Status of $e^+e^- \to 3j$ at NNLO

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Jet production in e^+e^--annihilation

Jets at LEP

- precision test of QCD: coupling constant, structure of gauge group, non-perturbative power corrections, . . .
- precise determination of α_s from 3–jet production rate and related event shape observables, extracted using NLO calculation

$$\alpha_s(M_Z) = 0.1202 \pm 0.0003(\text{stat}) \pm 0.0009(\text{sys}) \pm 0.0009(\text{had}) \pm 0.0047(\text{scale})$$

- error on α_s from jet observables dominated by theoretical uncertainty

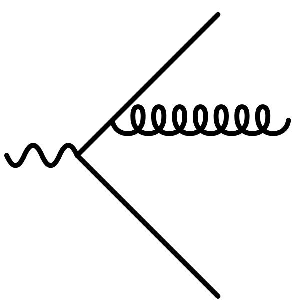
Jets at ILC

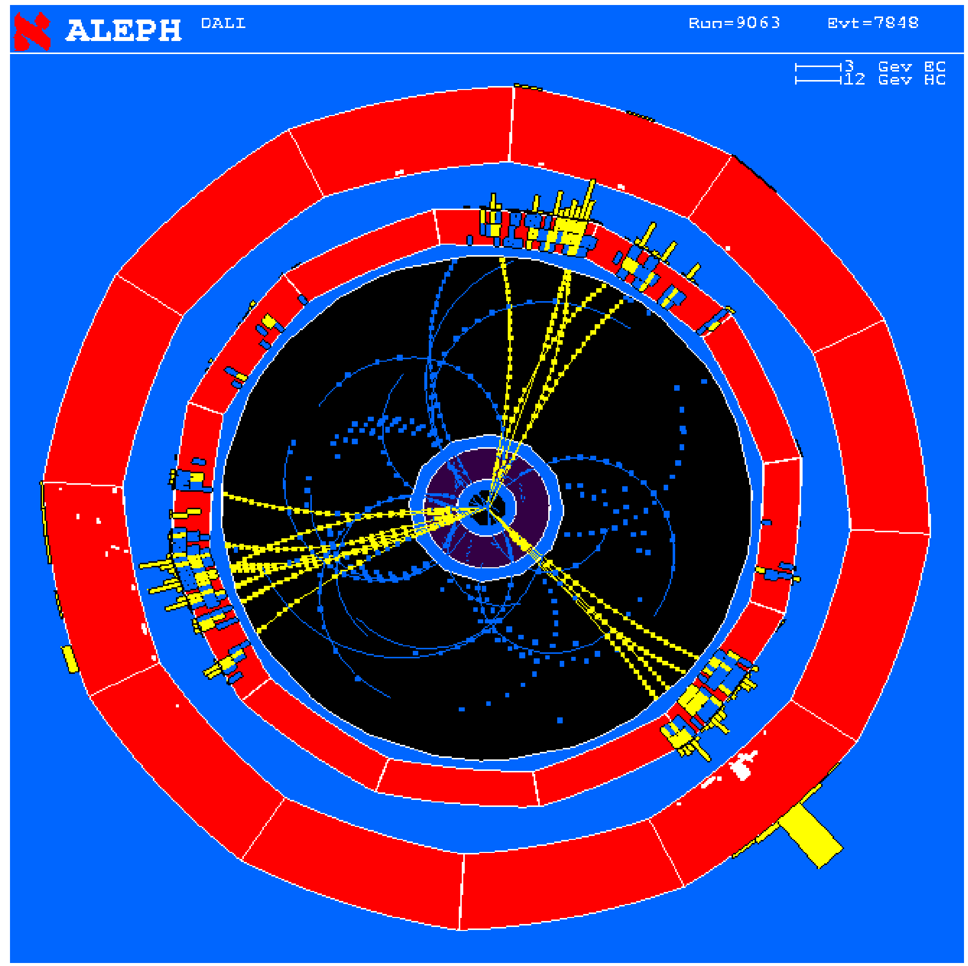
- measure the evolution of $\alpha_s(\sqrt{s})$; sensitive to new physics thresholds
- power corrections even less important than at LEP
- relative scale uncertainty comparable to LEP

