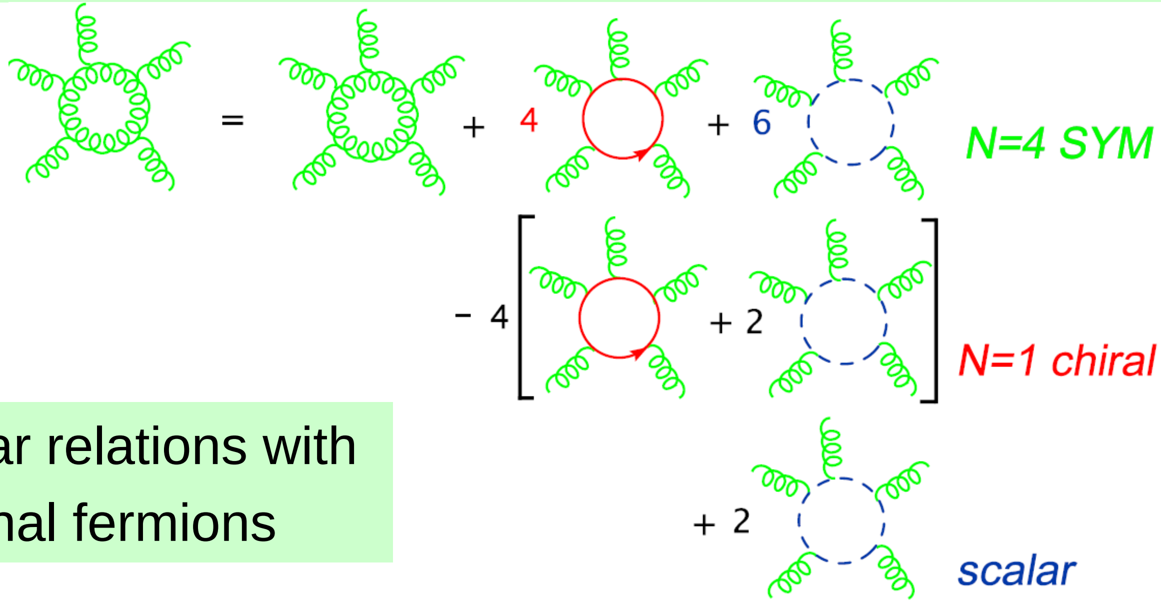


- Similar relations with external fermions too

# One loop rearrangement

Can rewrite gluon (and fermion) loop for *n*-gluon QCD amplitude as linear combinations of:

- N=4 SYM (simplest)
- N=1 chiral matter multiplet (next simplest)
- scalar (non-supersymmetric, but no spin-tangling)




Similar relations with external fermions

# Color-ordered amplitudes

Decompose tree-level $n$-gluon amplitudes as

$$\mathcal{A}_n^{\text{tree}}(\{k_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) \times A_n^{\text{tree}}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n}))$$

$A_n^{\text{tree}}$ color-ordered, only receive contributions from cyclicly-ordered Feynman diagrams, so poles in fewer kinematic variables

Mangano, Parke (1986)

Similarly decompose 1-loop $n$-gluon amplitudes as

$$\mathcal{A}_n^{1-\text{loop}}(\{k_i,\lambda_i,a_i\}) = g^n N_c \sum_{\sigma\in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) \times A_{n;1}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

Subleading-color terms, coeff's of Tr(..) Tr(..), not independent; sums of perm's of color-ordered $A_{n;1}$

Bern, Dunbar, LD, Kosower, hep-ph/9403226

# Spinor variables

Use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½), **not** 4-vectors $k_i^\mu$ (spin 1)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

Reconstruct $k_i^\mu$ from $u_\alpha(k_i)$ using positive-energy Dirac projector:

$$k_i^\mu (\sigma_\mu)_{\alpha\dot{\alpha}} = (\not{k}_i)_{\alpha\dot{\alpha}} = u_+(k_i)\bar{u}_+(k_i) = (\lambda_i)_\alpha (\tilde{\lambda}_i)_{\dot{\alpha}}$$

Singular 2 x 2 matrix:

$$\det(\not{k}_i) = \begin{vmatrix} k_t + k_z & k_x - ik_y \\ k_x + ik_y & k_t - k_z \end{vmatrix}$$
$$= k_t^2 - k_x^2 - k_y^2 - k_z^2 = 0$$

also shows $(\not{k}_i)_{\alpha\dot{\alpha}} = (\lambda_i)_\alpha (\tilde{\lambda}'_i)_{\dot{\alpha}}$

even for complex momenta

Gluon polarizations also in terms of spinors: $\varepsilon_\mu^\pm(k,\eta) = \pm \frac{\langle k^\pm | \gamma_\mu | \eta^\pm \rangle}{\sqrt{2}\langle k^\mp | \eta^\pm \rangle}$

# Spinor products

Instead of Lorentz products: $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products:

$$\bar{u}_-(k_i)u_+(k_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\, j\rangle$$

$$\bar{u}_+(k_i)u_-(k_j) = \varepsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_j)_{\dot{\beta}} = [i\, j]$$

These are **complex square roots** of Lorentz products:

$$\langle i\, j\rangle[j\, i] = \frac{1}{2}\mathrm{Tr}\left[\not{k}_i\ \not{k}_j\right] = 2k_i \cdot k_j = s_{ij}$$

$$\boxed{\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}}$$

# Supersymmetry Ward identities

Grisaru, Pendleton, van Nieuwenhuizen (1977)

In any unbroken supersymmetric theory, $Q|0\rangle = 0$, so

$$0 = \langle 0|[Q, \Phi_1 \Phi_2 \cdots \Phi_n]|0\rangle = \sum_{i=1}^{n} \langle 0|\Phi_1 \cdots [Q, \Phi_i] \cdots \Phi_n|0\rangle$$

**Leads to powerful S-matrix identities:**

$$A_n^{\mathrm{SUSY}}(1^\pm, 2^+, 3^+, 4^+, \ldots, n^+) = 0$$

$$A_n^{\mathrm{SUSY}}(1_f^-, 2_f^+, 3^-, 4^+, \ldots, n^+) = \frac{\langle 2\,3\rangle}{\langle 1\,3\rangle} \times A_n^{\mathrm{SUSY}}(1^-, 2^+, 3^-, 4^+, \ldots, n^+)$$

$$\frac{A_n^{\mathcal{N}=4\ \mathrm{SUSY}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle i\, j\rangle^4}$$ indep. of $i, j$ *etc.*

- Results hold order by order in perturbation theory.
- At tree-level, can be applied directly to QCD.

# Twistor Space

Start in **spinor space**: Amplitudes $A(k_i) \;\Rightarrow\; A(\lambda_i, \tilde{\lambda}_j)$

**Twistor transform = "half Fourier transform"**:

Fourier transform $\tilde{\lambda}_i$, but not $\lambda_i$ , for each leg $i$

$$\tilde{\lambda}_{\dot{a}} = i \frac{\partial}{\partial \mu^{\dot{a}}} \qquad\qquad \mu^{\dot{a}} = -i \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}$$

**Twistor space coordinates**: $(\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}})$ for each $i$

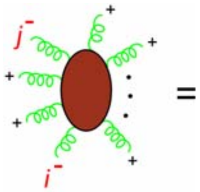
$$\sim (\xi\lambda_1, \xi\lambda_2, \xi\mu^{\dot{1}}, \xi\mu^{\dot{2}})$$

Amplitudes $A(k_i) \;\Rightarrow\; A(\lambda_i, \tilde{\lambda}_i) \;\Rightarrow\; A(\lambda_i, \mu_i)$

# Twistor Transform in QCD

Witten, hep-th/0312171

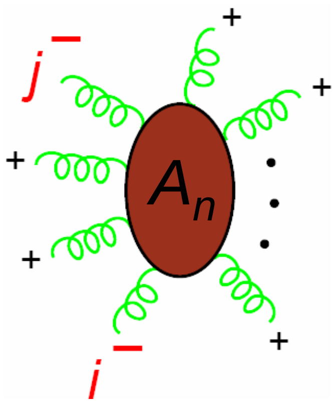
Parke-Taylor (1986) $\quad n_- = 2$ (MHV)



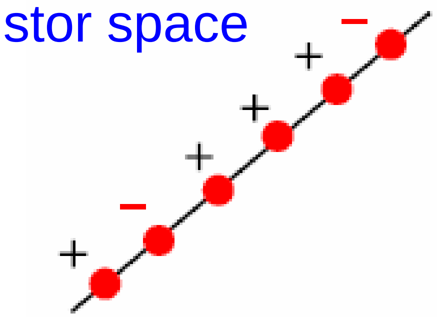

$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}\,\delta(\textstyle\sum_i k_i) \;=\; \int d^4x\, A(\lambda_i)\,\exp(ix\lambda_i\tilde{\lambda}_i)$$

$$\int d\tilde{\lambda}\,\exp(i\mu\tilde{\lambda})\exp(ix\lambda\tilde{\lambda}) \;\Rightarrow\; A(\lambda,\mu) \;\propto\; \delta(\mu + x\lambda)$$

linear constraints for each ***i*** imply
all points lie on single line in twistor space




