# Antenna Subtraction at NNLO

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Loopfest IV – Snowmass 2005

# Outline

- Jet observables
- Jets in perturbation theory
- Antenna subtraction at NLO
- Antenna subtraction at NNLO
  - Double real radiation
- Different antenna types

# Jet Observables

## Experimentally:

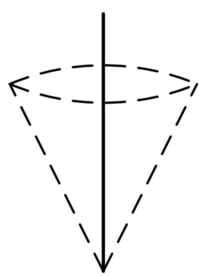
- major testing ground of QCD in $e^+e^-$ annihilation
- measurement of the 3–Jet production rate and related event shape observables allows a precise determination of $\alpha_s$
- current error on $\alpha_s$ from jet observables dominated by theoretical uncertainty:

$$\begin{aligned} \alpha_s(M_Z) &= 0.1202 \pm 0.0003(\text{stat}) \pm 0.0009(\text{sys}) \\ &\quad \pm 0.0009(\text{had}) \pm 0.0047(\text{scale}) \end{aligned}$$

# Jet Observables

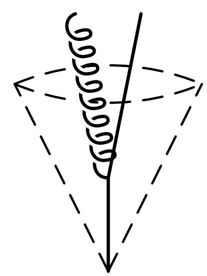
## Theoretically:

- Partons are combined into jets using the same jet algorithm (recombination procedure) as in experiment



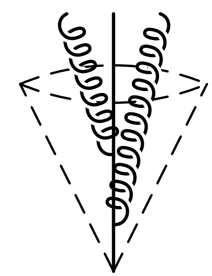
LO

each parton forms 1 jet on its own



NLO

2 partons in 1 jet, 1 parton experimentally unresolved



NNLO

3 partons in 1 jet, 2 partons experimentally unresolved

Current state-of-the-art: NLO

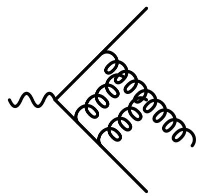
Need for NNLO:

- reduce error on $\alpha_s$
- better matching of parton level and hadron level jet algorithm

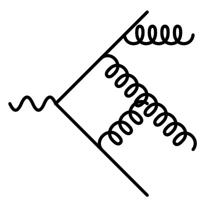
# Jets in Perturbation Theory

## Ingredients to NNLO $m$-jet:

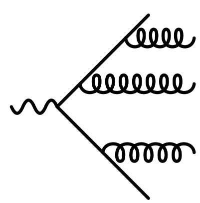
- Two-loop matrix elements

$|\mathcal{M}|^2_{\text{2-loop},m\text{ partons}}$  explicit infrared poles from loop integrals

- One-loop matrix elements

$|\mathcal{M}|^2_{\text{1-loop},m+1\text{ partons}}$  explicit infrared poles from loop integral and implicit infrared poles due to single unresolved radiation

- Tree level matrix elements

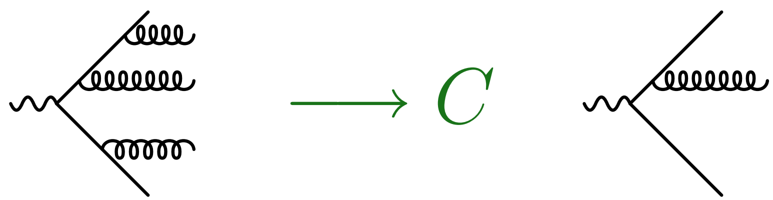
$|\mathcal{M}|^2_{\text{tree},m+2\text{ partons}}$  implicit infrared poles due to double unresolved radiation

Infrared Poles cancel in the sum

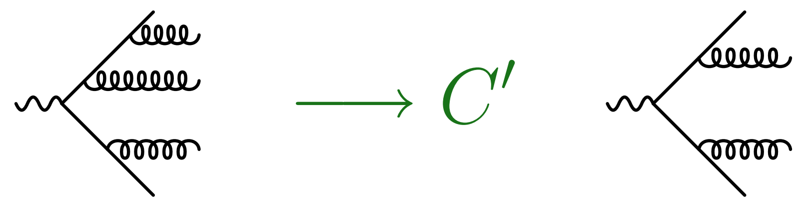
# Real Corrections at NNLO

## Infrared subtraction terms

$m+2$ partons $\rightarrow m$ jets:




$m+2 \rightarrow m+1$ pseudopartons $\rightarrow m$ jets:




- Double unresolved configurations:
  - triple collinear
  - double single collinear
  - soft/collinear
  - double soft

- Single unresolved configurations:
  - collinear
  - soft

J. Campbell, E.W.N. Glover; S. Catani, M. Grazzini

Issue: find subtraction functions which

- approximate full $m+2$ matrix element in all singular limits
- are sufficiently simple to be integrated analytically

# NLO Antenna Subtraction

Structure of NLO $m$-jet cross section (subtraction formalism):

Z. Kunszt, D. Soper

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} \left(d\sigma^R_{NLO} - d\sigma^S_{NLO}\right) + \left[\int_{d\Phi_{m+1}} d\sigma^S_{NLO} + \int_{d\Phi_m} d\sigma^V_{NLO}\right]$$

- $d\sigma^S_{NLO}$: local counter term for $d\sigma^R_{NLO}$
- $d\sigma^R_{NLO} - d\sigma^S_{NLO}$: free of divergences, can be integrated numerically

Building block of $d\sigma^S_{NLO}$: NLO-Antenna function $X^0_{ijk}$

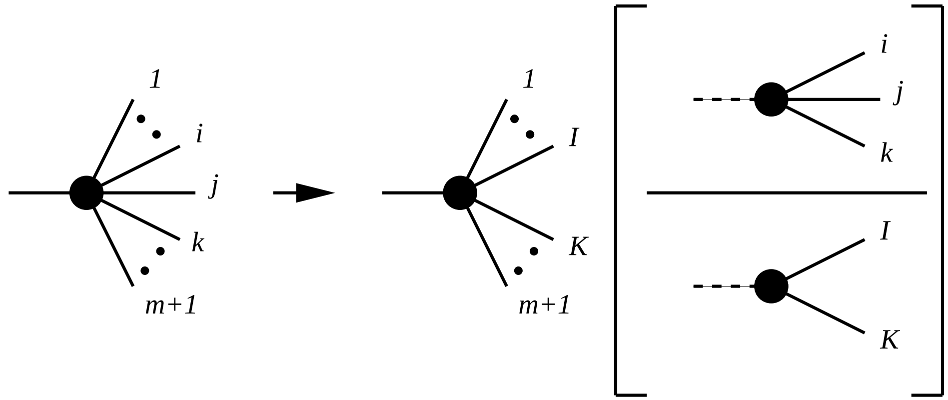
Normalised and colour-ordered 3-parton matrix element with 2 radiators and 1 radiated parton in between

(J. Campbell, M. Cullen, E.W.N. Glover; D. Kosower)

$$d\sigma^S_{NLO} = \mathcal{N} \sum_{m+1} d\Phi_{m+1}(p_1, \ldots, p_{m+1}; q) \frac{1}{S_{m+1}}$$
$$\times \sum_j X^0_{ijk} \, |\mathcal{M}_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_K, \ldots, p_{m+1})|^2 \, J^{(m)}_m(p_1, \ldots, \tilde{p}_I, \tilde{p}_K, \ldots, p_{m+1})$$

Dipole formalism (S. Catani, M. Seymour): two dipoles = one antenna

# NLO Antenna Subtraction




$$X^0_{ijk} = S_{ijk,IK} \frac{|M^0_{ijk}|^2}{|M^0_{IK}|^2} \qquad \mathrm{d}\Phi_{X_{ijk}} = \frac{\mathrm{d}\Phi_3}{P_2}$$

Phase space factorisation

$$\mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q) = \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1};q)\cdot \mathrm{d}\Phi_{X_{ijk}}(p_i,p_j,p_k;\tilde{p}_I+\tilde{p}_K)$$