Jet observables

$e^+e^- \to 3$ jets

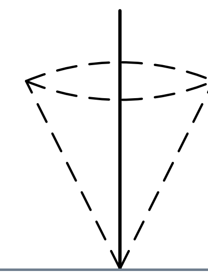
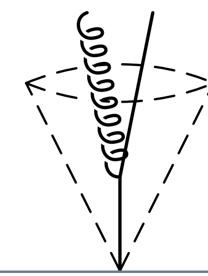
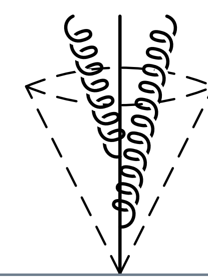
event at LEP





Jets

Jets in Perturbation Theory

LO	each parton forms one jet on its own	
NLO	up to two partons in one jet	
NNLO	up to three partons in one jet	

current state-of-the-art: NLO

Jets

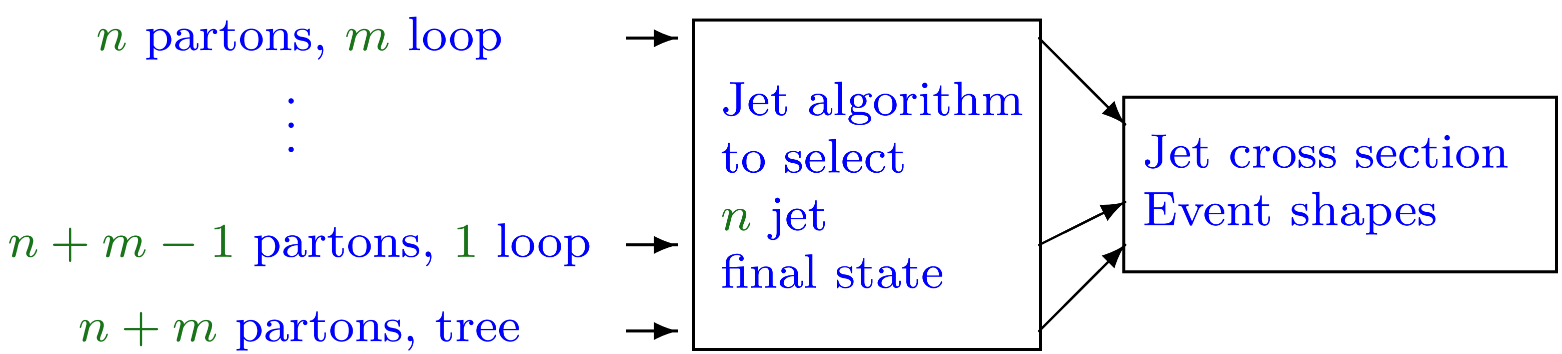
Results at NLO

- $e^+e^- \to 3$ Jets
  K. Ellis, D. Ross, A. Terrano
  G. Kramer et al.
  Z. Kunszt, P. Nason
  W. Giele, N. Glover
  S. Catani, M. Seymour
  D. Soper (purely numerical approach)
- $e^+e^- \to 4$ Jets
  L. Dixon, A. Signer
  Z. Nagy, Z. Trocsanyi
  J. Campbell, M. Cullen, N. Glover
  D. Kosower, S. Weinzierl
- $pp \to 2$ Jets
  S. Ellis, Z. Kunszt, D. Soper
  W. Giele, N. Glover, D. Kosower
- $ep \to (2+1)$ Jets
  E. Mirkes, D. Zeppenfeld
  S. Catani, M. Seymour
  D. Graudenz
- $ep \to (3+1)$ Jets, $pp \to 3$ Jets
  Z. Nagy
- $pp \to V + 1$ Jet
  W. Giele, N. Glover, D. Kosower
- $pp \to V + 2$ Jets
  J. Campbell, R.K. Ellis

Jets in perturbation theory

General structure:

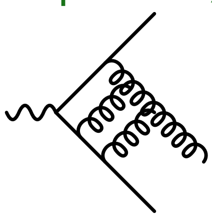
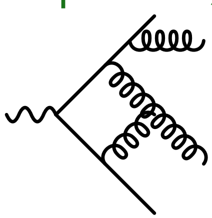
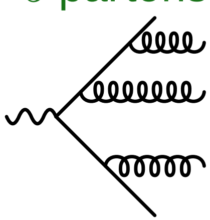
n jets, m–th order in perturbation theory



- Jet algorithm acts differently on different partonic final states
  $\longrightarrow$ Optical theorem can not be applied
- Divergencies from soft and collinear real and virtual contributions must be extracted before application of jet algorithm

Jet physics at NNLO

Structure of $e^+e^- \to 3$ jets at NNLO:

Subprocess	partonic final state	partons in jets
$\gamma^* \to 3$ partons, 2 loop e.g. 	3 partons	(1) (1) (1)
$\gamma^* \to 4$ partons, 1 loop e.g. 	4 partons (3+1) partons	(2) (1) (1) (1) (1) (1)
$\gamma^* \to 5$ partons, tree e.g. 	5 partons (4+1) partons (3+2) partons	(3) (1) (1) (2) (2) (1) (2) (1) (1) (1) (1) (1)

Partons in red are soft or collinear: theoretically unresolved.

Virtual corrections at NNLO

Virtual two-loop matrix elements are available for:

- Bhabha-Scattering: $e^+e^- \to e^+e^-$
  Z. Bern, L. Dixon, A. Ghinculov
- Hadron-Hadron 2-Jet production: $qq' \to qq'$, $q\bar{q} \to q\bar{q}$, $q\bar{q} \to gg$, $gg \to gg$
  C. Anastasiou, N. Glover, C. Oleari, M. Yeomans-Tejeda
  Z. Bern, A. De Freitas, L. Dixon [SUSY-YM]
- Photon pair production at LHC: $gg \to \gamma\gamma$, $q\bar{q} \to \gamma\gamma$
  Z. Bern, A. De Freitas, L. Dixon
  C. Anastasiou, N. Glover, M. Yeomans-Tejeda
- Three-jet production: $e^+e^- \to \gamma^* \to q\bar{q}g$
  L. Garland, N. Glover, A.Koukoutsakis, E. Remiddi, TG
  S. Moch, P. Uwer, S. Weinzierl
- DIS (2+1) jet production: $\gamma^* g \to q\bar{q}$, Hadronic (V+1) jet production: $qg \to Vq$
  E. Remiddi, TG

Real corrections at NNLO

Double real radiation

$$\mathrm{d}\sigma^{(m+2)} = |\mathcal{M}_{m+2}|^2 \mathrm{d}\Phi_{m+2} J_m^{(m+2)}(p_1,\ldots,p_{m+2}) \sim \frac{1}{\epsilon^4}$$