$\Rightarrow$



# Twistor implications in spinor space

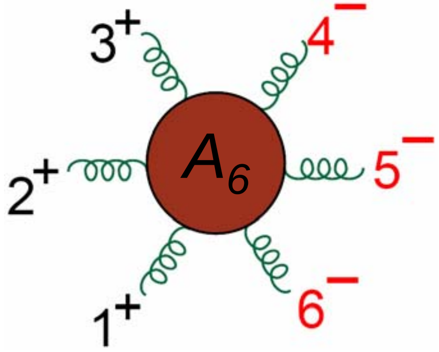
Witten, hep-th/0312171

- Vanishing relations on curves in twistor space $\Longrightarrow$ differential equations in $(\lambda_i, \tilde{\lambda}_j)$ space.
- $i, j, k$ have collinear support if $A$ annihilated by $C_{ijkL} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K \to \langle i\, j \rangle \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{a}}} + \langle j\, k \rangle \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} + \langle k\, i \rangle \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}}$ for $L = \dot{a}$.
- $i, j, k, l$ are coplanar if $A$ annihilated by $K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L \to \langle i\, j \rangle \, \epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \tilde{\lambda}_k^{\dot{a}} \partial \tilde{\lambda}_l^{\dot{b}}} + 5$ perms

# More Twistor Magic

Using **collinear**/**coplanar** differential operators, find:

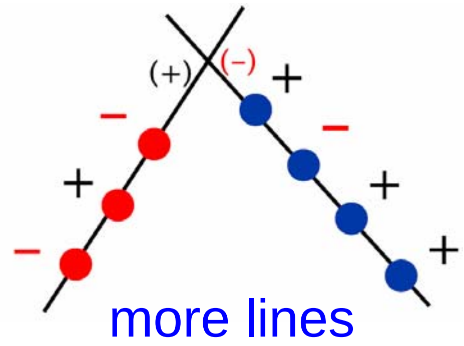
Witten, hep-th/0312171




Mangano, Parke, Xu (1988) $n_- = 3$ (NMHV)

$$= \frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

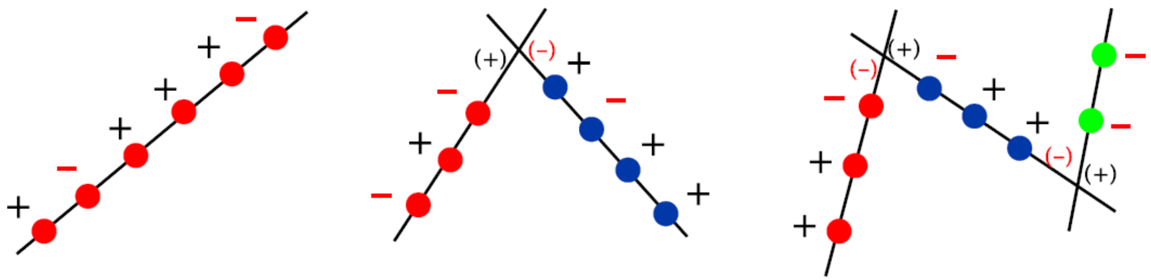
$\Rightarrow$



# Twistor magic from twistor strings

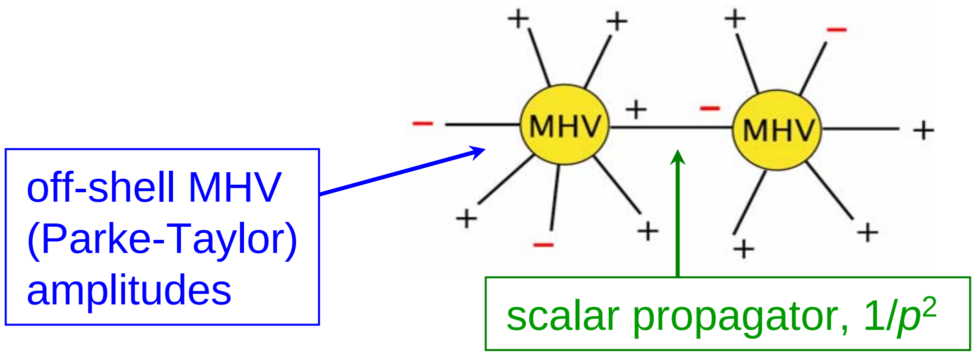
Original intuition from topological string:
$L$-loop amplitude with $n_-$ negative-helicity gluons should be supported on curve in twistor space with degree $d = n_- - 1 + L$, genus $g \leq L$.
MHV case: $n_- = 2$, $g = 0$ $\Rightarrow$ $d = 1$, a straight line.

"Experimentation" showed situation actually better than that for tree amplitudes:
Cachazo, Svrček, Witten (2004)
supported on $n_- - 1$ intersecting straight lines
(degenerate limit of the degree $d$ curve)



# MHV rules

Based on the "experimental" results, and an interpretation of the twistor string path integral, Cachazo, Svrcek, Witten, hep-th/0403047 proposed "MHV rules" for *n*-gluon scattering:




For example, if a + leg goes off-shell, use:

$$A_n^{\text{tree,MHV},ij}(1^*) = \frac{\langle i\, j\rangle^4}{\langle 1^*\, 2\rangle \ldots \langle n\, 1^*\rangle} = \frac{\langle i\, j\rangle^4}{\langle \eta^+ |\, 1\, | 2^+\rangle \ldots \langle n^- |\, 1\, | \eta^-\rangle}$$

where $\eta^2 = 0$ is arbitrary. Results independent of $\eta$, agree numerically with Feynman diagram computations

# MHV rules for trees

Rules quite efficient, extended to many collider applications

- massless quarks

Georgiou, Khoze, hep-th/0404072;
Wu, Zhu, hep-th/0406146;
Georgiou, Glover, Khoze, hep-th/0407027

- Higgs bosons (*Hgg* coupling)

LD, Glover, Khoze, hep-th/0411092;
Badger, Glover, Khoze, hep-th/0412275

- vector bosons ($W,Z,\gamma^*$)

Bern, Forde, Kosower, Mastrolia, hep-th/0412167

- Related approach to QCD + massive quarks more directly from field theory

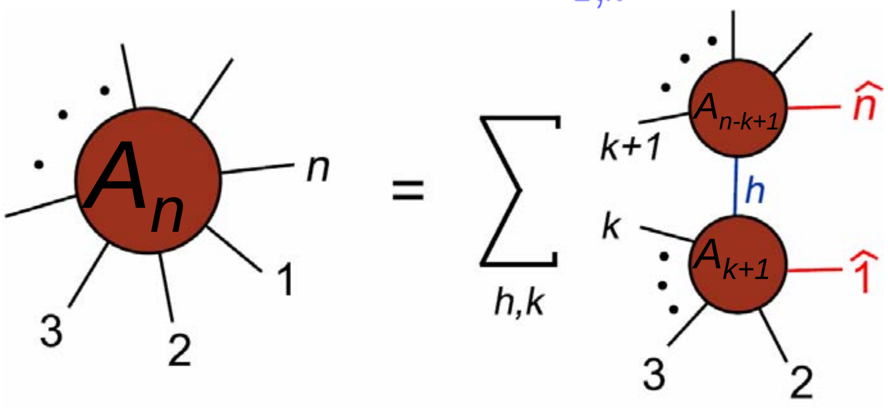
Schwinn, Weinzierl, hep-th/0503015

# Even better than MHV rules

**On-shell recursion relations**

Britto, Cachazo, Feng, hep-th/0412308

$$A_n(1,2,\ldots,n) \;=\; \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \times \frac{i}{K_{1,k}^2} A_{n-k+1}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$

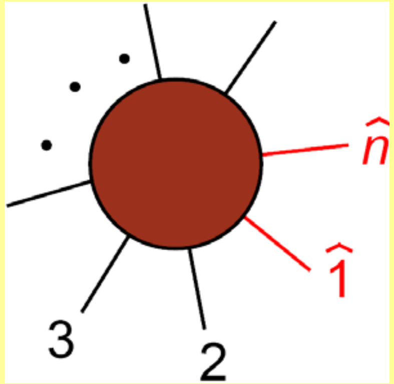



$A_{k+1}$ and $A_{n-k+1}$ are on-shell tree amplitudes with fewer legs, evaluated with 2 momenta shifted by a **complex** amount

# Proof of on-shell tree recursion

Britto, Cachazo, Feng, Witten, hep-th/0501052

- Consider a family of on-shell amplitudes $A_n(z)$ depending on a complex parameter $z$ which shifts the momenta.
- Best described using spinor variables.
- For example, the (n,1) shift:




$$\lambda_1 \to \hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_1 \to \tilde{\lambda}_1$$

$$\lambda_n \to \lambda_n \qquad \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