Integrated subtraction term (analytically)

$$|\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijk}}\, X^0_{ijk} \sim |\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_3 |M^0_{ijk}|^2$$

can be combined with $\mathrm{d}\sigma^V_{NLO}$

# NNLO Infrared Subtraction

Structure of NNLO $m$-jet cross section:

$$
\begin{aligned}
\mathrm{d}\sigma_{NNLO} \quad = \quad & \int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\sigma^{R}_{NNLO} - \mathrm{d}\sigma^{S}_{NNLO}\right) + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^{S}_{NNLO} \\
& + \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^{V,1}_{NNLO} - \mathrm{d}\sigma^{VS,1}_{NNLO}\right) + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{NNLO} \\
& + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V,2}_{NNLO} \,,
\end{aligned}
$$

- $\mathrm{d}\sigma^{S}_{NNLO}$: real radiation subtraction term for $\mathrm{d}\sigma^{R}_{NNLO}$
- $\mathrm{d}\sigma^{VS,1}_{NNLO}$: one-loop virtual subtraction term for $\mathrm{d}\sigma^{V,1}_{NNLO}$
- $\mathrm{d}\sigma^{V,2}_{NNLO}$: two-loop virtual corrections

# Double Real Subtraction

Tree-level real radiation contribution to $m$ jets at NNLO

$$\mathrm{d}\sigma^{R}_{NNLO} = \mathcal{N} \sum_{m+2} \mathrm{d}\Phi_{m+2}(p_1, \ldots, p_{m+2}; q) \frac{1}{S_{m+2}}$$
$$\times \quad |\mathcal{M}_{m+2}(p_1, \ldots, p_{m+2})|^2 \; J_m^{(m+2)}(p_1, \ldots, p_{m+2})$$

- $\mathrm{d}\Phi_{m+2}$: full $m+2$-parton phase space
- $J_m^{(m+2)}$: ensures $m+2$ partons $\rightarrow$ $m$ jets
  $\longrightarrow$ two partons must be experimentally unresolved

Up to two partons can be theoretically unresolved (soft and/or collinear)

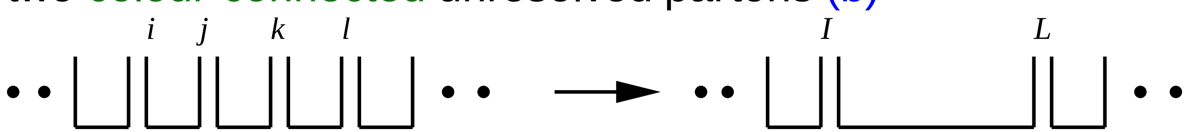
Building blocks of subtraction terms:

- products of two three-parton antenna functions
- single four-parton antenna function

# Double Real Subtraction

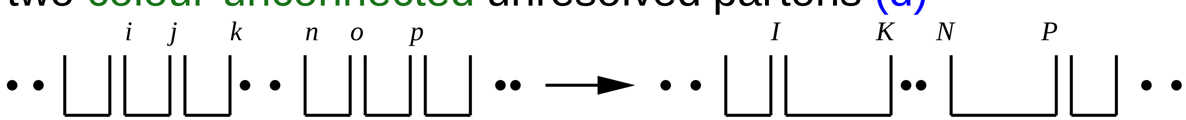
Distinct Configurations for $m+2$ partons $\to$ $m$ jets: Colour connections

- one unresolved parton (a)
  - three parton antenna function $X^0_{ijk}$ can be used (as at NLO)
  - this will not yield a finite contribution in all single unresolved limits
- two colour-connected unresolved partons (b)




  - four-parton antenna function $X^0_{ijkl}$
- two almost colour-unconnected unresolved partons (common radiator) (c)


  - strongly ordered product of non-independent three-parton antenna functions
- two colour-unconnected unresolved partons (d)