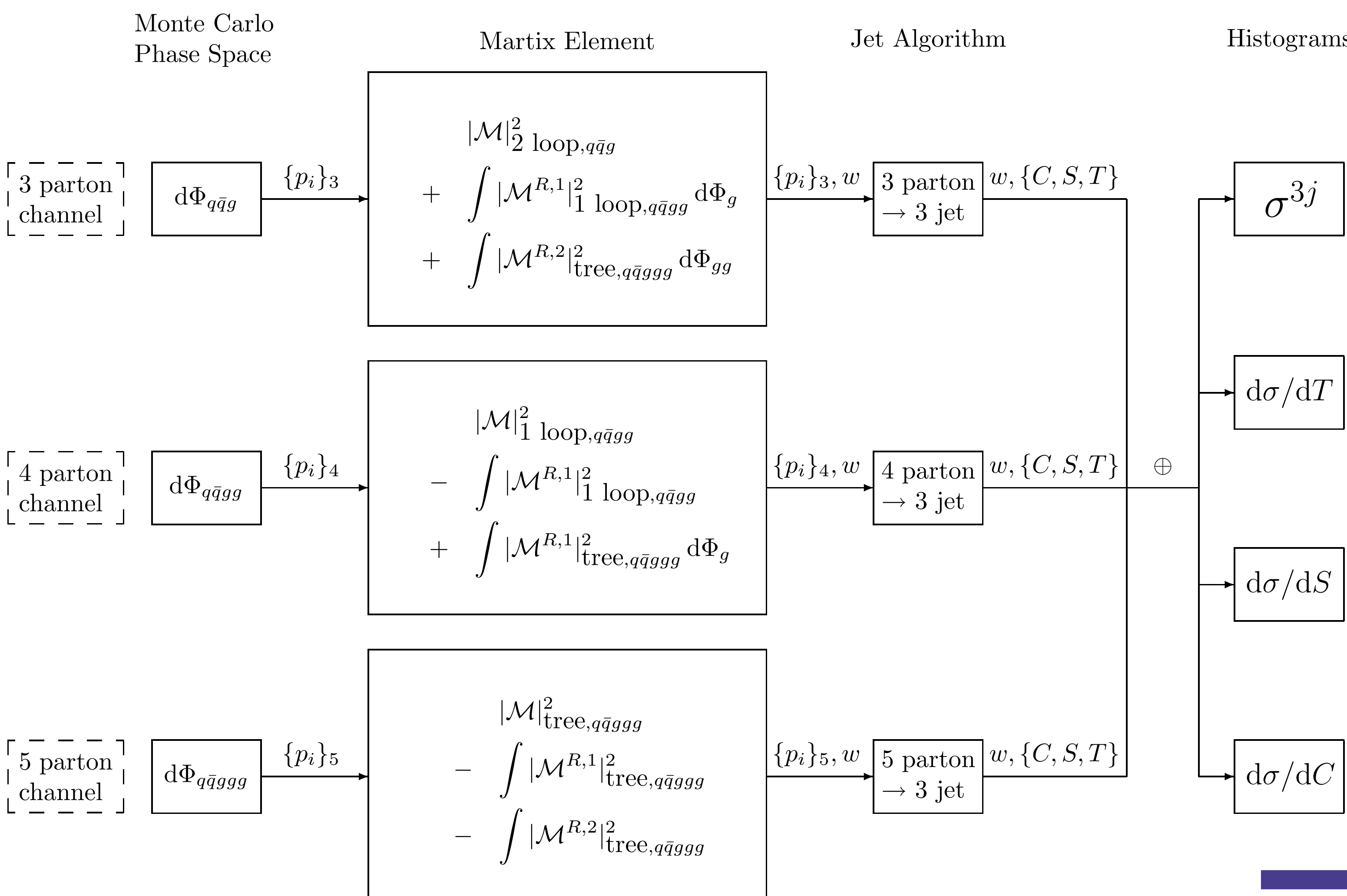
with $J_m^{(m+2)}$ jet definition for combining m+2 partons into m jets

Two approaches:

- Direct evaluation ($\longrightarrow$ talk of K. Melnikov)
  C. Anastasiou, K. Melnikov, F. Petriello
  - expand $|\mathcal{M}_{m+2}|^2 \mathrm{d}\Phi_{m+2}$ in distributions
  - decompose $\mathrm{d}\Phi_{m+2}$ into sectors corresponding to different singular configurations (Iterated sector decomposition)
    T. Binoth, G. Heinrich
  - compute sector integrals numerically
    tested on $e^+e^- \to 2j$, $pp \to H + X$, $\mu \to e + \nu + \bar{\nu} + X$
- Evaluation with subtraction term (this talk)
  tested on $e^+e^- \to 2j$, $e^+e^- \to 3j$ $(1/N^2)$

Real corrections at NNLO

Structure of $e^+e^- \to 3$ jets program:



Antenna subtraction

Antenna function encapsulates all unresolved parton emission between two hard partons