- On-shell condition: $(\hat{k}_1)^\mu(\hat{k}_1)_\mu \;=\; (\hat{k}_1)^{\alpha\dot{\alpha}}(\hat{k}_1)_{\dot{\alpha}\alpha} \;=\; \langle(\lambda_1 + z\lambda_n)(\lambda_1 + z\lambda_n)\rangle[1\,1] = 0$

  similarly, $\hat{k}_n^2 \;=\; 0$
- Momentum conservation:

$$\hat{k}_1 + \hat{k}_n \;=\; (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 + \lambda_n(\tilde{\lambda}_n - z\tilde{\lambda}_1) = k_1 + k_n$$

# MHV example

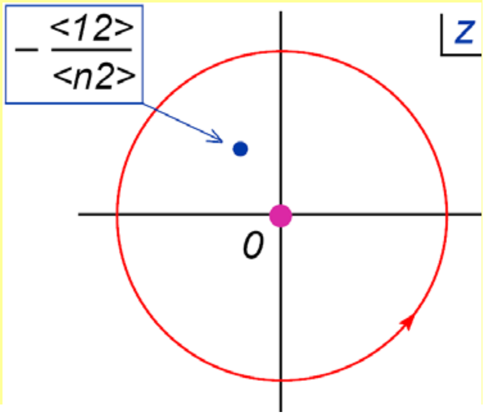
- Apply this shift to the Parke-Taylor (MHV) amplitudes:

$$A_n(z=0) = A_n^{jn,\,\mathrm{MHV}} \;=\; \frac{\langle j\, n\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

- Under the (n,1) shift: $\lambda_1 \to \lambda_1 + z\lambda_n \qquad \tilde\lambda_n \to \tilde\lambda_n - z\tilde\lambda_1$

$$\begin{aligned}\langle n\,1\rangle &= \lambda_n\lambda_1 \to \lambda_n(\lambda_1 + z\lambda_n) = \langle n\,1\rangle + z\langle n\,n\rangle = \langle n\,1\rangle \\ \langle 1\,2\rangle &= \lambda_1\lambda_2 \to (\lambda_1 + z\lambda_n)\lambda_2 = \langle 1\,2\rangle + z\langle n\,2\rangle\end{aligned}$$

- So $$A_n(z) \;=\; \frac{\langle j\, n\rangle^4}{(\langle 1\,2\rangle + z\langle n\,2\rangle)\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$




- Consider: $$\frac{1}{2\pi i}\oint_C dz \frac{A_n(z)}{z}$$

- 2 poles, opposite residues

# MHV example (cont.)

- MHV amplitude obeys: $$A_n(0) = - \underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{A_n(z)}{z}$$

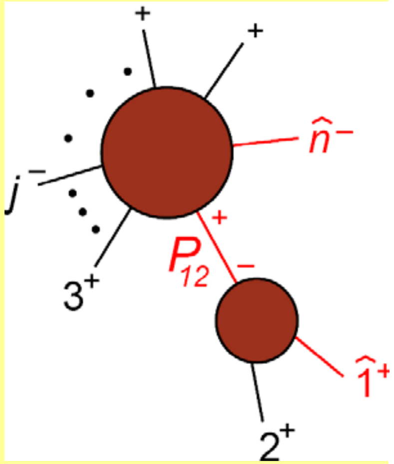
- Compute residue using factorization

- At $z = -\frac{\langle 1\,2\rangle}{\langle n\,2\rangle} = -\frac{\langle 1\,2\rangle[2\,1]}{\langle n\,2\rangle[2\,1]} = -\frac{s_{12}}{\langle n^-|(1+2)|1^-\rangle}$

  kinematics are complex collinear

$$\langle \hat{1}\,2\rangle = \langle 1\,2\rangle + z\langle n\,2\rangle = 0 \qquad [\hat{1}\,2] = [1\,2] \neq 0$$

$$s_{\hat{1}2} = \langle \hat{1}\,2\rangle[2\,\hat{1}] = 0$$




- so $$- \underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{A_n(z)}{z} = A_{n-1}(\hat{P}^+_{12}, 3^+, \ldots, j^-, \ldots, n^-)$$
$$\times \left[ - \underset{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}}{\text{Res}} \frac{1}{z} \frac{1}{\hat{P}^2_{12}(z)} \right] A_3(\hat{1}^+, 2^+, -\hat{P}^-_{12})$$

note
$A_3(+,+,+) = 0$

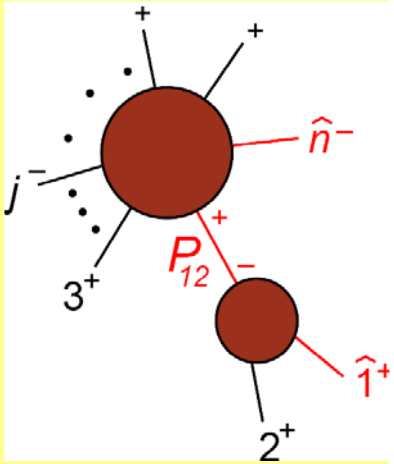
# Evaluate ingredients

- Since $\hat{P}_{12}^2(z) \;=\; (k_1 + k_2 + z\lambda_n\tilde{\lambda}_1)^2 = s_{12} + z\langle n^-|(1+2)|1^-\rangle$

$$-\operatorname*{Res}_{z=-\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}} \frac{1}{z}\frac{1}{\hat{P}_{12}^2(z)} \;=\; -\frac{\langle n^-|(1+2)|1^-\rangle}{s_{12}}\frac{1}{\langle n^-|(1+2)|1^-\rangle} = \frac{1}{s_{12}}$$

- So

$$\boxed{A_n(0) \;=\; A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)\frac{1}{s_{12}}A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)}$$




- Check this explicitly:

$$\begin{aligned} A_n(0) \;&=\; \frac{\langle j\,\hat{n}\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,\hat{n}\rangle\langle \hat{n}\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[\hat{1}\,2]^3}{[2\,\hat{P}][\hat{P}\,\hat{1}]} \\ &=\; \frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]} \end{aligned}$$

# MHV check (cont.)

- Using $$\begin{aligned}\langle n\,\hat{P}\rangle[\hat{P}\,2] &= \langle n^-|(1+2)|2^-\rangle + z\langle n\,n\rangle[1\,2] = \langle n\,1\rangle[1\,2]\\ \langle 3\,\hat{P}\rangle[\hat{P}\,1] &= \langle 3^-|(1+2)|1^-\rangle + z\langle 3\,n\rangle[1\,1] = \langle 3\,2\rangle[2\,1]\end{aligned}$$

one confirms

$$\begin{aligned}A_n(0) &= \frac{\langle j\,n\rangle^4}{\langle\hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}\\ &= \frac{\langle j\,n\rangle^4[1\,2]^3}{(\langle 1\,2\rangle[2\,1])([1\,2]\langle 2\,3\rangle)(\langle n\,1\rangle[1\,2])\langle 3\,4\rangle\cdots\langle n-1,n\rangle}\\ &= \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle}\\ &= A_n^{jn,\,\mathrm{MHV}}\end{aligned}$$

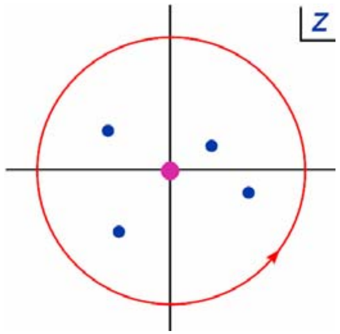
So MHV amplitudes from *n=4* on are derived recursively

# The general case

**Same analysis as above – Cauchy’s theorem + amplitude factorization**

Let complex momentum shift depend on *z*. Use analyticity in *z*.

$$\hat\lambda_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde\lambda}_1 = \tilde\lambda_1$$
$$\hat\lambda_n = \lambda_n \qquad \hat{\tilde\lambda}_n = \tilde\lambda_n - z\tilde\lambda_1 \qquad \Rightarrow \quad A(0) \;\to\; A(z)$$

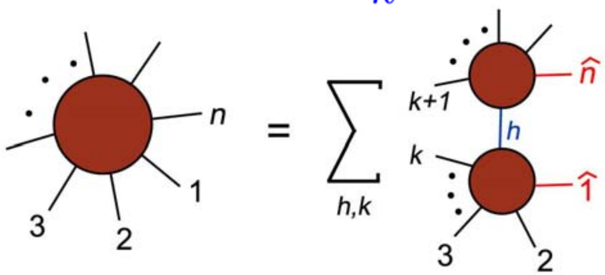



Cauchy: If $A(\infty) = 0$ then

$$0 = \frac{1}{2\pi i}\oint dz\,\frac{A(z)}{z} = A(0) + \sum_k \mathrm{Res}\Big[\frac{A(z)}{z}\Big]\Big|_{z=z_k}$$

poles in *z*: physical factorizations $\hat K^2_{1,k} = 0$

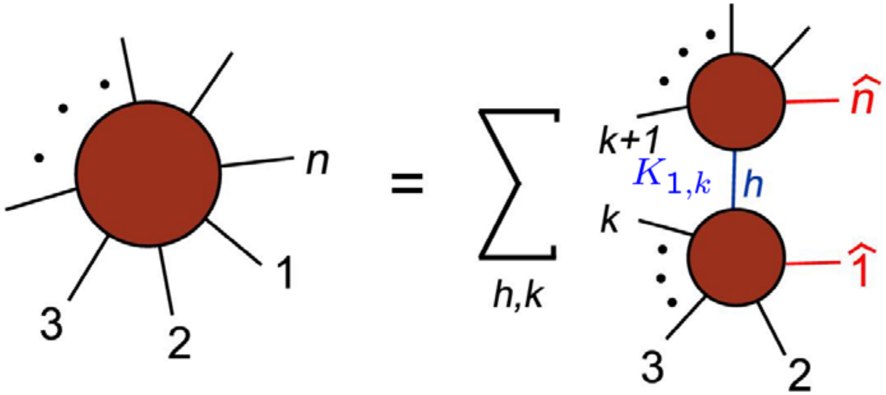
residue at $z_k = -\frac{K^2_{1,k}}{\langle n^- | \not{K}_{1,k} | 1^-\rangle}$ = [$k^{\text{th}}$ term]