  - product of independent three-parton antenna functions

# Double Real Subtraction

## Two colour-connected unresolved partons

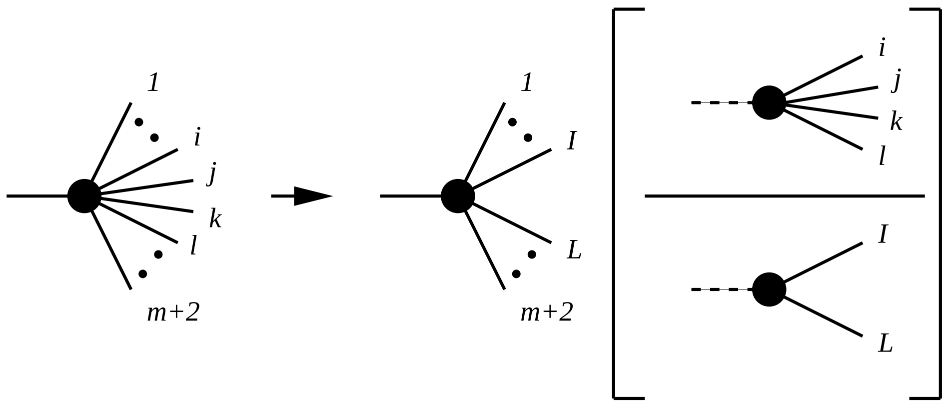
$$
\begin{aligned}
\mathrm{d}\sigma^{S,b}_{NNLO} &= \mathcal{N} \sum_{m+2} \mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q)\frac{1}{S_{m+2}} \\
&\times \quad \Bigg[ \sum_{jk} \left( X^0_{ijkl} - X^0_{ijk}X^0_{IKl} - X^0_{jkl}X^0_{iJL} \right) \\
&\times \quad |\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_L,\ldots,p_{m+2})|^2\, J^{(m)}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_L,\ldots,p_{m+2}) \Bigg]
\end{aligned}
$$

- $X^0_{ijkl}$: four-parton tree-level antenna function contains all double unresolved $p_j, p_k$ limits of $|\mathcal{M}_{m+2}|^2$, but is also singular in single unresolved limits of $p_j$ or $p_k$
- $X^0_{ijk}X^0_{IKl}$: cancels single unresolved limit in $p_j$ of $X^0_{ijkl}$
- $X^0_{jkl}X^0_{iKL}$: cancels single unresolved limit in $p_k$ of $X^0_{ijkl}$
- Triple-collinear, soft-collinear, double soft limits: $X^0_{ijk}X^0_{IKl}, X^0_{jkl}X^0_{iKL} \to 0$
- Double single collinear limit: $X^0_{ijk}X^0_{IKl}, X^0_{jkl}X^0_{iKL} \neq 0$

  cancels with double single collinear limit of $\mathrm{d}\sigma^{S,a}_{NNLO}$

# Double Real Subtraction

## Two colour-connected unresolved partons




$$X^0_{ijkl} = S_{ijkl,IL} \frac{|M^0_{ijkl}|^2}{|M^0_{IL}|^2}$$

$$\mathrm{d}\Phi_{X_{ijkl}} = \frac{\mathrm{d}\Phi_4}{P_2}$$

Phase space factorisation

$$\mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q) = \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_L,\ldots,p_{m+2};q)\cdot\mathrm{d}\Phi_{X_{ijkl}}(p_i,p_j,p_k,p_l;\tilde{p}_I+\tilde{p}_L)$$

Integrated subtraction term (analytically)

$$|\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijkl}}\, X^0_{ijkl} \sim |\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_4 |M^0_{ijkl}|^2$$