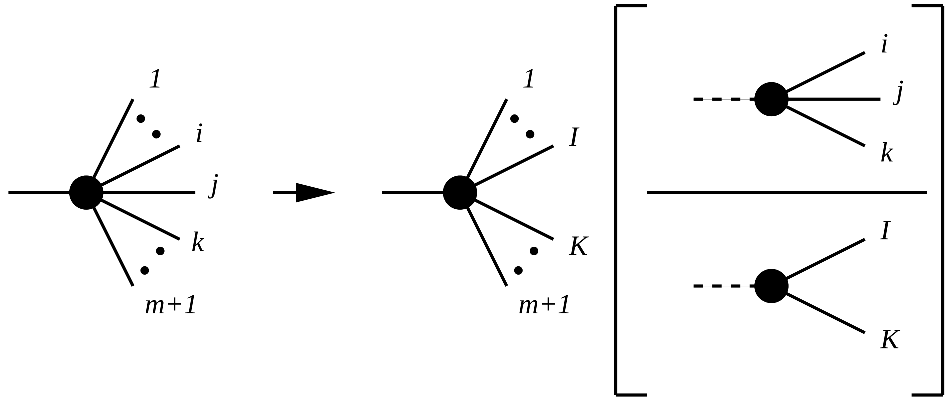
NLO: (D. Kosower; J. Campbell, M. Cullen, E.W.N. Glover)

- one unresolved parton at tree-level (three-parton tree-level antenna function)

NNLO: A. Gehrmann–De Ridder, E.W.N. Glover, TG
($\longrightarrow$ talk of A. Gehrmann–De Ridder)

- one unresolved parton at one-loop (three-parton one-loop antenna function)
- two unresolved partons at tree-level (four-parton tree-level antenna functions)

Antenna subtraction at NLO



$$X^0_{ijk} = S_{ijk,IK} \frac{|M^0_{ijk}|^2}{|M^0_{IK}|^2} \qquad \mathrm{d}\Phi_{X_{ijk}} = \frac{\mathrm{d}\Phi_3}{P_2}$$

Phase space factorisation

$$\mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q) = \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1};q) \cdot \mathrm{d}\Phi_{X_{ijk}}(p_i,p_j,p_k;\tilde{p}_I+\tilde{p}_K)$$

Integrated subtraction term (analytically)

$$|\mathcal{M}_m|^2 \, J_m^{(m)} \, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijk}} \, X^0_{ijk} \sim |\mathcal{M}_m|^2 \, J_m^{(m)} \, \mathrm{d}\Phi_m \, \mathcal{X}^0_{ijk}$$

can be combined with $\mathrm{d}\sigma^V_{NLO}$

Antenna subtraction

	tree level	one loop
quark-antiquark		
$qg\bar{q}$	$A_3^0(q,g,\bar{q})$	$A_3^1(q,g,\bar{q}),\ \tilde{A}_3^1(q,g,\bar{q}),\ \hat{A}_3^1(q,g,\bar{q})$
$qgg\bar{q}$	$A_4^0(q,g,g,\bar{q}),\ \tilde{A}_4^0(q,g,g,\bar{q})$	
$qq'\bar{q}'\bar{q}$	$B_4^0(q,q',\bar{q}',\bar{q})$	
$qq\bar{q}\bar{q}$	$C_4^0(q,q,\bar{q},\bar{q})$	
quark-gluon		
qgg	$D_3^0(q,g,g)$	$D_3^1(q,g,g),\ \hat{D}_3^1(q,g,g)$
$qggg$	$D_4^0(q,g,g,g)$	
$qq'\bar{q}'$	$E_3^0(q,q',\bar{q}')$	$E_3^1(q,q',\bar{q}'),\ \tilde{E}_3^1(q,q',\bar{q}'),\ \hat{E}_3^1(q,q',\bar{q}')$
$qq'\bar{q}'g$	$E_4^0(q,q',\bar{q}',g),\ \tilde{E}_4^0(q,q',\bar{q}',g)$	
gluon-gluon		
ggg	$F_3^0(g,g,g)$	$F_3^1(g,g,g),\ \hat{F}_3^1(g,g,g)$
$gggg$	$F_4^0(g,g,g,g)$	
$gq\bar{q}$	$G_3^0(g,q,\bar{q})$	$G_3^1(g,q,\bar{q}),\ \tilde{G}_3^1(g,q,\bar{q}),\ \hat{G}_3^1(g,q,\bar{q})$
$gq\bar{q}g$	$G_4^0(g,q,\bar{q},g),\ \tilde{G}_4^0(g,q,\bar{q},g)$	
$q\bar{q}q'\bar{q}'$	$H_4^0(q,\bar{q},q',\bar{q}')$	