# Momentum shift

Shift for $k^{\text{th}}$ term comes from setting $z = z_k$, where

$$z_k = -\frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}$$




is the solution to

$$\hat{K}_{1,k}^2(z) = 0 = (K_{1,k} + z\lambda_n\tilde{\lambda}_1)^2 = K_{1,k}^2 + z\lambda_n^a (K_{1,k})_{a\dot{a}} \tilde{\lambda}_1^{\dot{a}}$$

plugging in, shift is:

$$\hat{\lambda}_1 = \lambda_1 - \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$$

$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n + \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle} \tilde{\lambda}_1$$

# To show: $A(\infty) = 0$

Britto, Cachazo, Feng, Witten, hep-th/0501052

**Propagators:**

$$\frac{1}{\hat{K}_{1,k}^2(z)} = \frac{1}{K_{1,k}^2 + z\lambda_n^a (K_{1,k})_{a\dot{a}} \tilde{\lambda}_1^{\dot{a}}} \sim \frac{1}{z}$$

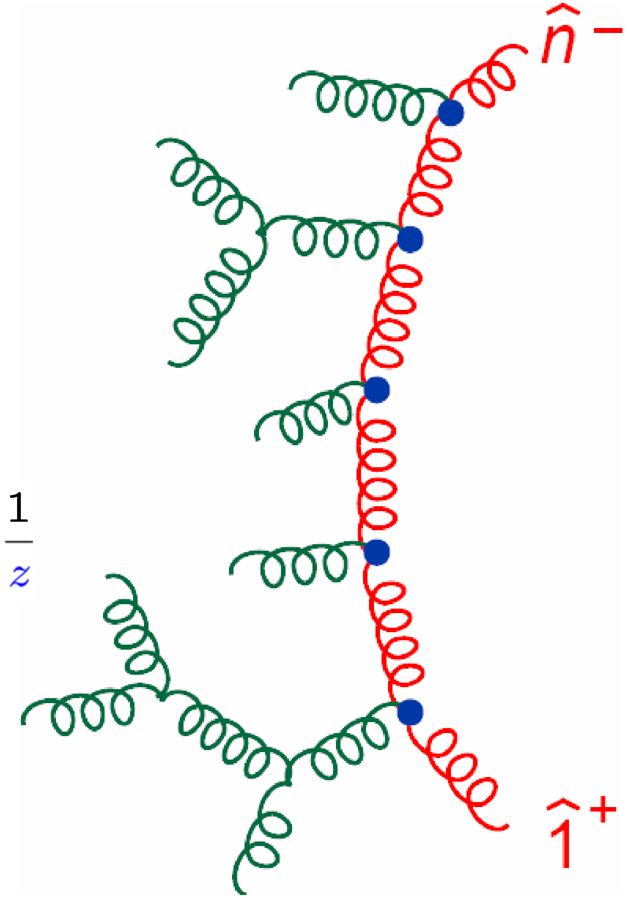
**3-point vertices:** $\propto \ \hat{k}^\mu(z) \propto z$

**Polarization vectors:**

$$\not{\epsilon}_1^+ \ \propto \ \frac{\tilde{\lambda}_1 \lambda_q}{\langle \lambda_1 \, \lambda_q \rangle} \ \propto \ \frac{1}{z} \qquad \not{\epsilon}_n^- \ \propto \ \frac{\lambda_n \tilde{\lambda}_q}{\langle \tilde{\lambda}_n \, \tilde{\lambda}_q \rangle} \ \propto \ \frac{1}{z}$$

**Total:**

$$\frac{1}{z} \times \left( z \frac{1}{z} \right)^r z \times \frac{1}{z} = \frac{1}{z}$$



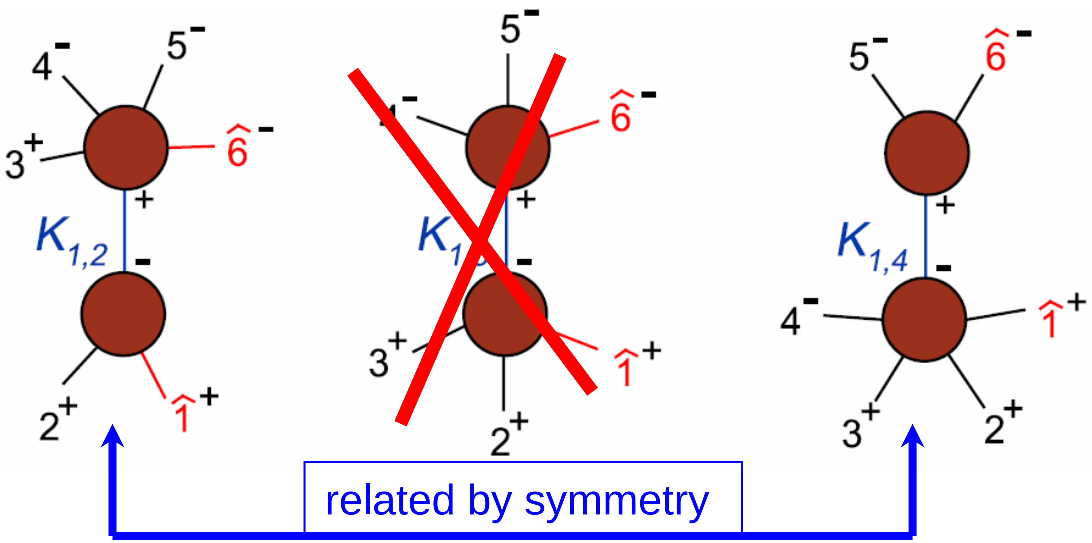
# A 6-gluon example

220 Feynman diagrams for *gggggg*

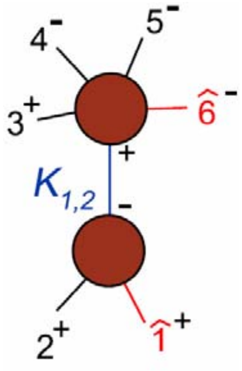
Helicity + color + MHV results + symmetries

$\Rightarrow$ only $\boxed{A_6(1^+,2^+,3^+,4^-,5^-,6^-)}$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

3 BCF diagrams $\rightarrow$ 2 $\rightarrow$ 1



# The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram




$$= -\frac{i}{s_{12}} \frac{[\hat{1}\,2]^3}{[2\,\hat{K}][\hat{K}\,\hat{1}]} \frac{[\hat{K}\,3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\,\hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\,2]^3}{([2\,\hat{K}]\langle\hat{K}\,6\rangle)(\langle 6\,\hat{K}\rangle[\hat{K}\,1])} \frac{(\langle 6\,\hat{K}\rangle[\hat{K}\,3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle)}$$

$$= i\,\frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\begin{aligned} \langle 6\,\hat{K}\rangle[\hat{K}\,a] &= \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a] \\ &= \langle 6^-|(1+2)|a^-\rangle \end{aligned}$$

$$[5\,\hat{6}] = [5\,6] + \frac{s_{12}[5\,1]}{\langle 6\,2\rangle[2\,1]} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\,2\rangle}$$

$$[\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

# Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\langle 3\,4\rangle[5\,6][6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s despite (because of?) spurious singularities $\langle 2^-|(6+1)|5^-\rangle$

Mangano, Parke, Xu (1988)

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{([1\,2]\langle 4\,5\rangle\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\langle 5\,6\rangle\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

Relative simplicity even more striking for *n>6*

Bern, Del Duca, LD, Kosower (2004)

# On-shell recursion at tree-level

Rules even more efficient, and easily extendable than MHV rules:

- massless quarks

Luo, Wen, hep-th/0501121, 0502009

- **massive** scalars

Badger, Glover, Khoze, Svrcek, hep-th/0504159;
Forde, Kosower, hep-th/0507292

- **massive vector bosons** and **fermions**

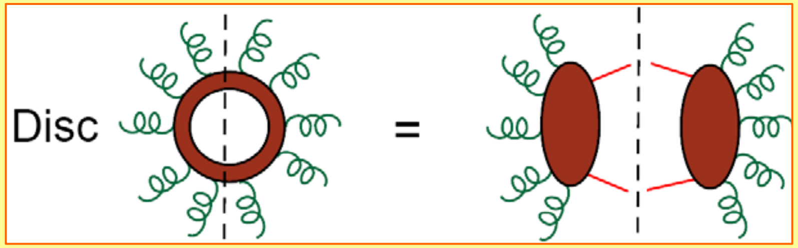
Badger, Glover, Khoze,
hep-th/0507161

# Unitarity

- Unitarity is an efficient method for determining imaginary parts of loop amplitudes:

$$S = 1 + iA$$
$$S^\dagger S = 1 \quad \Rightarrow \quad 1 = (1 - iA^\dagger)(1 + iA)$$
$$\Rightarrow \quad -i(A - A^\dagger) = 2\,\mathrm{Im}\,A = \mathrm{Disc}\,A = A^\dagger A$$

- Efficient because it recycles trees into loops

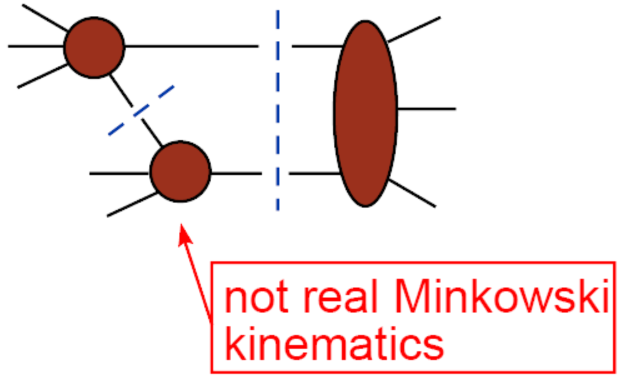



- Only thing missing: rational functions
- Can get these using on-shell recursion relations

# Generalized unitarity

Eden, Landshoff, Olive, Polkinghorne (1966); Bern, LD, Kosower, hep-ph/9708239; Britto, Cachazo, Feng, hep-th/0412103

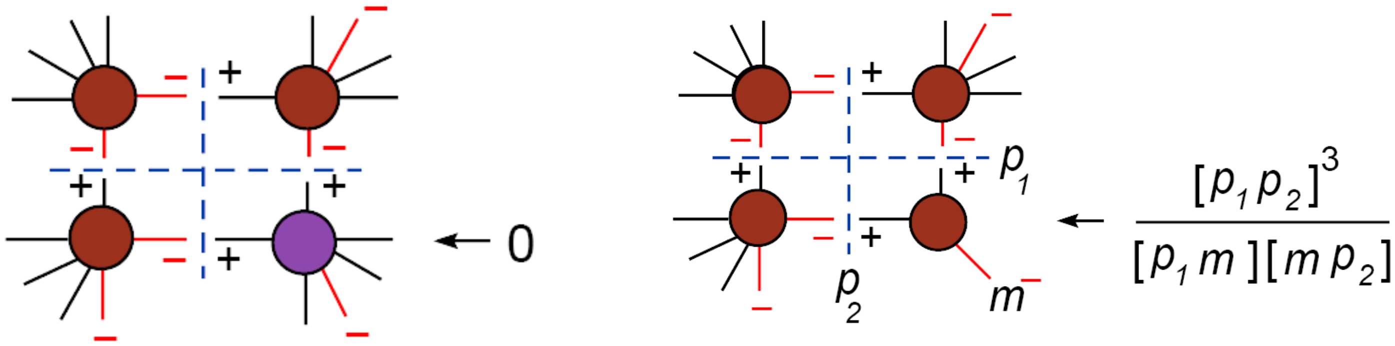
- Triangle and box integrals have 3 or 4 propagators "on shell". Can extract from more restrictive cut kinematics, such as:




- Get a product of 3 or 4 simpler tree amplitudes, compared with the ordinary cut.

# Generalized unitarity (cont.)

- For example, use quadruple cut to show all 4-mass box integrals vanish in all NMHV amplitudes. Bern, Del Duca, LD, Kosower, hep-th/0410224
  (Have $3+4=7$ negative helicities; need $2\times 4=8$.)



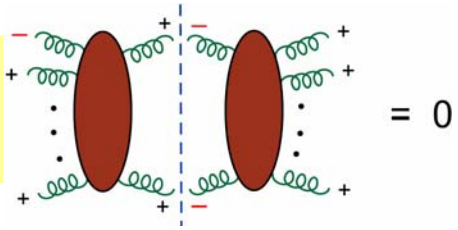

- 3-mass boxes do not vanish, because 3-point "amplitude" can be $(++-)$ (in complex kinematics).
- Computation of $c^{3\mathrm{m}}$ from quadruple cut can be done algebraically because all 4 components of loop momentum are frozen by the 4 on-shell constraints Britto, Cachazo, Feng, hep-th/0412103

# On-shell recursion at one loop

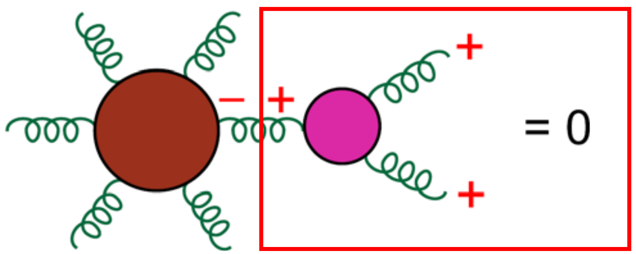
Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005

- **Same techniques** can be used to compute one-loop amplitudes
 -- which are much harder to obtain by other methods than are trees.

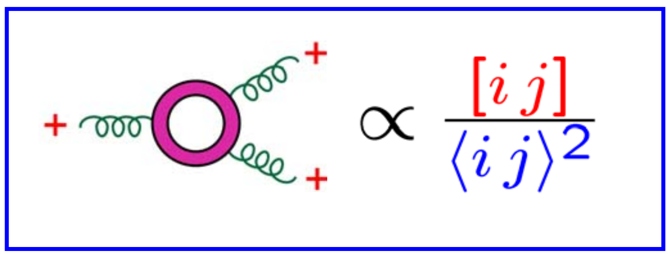
- First consider special tree-like one-loop amplitudes with no cuts, only poles: $A_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$



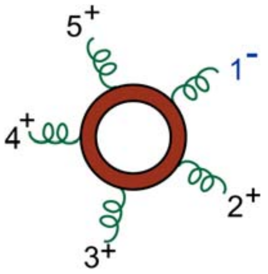

- **New features** arise compared with tree case due to different collinear behavior of loop amplitudes:




but



# A one-loop pole analysis

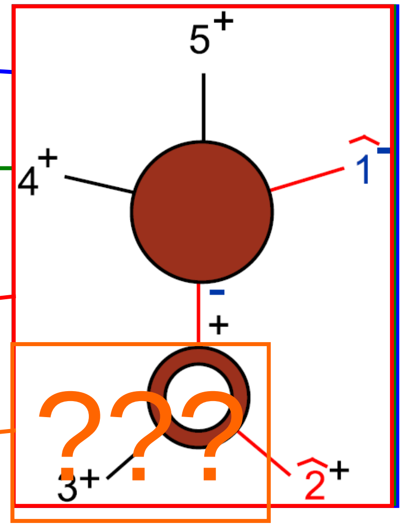



$$= -\frac{[2\,5]^3}{[5\,1][1\,2]\langle 3\,4\rangle^2} + \frac{\langle 1\,4\rangle^3[4\,5]\langle 3\,5\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle^2\langle 4\,5\rangle^2} + \frac{\langle 1\,3\rangle^3[2\,3]\langle 2\,4\rangle}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$

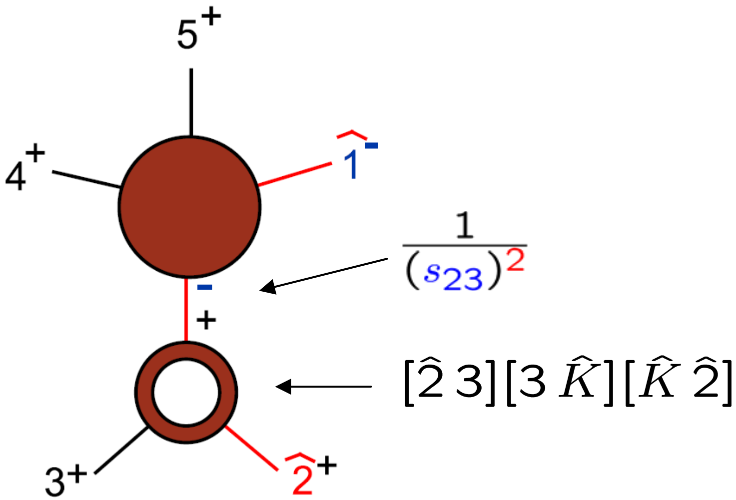
Bern, LD, Kosower, hep-ph/9302280

under shift $\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 - z\tilde{\lambda}_2 \qquad \hat{\lambda}_2 = \lambda_2 + z\lambda_1$ plus partial fraction

$$\Rightarrow \quad -\frac{[2\,5]^3}{([5\,1] - z[5\,2])[1\,2]\langle 3\,4\rangle^2} + \frac{\langle 1\,4\rangle^3[4\,5]\langle 3\,5\rangle}{\langle 1\,2\rangle(\langle 2\,3\rangle + z\langle 1\,3\rangle)\langle 3\,4\rangle^2\langle 4\,5\rangle^2} - \frac{\langle 1\,3\rangle^2[2\,3]\langle 1\,2\rangle\langle 3\,4\rangle}{(\langle 2\,3\rangle + z\langle 1\,3\rangle)^2\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle} - \frac{\langle 1\,3\rangle^2[2\,3]\langle 1\,4\rangle}{(\langle 2\,3\rangle + z\langle 1\,3\rangle)\langle 3\,4\rangle^2\langle 4\,5\rangle\langle 5\,1\rangle}$$