Four-particle inclusive phase space integrals are known

T. Gehrmann, G. Heinrich, AG

# Colour-ordered antenna functions

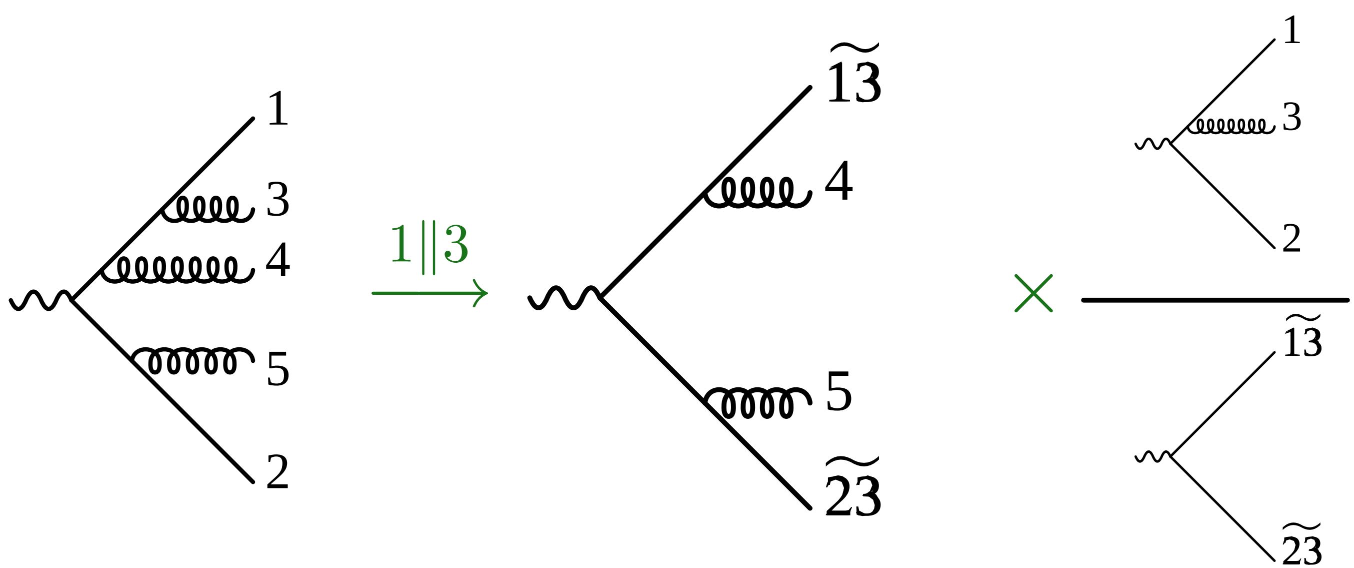
## Antenna functions

- colour-ordered pair of hard partons (radiators) with radiation in between
  - hard quark-antiquark pair
  - hard quark-gluon pair
  - hard gluon-gluon pair
- three-parton antenna $\longrightarrow$ one unresolved parton
- four-parton antenna $\longrightarrow$ two unresolved partons
- can be at tree level or at one loop
- all three-parton and four-parton antenna functions can be derived from physical matrix elements, normalised to two-parton matrix elements

# Antenna functions

## Quark-antiquark

consider subleading colour (gluons photon-like)



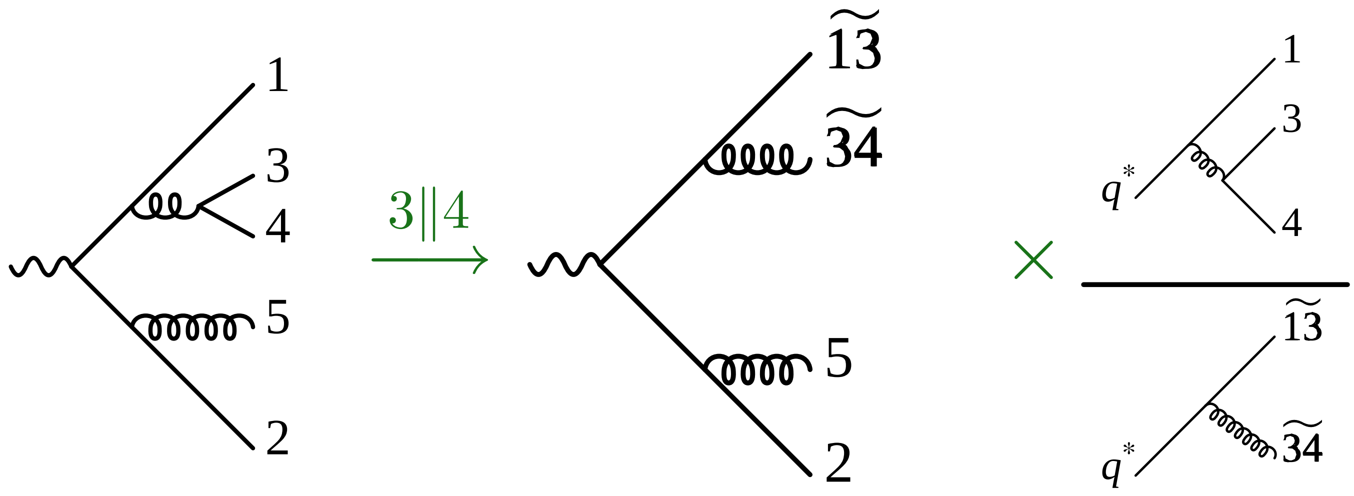

$$|M_{q\bar{q}ggg}|^2(1,3,4,5,2) \xrightarrow{1\|3} |M_{q\bar{q}gg}|^2(\widetilde{13},4,5,\widetilde{23}) \times X_{132}$$