$\mathcal{O}(1/N^2)$ contributions to $e^+e^- \to 3j$

Partonic channels

- $\gamma^* \to q\bar{q}ggg$ and $\gamma^* \to q\bar{q}q\bar{q}g$ at tree-level
  K. Hagiwara, D. Zeppenfeld; F.A. Berends, W.T. Giele, H. Kuijf;
  N. Falck, D. Graudenz, G. Kramer
- $\gamma^* \to q\bar{q}gg$ and $\gamma^* \to q\bar{q}q\bar{q}$ at one loop
  Z. Bern, L. Dixon, D. Kosower, S. Weinzierl;
  J. Campbell, D.J. Miller, E.W.N. Glover; Z. Nagy, Z. Trocsanyi
- $\gamma^* \to q\bar{q}g$ at two loops
  L. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis, E. Remiddi

$\mathcal{O}(1/N^2)$ receives contributions from all partonic processes (also identical $q\bar{q}q\bar{q}$)

- Discuss here: subtraction terms for $\gamma^* \to q\bar{q}ggg$, $\gamma^* \to q\bar{q}gg$ in $\mathcal{O}(1/N^2)$
- Gluons are photon-like (no self-coupling)

Five-parton contributions

Matrix element for $\gamma^* \to q\bar{q}ggg$

$$\left|M^0_{q\bar{q}3g}\right|^2 = N_5 \, \frac{1}{N^2} \bar{A}^0_5(1_q, 3_g, 4_g, 5_g, 2_{\bar{q}})$$

with

$$N_n = 4\pi\alpha \sum_q e_q^2 \left(g^2\right)^{(n-2)} \left(N^2 - 1\right) \left|\mathcal{M}^0_{q\bar{q}}\right|^2$$

$$\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = 4(1-\epsilon)q^2$$

NNLO real radiation contribution to three jet final state

$$\mathrm{d}\sigma^R_{NNLO,\bar{A}} = \frac{N_5}{N^2} \, \mathrm{d}\Phi_5(p_1, \ldots, p_5; q) \, \frac{1}{3!} \, \bar{A}^0_5(1_q, 3_g, 4_g, 5_g, 2_{\bar{q}}) J^{(5)}_3(p_1, \ldots, p_5)$$

Only one possible hard radiator pair: $(q - \bar{q})$ for this colour factor

Five-parton contributions

Single unresolved parton subtraction

$$
\begin{aligned}
\mathrm{d}\sigma^{S,a}_{NNLO,\bar{A}} &= \frac{N_5}{N^2}\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\,\frac{1}{3!} \\
&\times \sum_{i,j,k\in P_C(3,4,5)} A_3^0(1_q,i_g,2_{\bar{q}})\,\tilde{A}_4^0(\widetilde{(1i)}_q,j_g,k_g,\widetilde{(2i)}_{\bar{q}})\;J_3^{(4)}(\widetilde{p_{1i}},p_j,p_k,\widetilde{p_{2i}})
\end{aligned}
$$

Colour connected double unresolved subtraction

$$
\begin{aligned}
\mathrm{d}\sigma^{S,b}_{NNLO,\bar{A}} &= N_5\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\,\frac{1}{3!}\sum_{i,j,k\in P_C(3,4,5)}\Bigg(\tilde{A}_4^0(1_q,i_g,j_g,2_{\bar{q}}) \\
&\quad -A_3^0(1_q,i_g,2_{\bar{q}})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}}) - A_3^0(1_q,j_g,2_{\bar{q}})\,A_3^0(\widetilde{(1j)}_q,i_g,\widetilde{(2j)}_{\bar{q}})\Bigg) \\
&\quad \times A_3^0(\widetilde{(1ij)}_q,k_g,\widetilde{(2ij)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{1ij}},p_k,\widetilde{p_{2ij}})
\end{aligned}
$$

$\mathrm{d}\sigma^{R}_{NNLO,\bar{A}} - \mathrm{d}\sigma^{S,a}_{NNLO,\bar{A}} - \mathrm{d}\sigma^{S,b}_{NNLO,\bar{A}}$ is finite and can be integrated numerically over $\mathrm{d}\Phi_5$