# Underneath the double pole




Missing diagram should be related, but suppressed by factor of $s_{23}$

Don't know collinear behavior at this level, must guess the correct suppression factor:

$$s_{23}\,\mathcal{S}(a,\hat{K}^+,b)\,\mathcal{S}(c,(-\hat{K})^-,d)$$

in terms of universal eikonal factors for soft gluon emission

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

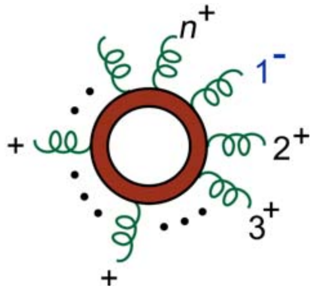
Here, multiplying 3rd diagram by

$$s_{23}\,\mathcal{S}(\hat{1},\hat{K}^+,4)\,\mathcal{S}(3,(-\hat{K})^-,\hat{2})$$

gives the correct missing term!

# A one-loop all-*n* recursion relation

Same suppression factor works in the case of *n* external legs!



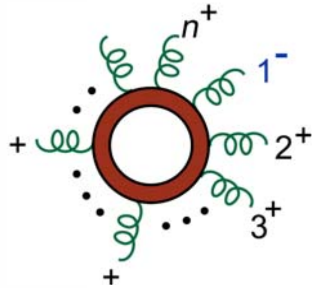

$$
\begin{aligned}
&A_n^{(1)}(1^-,2^+,\ldots,n^+) \\
&\quad = A_{n-1}^{(1)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^+)\,\frac{i}{K_{23}^2}\,A_3^{(0)}(\hat{2}^+,3^+,-\hat{K}_{23}^-) \\
&\quad + \sum_{j=4}^{n-1} A_{n-j+2}^{(0)}((j+1)^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{2\ldots j}^-)\,\frac{i}{K_{2\ldots j}^2}\,A_j^{(1)}(\hat{2}^+,3^+,\ldots,j^+,-\hat{K}_{2\ldots j}^+) \\
&\quad + A_{n-1}^{(0)}(4^+,5^+,\ldots,n^+,\hat{1}^-,\hat{K}_{23}^-)\,\frac{i}{(K_{23}^2)^2}\,V_3^{(1)}(\hat{2}^+,3^+,-\hat{K}_{23}^+) \\
&\qquad\qquad \times \Big(1 + K_{23}^2\,\mathcal{S}^{(0)}(\hat{1},\hat{K}_{23}^+,4)\,\mathcal{S}^{(0)}(3,-\hat{K}_{23}^-,\hat{2})\Big)
\end{aligned}
$$

Know it works because results agree with Mahlon, hep-ph/9312276, though much shorter formulae are obtained from this relation

# Solution to recursion relation

hep-ph/0505055

$$A_n^{(1)}(1^-,2^+,3^+,\ldots,n^+) = \frac{i}{3}\frac{T_1+T_2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,$$




where

$$T_1 = \sum_{l=2}^{n-1}\frac{\langle 1\,l\rangle\,\langle 1\,(l+1)\rangle\,\langle 1^-|\,\not{K}_{l,l+1}\not{K}_{(l+1)\cdots n}\,|1^+\rangle}{\langle l\,(l+1)\rangle}\,,$$

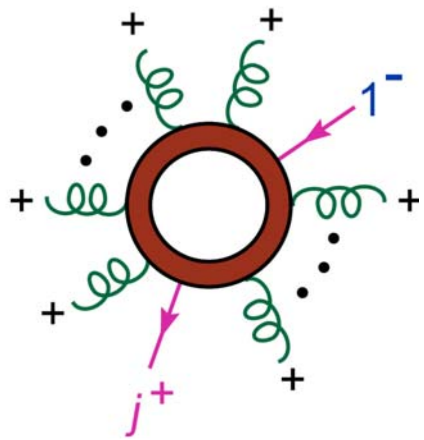
$$\begin{aligned} T_2 = \sum_{l=3}^{n-2}\sum_{p=l+1}^{n-1} & \frac{\langle (l-1)\,l\rangle}{\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|(l-1)^+\rangle\,\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|l^+\rangle} \\ & \times\frac{\langle p\,(p+1)\rangle}{\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|p^+\rangle\,\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|(p+1)^+\rangle} \\ & \times\left\langle 1^-\right|\not{K}_{l\cdots p}\not{K}_{(p+1)\cdots n}\left|1^+\right\rangle^3 \\ & \times\frac{\langle 1^-|\,\not{K}_{2\cdots(l-1)}[\mathcal{F}(l,p)]^2\not{K}_{(p+1)\cdots n}\,|1^+\rangle}{s_{l\cdots p}}\,. \end{aligned}$$

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1}\sum_{m=i+1}^{p}\not{k}_i\not{k}_m$$

# External fermions too

hep-ph/0505055

Can similarly write down recursion relations
for the finite, cut-free amplitudes with 2 external fermions:




and the solutions are just as compact

# Loop amplitudes with cuts

- Recently extended same recursive technique (combined with unitarity) to loop amplitudes with cuts (hep-ph/0507005)
- Here rational-function terms contain "spurious singularities", e.g. $\sim \frac{\ln(r)+1-r}{(1-r)^2}, \quad r = s_2/s_1$
- accounting for them properly yields simple "overlap diagrams" in addition to recursive diagrams
- No loop integrals required to bootstrap the rational functions from the cuts and lower-point amplitudes
- Tested method on 5-point amplitudes, used it to compute $A_6(1^-,2^-,3^+,4^+,5^+,6^+)$, $A_7(1^-,2^-,3^+,4^+,5^+,6^+,7^+)$

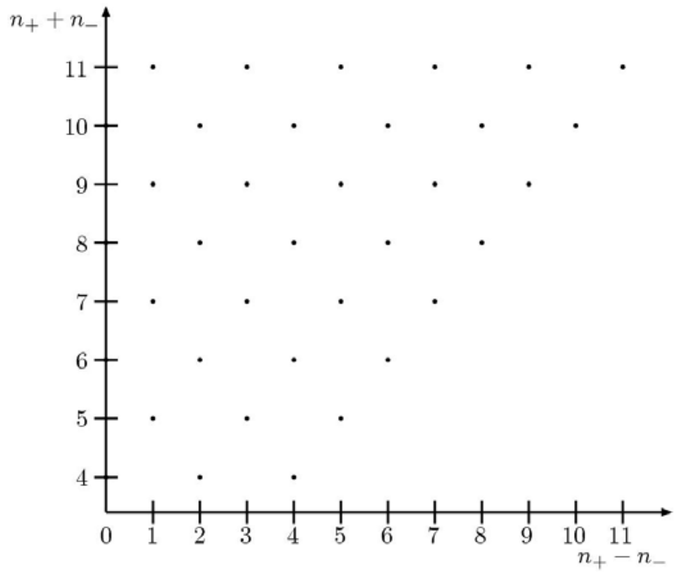
# Conclusions

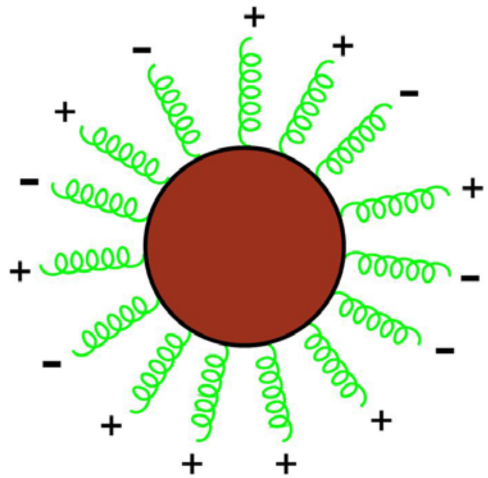
- MHV rules, and especially on-shell recursion relations a very efficient way to compute multi-leg tree amplitudes in gauge theory
- Development a spinoff from twistor string theory
- Also much progress on loops in supersymmetric theories using (generalized) unitarity
- Quite recently, new loop amplitudes in QCD, needed for colliders, are beginning to fall to twistor-inspired recursive approaches
- Prospects look very good for attacking a wide range of multi-parton processes in this way