with

$$X_{132} = \frac{|M_{q\bar{q}g}|^2}{|M_{q\bar{q}}|^2} \equiv A_3^0(1_q,3_g,2_{\bar{q}})$$

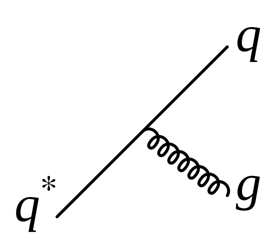
# Antenna functions

## Quark-gluon




$$|M_{q\bar{q}q\bar{q}g}|^2(1,3,4,5,2) \xrightarrow{3\|4} |M_{q\bar{q}gg}|^2(\widetilde{13},\widetilde{34},5,2) \times X_{134}$$

with hard radiators:
quark $(\widetilde{13})$ and gluon $(\widetilde{34})$




$$\begin{array}{rcl} q^* & : & \text{spin } 1/2, \text{colour triplet} \\ q(\widetilde{13}) & : & \text{spin } 1/2, \text{colour triplet} \\ g(\widetilde{34}) & : & \text{spin } 1, \text{colour octet} \end{array}$$

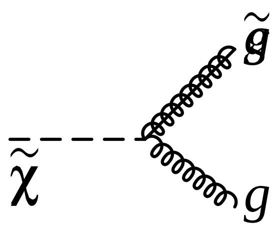
Off-shell matrix element: violates $SU(3)$ gauge invariance

# Antenna functions

## Quark-gluon

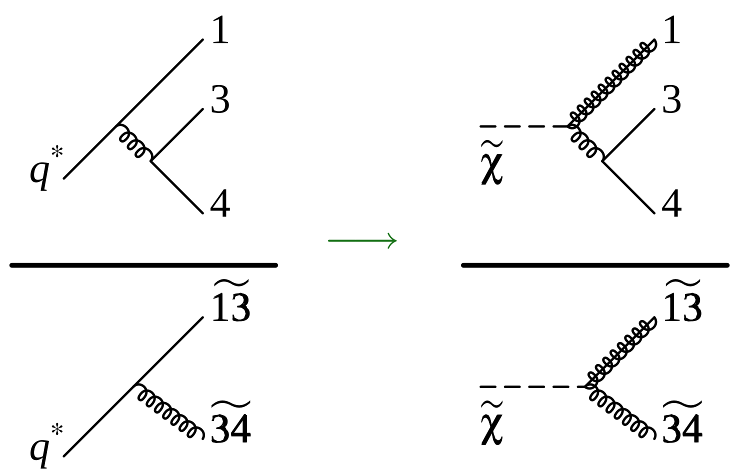
Construct colour-ordered $qg$ antenna function from $SU(3)$ gauge-invariant decay:

neutralino $\rightarrow$ gluino + gluon (AG, T. Gehrmann, E.W.N. Glover)




$$\begin{array}{lcl} \tilde{\chi} & : & \text{spin } 1/2, \text{colour singlet} \\ \tilde{g} & : & \text{spin } 1/2, \text{colour octet} \\ g & : & \text{spin } 1, \text{colour octet} \end{array}$$

Gluino $\tilde{g}$ mimics quark and antiquark (same Dirac structure), but is octet in colour space