Five-parton contributions

Numerical implementation

- phase space for subtraction terms factorises into antenna phase space and matrix element phase space
- use homogeneous numerical mapping of antenna phase space
  D. Kosower
  - all collinear limits are treated symmetrically
  - requires ordered decomposition of antenna functions

    Subleading colour $\tilde{A}_4^0(1_q, 3_g, 4_g, 2_{\bar{q}})$ has double single collinear limits $1 \parallel 3, 2 \parallel 4$ and $1 \parallel 4, 2 \parallel 3$; must be separated using partial fractioning
- checked proper local subtraction using phase space trajectories into single and double unresolved limits

Four-parton contributions

One-loop $\mathcal{O}(1/N^2)$ contribution to $\gamma^* \to q\bar{q}gg$

$$\mathrm{d}\sigma^{V,1}_{NNLO,\bar{A}} = \frac{N_4}{N^2}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\,\frac{1}{2!}\,\tilde{A}_4^{1,b}(1_q,3_g,4_g,2_{\bar{q}})\;J_3^{(4)}(p_1,\ldots,p_4)$$

Three types of subtraction terms (explicit poles, implicit unresolved poles, compensation)

$$\begin{aligned}
\mathrm{d}\sigma^{VS,1,a}_{NNLO,\bar{A}} &= -\mathrm{d}\sigma^{S,a}_{NNLO,\bar{A}} \\
&= -\frac{N_4}{N^2}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\,\frac{1}{2!} \\
&\quad \times\mathcal{A}_3^0(s_{12})\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}})J_3^{(4)}(p_1,p_2,p_3,p_4)
\end{aligned}$$

with

- $\mathcal{A}_3^0 = \int_{\Phi_D} A_3^0$: integrated three-parton antenna function
- $\tilde{A}_4^0$: four-parton tree-level matrix element

Four-parton contributions

Subtraction for single unresolved one-loop contributions

$$
\begin{aligned}
\mathrm{d}\sigma^{VS,1,b}_{NNLO,\bar{A}} &= \frac{N_4}{N^2}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\frac{1}{2!}\sum_{i,j\in P(3,4)} J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}}) \\
&\times \Bigg(\tilde{A}_3^1(1_q,i_g,2_{\bar{q}})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}}) \\
&\quad + A_3^0(1_q,i_g,2_{\bar{q}})\left[\tilde{A}_3^1(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}}) + \mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\right]\Bigg)
\end{aligned}
$$

- Structure: $X_3^1|M_{3,\mathrm{tree}}|^2 + X_3^0|M_{3,1l}|^2$ (Z. Bern, L. Dixon, D. Dunbar, D. Kosower)
- introduces spurious poles outside the limits, compensated by

$$
\begin{aligned}
\mathrm{d}\sigma^{VS,1,c}_{NNLO,\bar{A}} &= \frac{N_4}{N^2}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\frac{1}{2!} \\
&\times \sum_{i,j\in P(3,4)} \mathcal{A}_3^0(s_{12})A_3^0(1_q,i_g,2_{\bar{q}})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})
\end{aligned}
$$

$\mathrm{d}\sigma^{V,1}_{NNLO,\bar{A}} - \mathrm{d}\sigma^{VS,1}_{NNLO,\bar{A}}$ is free of $1/\epsilon$ poles, and can be integrated numerically over $\mathrm{d}\Phi_4$

Three-parton contributions

Two-loop three-parton matrix element plus integrated five-parton and four-parton subtraction terms

$$\mathrm{d}\sigma^{V,2}_{NNLO} + \mathrm{d}\sigma^{S}_{NNLO} + \mathrm{d}\sigma^{VS,1}_{NNLO}$$

with

$$\mathrm{d}\sigma^{S}_{NNLO,\bar{A}} - \mathrm{d}\sigma^{T}_{NNLO,\bar{A}} = \frac{1}{N^2}\,\frac{1}{2}\tilde{\mathcal{A}}_4^0(s_{12})A_3^0(1_q,3_g,2_{\bar{q}})\,\mathrm{d}\sigma_3\ ,$$