# Some other reviews

- V.V. Khoze, hep-th/0408233
- F. Cachazo, P. Svrcek, hep-th/0504194 (Trieste lectures)
- N. Glover, talk at SUSY2005
  http://susy-2005.dur.ac.uk/PLENARY/WED/GLOVERsusy.pdf

# March of the *n*-gluon helicity amplitudes




$n_+$ positive-helicity gluons

$n_-$ negative-helicity gluons

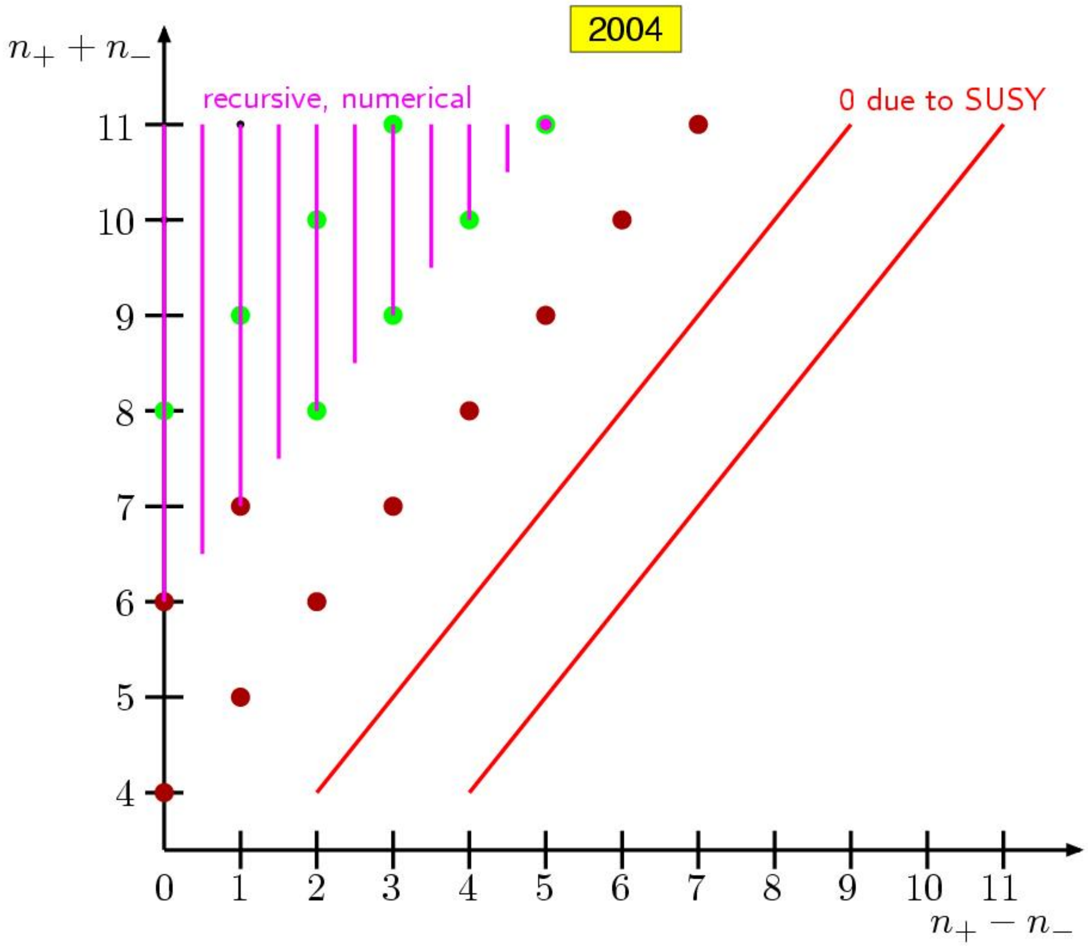
$n = n_+ + n_- \geq 4$

$n_+ \geq n_-$ by parity

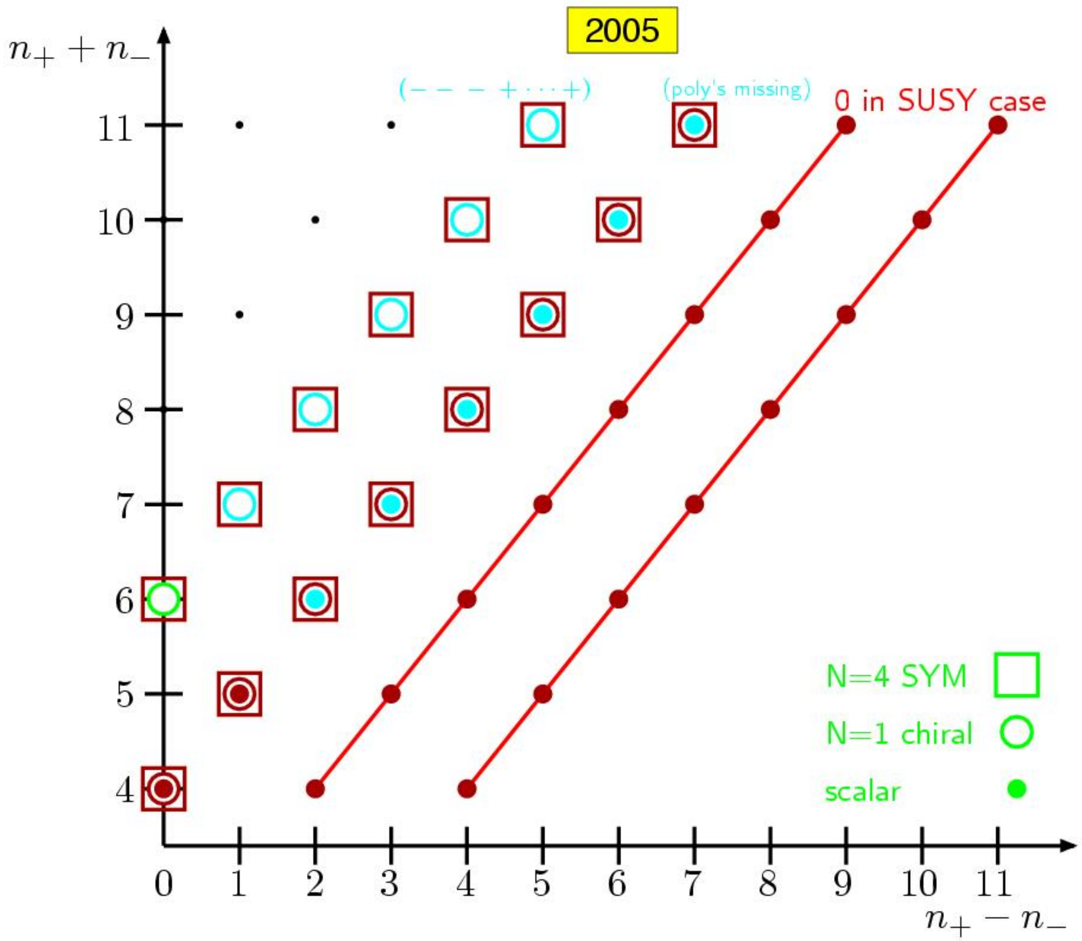
At 1-loop, QCD decomposable into

N=4 SYM, N=1 chiral, scalar contributions

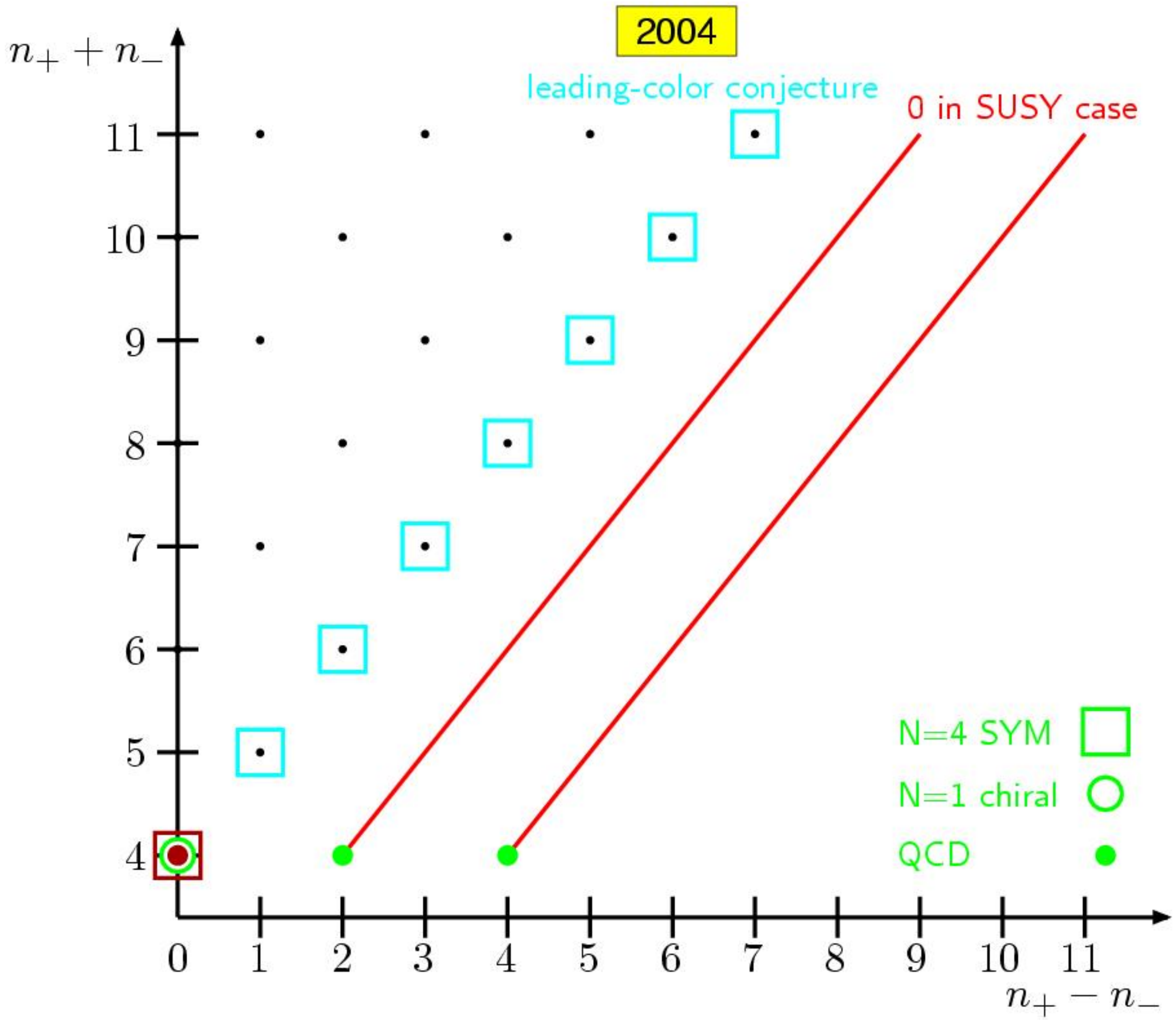
# March of the tree amplitudes



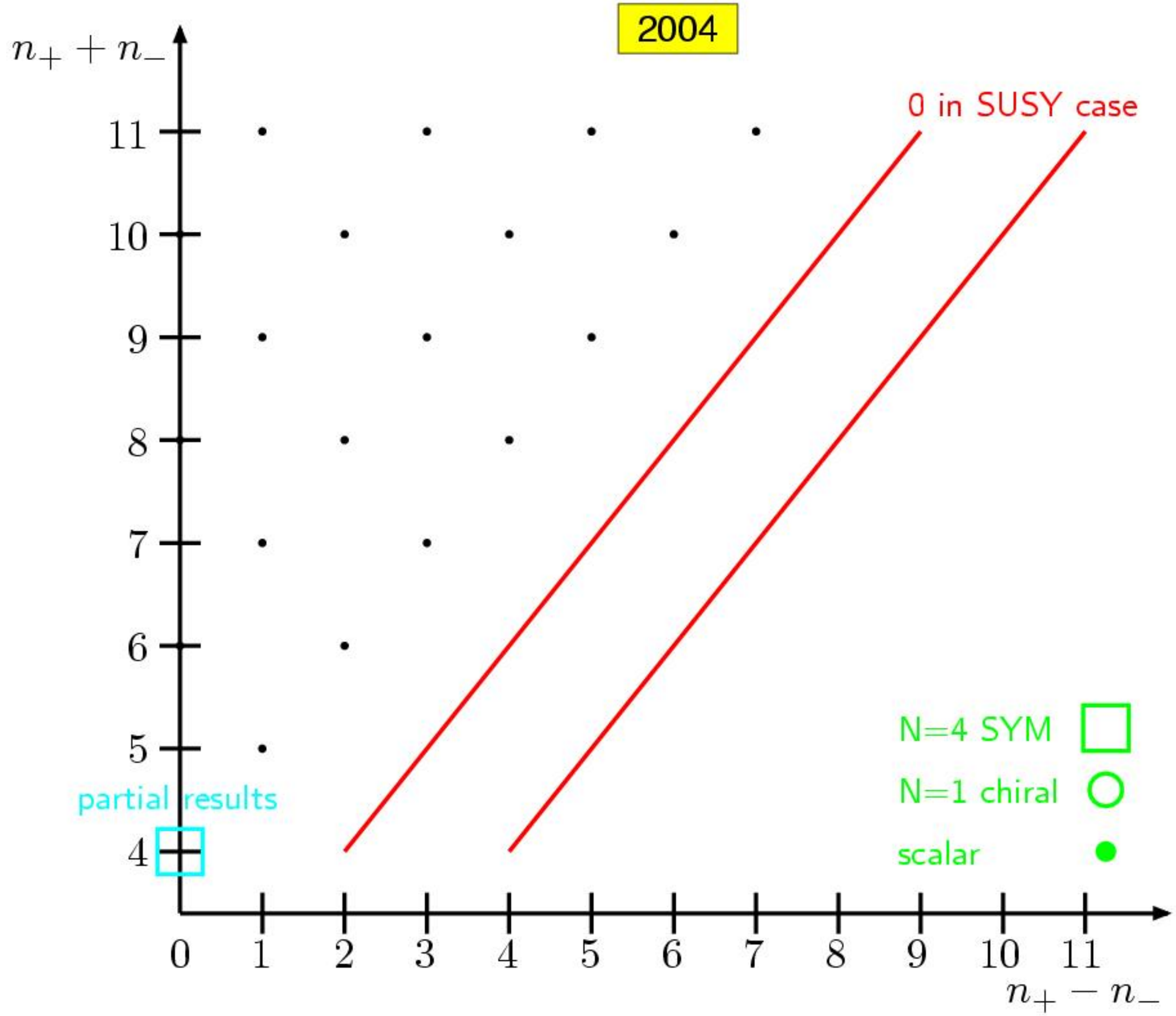
# March of the 1-loop amplitudes



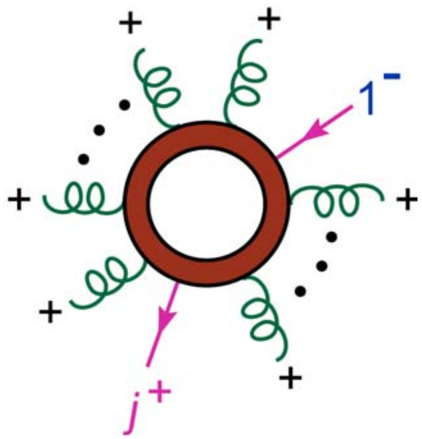
# March of the 2-loop amplitudes



# March of the 3-loop amplitudes



# Fermionic solutions




$$A_n^{L-s}(1_f^-, 2^+, \ldots, j_f^+, \ldots, n^+) = \frac{i}{2}\frac{\langle 1\,j\rangle\, \Sigma_{l=3}^{n-1}\langle 1^-|\,\not{K}_{2\cdots l}\not{k}_l\,|1^+\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

and