$\tilde{\chi} \rightarrow \tilde{g}g$ described by effective Lagrangian

H. Haber, D. Wyler

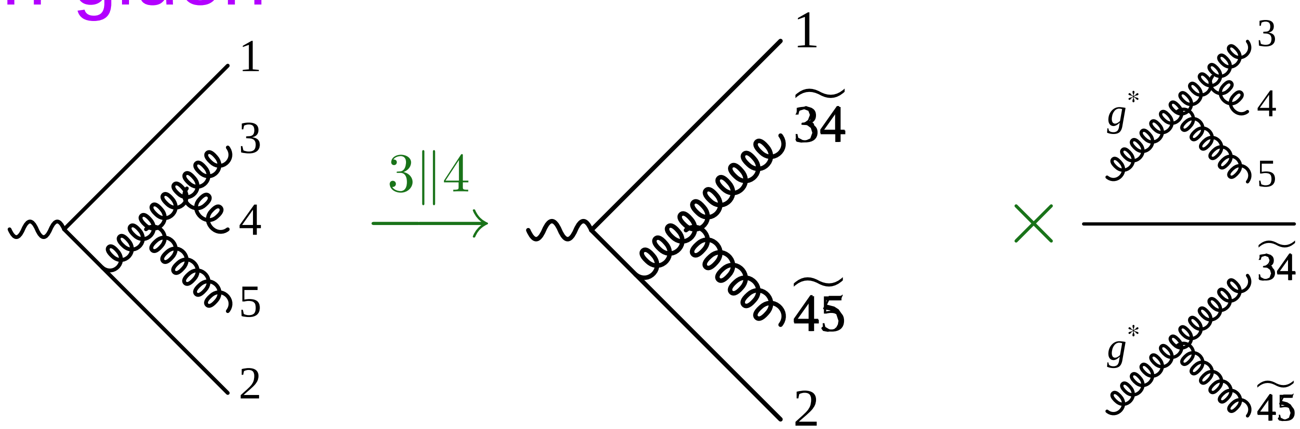
$$\mathcal{L}_{\text{int}} = i\eta\overline{\psi}^{a}_{\tilde{g}}\sigma^{\mu\nu}\psi_{\tilde{\chi}}F^{a}_{\mu\nu} + (\text{h.c.})$$

Antenna function

$$X_{134} = \frac{|M_{\tilde{g}q'\bar{q}'}|^2}{|M_{\tilde{g}g}|^2} \equiv E_3^0(1_q, 3_{q'}, 4_{\bar{q}'})$$

# Antenna functions

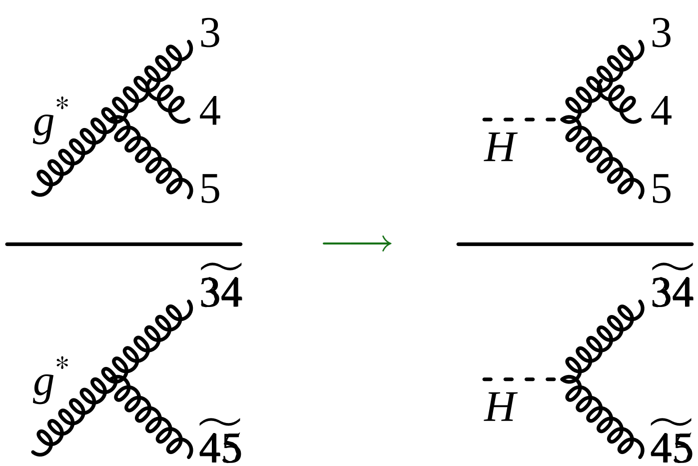
## Gluon-gluon




$$|M_{q\bar{q}ggggg}|^2(1,3,4,5,2) \xrightarrow{3\|4} |M_{q\bar{q}gg}|^2(1,\widetilde{34},\widetilde{45},2) \times X_{345}$$

$H \longrightarrow gg$ described by effective Lagrangian

F. Wilczek; M. Shifman, A. Vainshtein, V. Zakharov




$$\mathcal{L}_{\text{int}} = \frac{\lambda}{4} H F^a_{\mu\nu} F^{\mu\nu}_a$$

Antenna function

$$X_{345} = \frac{|M_{ggg}|^2}{|M_{gg}|^2} \equiv F_3^0(3_g, 4_g, 5_g)$$

# Antenna functions

| | tree level | one loop |
|---|---|---|
| quark-antiquark | | |
| $qg\bar{q}$ | $A_3^0(q,g,\bar{q})$ | $A_3^1(q,g,\bar{q}),\ \tilde{A}_3^1(q,g,\bar{q}),\ \hat{A}_3^1(q,g,\bar{q})$ |
| $qgg\bar{q}$ | $A_4^0(q,g,g,\bar{q}),\ \tilde{A}_4^0(q,g,g,\bar{q})$ | |
| $qq'\bar{q}'\bar{q}$ | $B_4^0(q,q',\bar{q}',\bar{q})$ | |
| $qq\bar{q}\bar{q}$ | $C_4^0(q,q,\bar{q},\bar{q})$ | |
| quark-gluon | | |
| $qgg$ | $D_3^0(q,g,g)$ | $D_3^1(q,g,g),\ \hat{D}_3^1(q,g,g)$ |
| $qggg$ | $D_4^0(q,g,g,g)$ | |
| $qq'\bar{q}'$ | $E_3^0(q,q',\bar{q}')$ | $E_3^1(q,q',\bar{q}'),\ \tilde{E}_3^1(q,q',\bar{q}'),\ \hat{E}_3^1(q,q',\bar{q}')$ |
| $qq'\bar{q}'g$ | $E_4^0(q,q',\bar{q}',g),\ \tilde{E}_4^0(q,q',\bar{q}',g)$ | |
| gluon-gluon | | |
| $ggg$ | $F_3^0(g,g,g)$ | $F_3^1(g,g,g),\ \hat{F}_3^1(g,g,g)$ |
| $gggg$ | $F_4^0(g,g,g,g)$ | |
| $gq\bar{q}$ | $G_3^0(g,q,\bar{q})$ | $G_3^1(g,q,\bar{q}),\ \tilde{G}_3^1(g,q,\bar{q}),\ \hat{G}_3^1(g,q,\bar{q})$ |
| $gq\bar{q}g$ | $G_4^0(g,q,\bar{q},g),\ \tilde{G}_4^0(g,q,\bar{q},g)$ | |
| $q\bar{q}q'\bar{q}'$ | $H_4^0(q,\bar{q},q',\bar{q}')$ | |

# Antenna functions

## Numerical implementation

- requires partonic emissions to be ordered
  - two hard radiators identified uniquely (not a priori the case for $qg$ and $gg$)
  - each unresolved parton can only be singular with its two adjacent partons
- need to separate
  - multiple antenna configurations in single antenna function
    (e.g. $F(3_g, 4_g, 5_g)$ contains three configurations: (345), (453), (534))
  - non-ordered emission (if gluons are photon-like)
- all ordered forms (obtained by partial fractioning) of a given antenna function have
  - the same phase space factorisation
  - different phase space mappings (D. Kosower)

# $e^+e^- \to 3$ jets at NNLO

## First applications of antenna subtraction

- NNLO corrections to $1/N^2$ colour factor in $e^+e^- \to 3$ jets
  (⟶ talk of T. Gehrmann)
  - constructed 5-parton and 4-parton subtraction terms
  - 5-parton channel numerically finite in all single and double unresolved regions
  - 4-parton channel free of explicit $1/\epsilon$ poles and numerically finite in all single unresolved regions
  - 3-parton channel free of explicit $1/\epsilon$ poles

$$\mathcal{P}oles\left(\mathrm{d}\sigma^{S}_{NNLO} + \mathrm{d}\sigma^{VS,1}_{NNLO} + \mathrm{d}\sigma^{V,2}_{NNLO}\right) = 0$$

- explicit infrared pole terms of $\mathrm{d}\sigma^{V,2}_{NNLO}$ can be expressed by integrated antenna functions for all colour factors

# Summary

## Main features of antenna subtraction at NNLO

- building blocks of subtraction terms: 3 and 4 parton antenna functions
- antenna functions are derived from physical $|\mathcal{M}|^2$
  - quark-antiquark: $\gamma^* \to q\bar{q} + X$
  - quark-gluon: $\tilde{\chi} \to \tilde{g}g + X$
  - gluon-gluon: $H \to gg + X$
- subtraction terms:
  - approximate correctly the full $|\mathcal{M}|^2$ (double real)
  - do not oversubtract
  - can be integrated analytically
- in progress: all colour factors in 3-jet rate
- possible extensions: lepton-hadron, hadron-hadron; same antenna functions, but different phase space