$$A_n^s(j_f^+) = \frac{i}{3}\frac{S_1 + S_2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,$$

where

$$S_1 = \sum_{l=j+1}^{n-1}\frac{\langle j\,l\rangle\,\langle 1\,(l+1)\rangle\,\langle 1^-|\,\not{K}_{l,l+1}\not{K}_{(l+1)\cdots n}\,|1^+\rangle}{\langle l\,(l+1)\rangle}\,,$$

$$\begin{aligned}
S_2 = \sum_{l=j+1}^{n-2}\sum_{p=l+1}^{n-1} & \frac{\langle (l-1)\,l\rangle}{\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|(l-1)^+\rangle\,\langle 1^-|\,\not{K}_{(p+1)\cdots n}\not{K}_{l\cdots p}\,|l^+\rangle} \\
& \times\frac{\langle p\,(p+1)\rangle}{\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|p^+\rangle\,\langle 1^-|\,\not{K}_{2\cdots(l-1)}\not{K}_{l\cdots p}\,|(p+1)^+\rangle} \\
& \times\left\langle 1^-\right|\,\not{K}_{l\cdots p}\not{K}_{(p+1)\cdots n}\left|1^+\right\rangle^2\left\langle j^-\right|\,\not{K}_{l\cdots p}\not{K}_{(p+1)\cdots n}\left|1^+\right\rangle \\
& \times\frac{\langle 1^-|\,\not{K}_{2\cdots(l-1)}[\mathcal{F}(l,p)]^2\not{K}_{(p+1)\cdots n}\,|1^+\rangle}{s_{l\cdots p}}\,,
\end{aligned}$$

$$\mathcal{F}(l,p) = \sum_{i=l}^{p-1}\sum_{m=i+1}^{p}\not{k}_i\not{k}_m$$