$$\mathrm{d}\sigma^{VS,1}_{NNLO,\bar{A}} + \mathrm{d}\sigma^{T}_{NNLO,\bar{A}} = \frac{1}{N^2}\Bigg(\tilde{\mathcal{A}}_3^1(s_{12})A_3^0(1_q,3_g,2_{\bar{q}})$$

$$+\mathcal{A}_3^0(s_{12})\left[\tilde{A}_3^1(1_q,3_g,2_{\bar{q}}) + \mathcal{A}_2^1(s_{123})A_3^0(1_q,3_g,2_{\bar{q}})\right]\Bigg)\mathrm{d}\sigma_3$$

Together with the contributions from $\gamma^* \to q\bar{q}q\bar{q}(g)$ find at $\mathcal{O}(1/N^2)$:

$$\mathcal{P}oles(\mathrm{d}\sigma^{S}_{NNLO} + \mathrm{d}\sigma^{VS,1}_{NNLO} + \mathrm{d}\sigma^{V}_{NNLO}) = 0$$

$\mathrm{d}\sigma^{V,2}_{NNLO} + \mathrm{d}\sigma^{VS,1}_{NNLO} + \mathrm{d}\sigma^{S}_{NNLO}$ is free of $1/\epsilon$ and can be integrated numerically over $\mathrm{d}\Phi_3$

$e^+e^- \to 3j$ **at NNLO**

Full colour structure

$$N^2; \quad N^0; \quad 1/N^2; \quad N_F\, N; \quad N_F/N; \quad N_F^2; \quad N_{F,\gamma}$$

Current status: (A. Gehrmann–De Ridder, E.W.N. Glover, G. Heinrich, TG)

- implemented $1/N^2$
- implemented N_F^2: contains only 4-parton and 3-parton contribution
- implemented $N_{F,\gamma}$: NLO-type without double unresolved and virtual unresolved
- all colour factors: inferred structure of antenna subtraction terms for 5-parton and 4-parton channels from infrared pole structure of 3-parton contribution

$$\mathcal{P}oles\left(\mathrm{d}\sigma^{S}_{NNLO} + \mathrm{d}\sigma^{VS,1}_{NNLO} + \mathrm{d}\sigma^{V,2}_{NNLO}\right) = 0$$

- remaining issues:
  - cancellations between 5-parton and 4-parton antenna subtraction terms $(\mathrm{d}\sigma^{T}_{NNLO})$ in remaining colour factors
  - numerical phase space mappings in remaining colour factors

$e^+e^- \to 3j$ **at NNLO**

Integrated subtraction terms

$$\mathrm{d}\sigma^S_{NNLO} - \mathrm{d}\sigma^T_{NNLO} = \mathcal{X}^S_{q\bar{q}g,NNLO}\, A_3^0(1_q,3_g,2_{\bar{q}})\,\mathrm{d}\sigma_3$$

$$\mathrm{d}\sigma^{VS,1}_{NNLO} + \mathrm{d}\sigma^T_{NNLO} = \Bigg\{\mathcal{X}^{VS}_{q\bar{q}g,NNLO}\, A_3^0(1_q,3_g,2_{\bar{q}}) + \mathcal{X}^{VS,tree}_{q\bar{q}g,NNLO}\, A_3^{(1\times 0)}(1_q,3_g,2_{\bar{q}}) + \mathcal{X}^{VS,\beta}_{q\bar{q}g,NNLO}\,\frac{\beta_0}{\epsilon}\, A_3^0(1_q,3_g,2_{\bar{q}})\Bigg\}\,\mathrm{d}\sigma_3$$

e.g.

$$\mathcal{X}^S_{q\bar{q}g,NNLO,N^2} = N^2\Bigg[\frac{1}{2}\,\mathcal{D}_4^0(s_{13}) + \frac{1}{2}\,\mathcal{D}_4^0(s_{23}) - \frac{1}{8}\left(\mathcal{D}_3^0(s_{13}) - \mathcal{D}_3^0(s_{23})\right)^2 - \frac{1}{2}\left(\tilde{\mathcal{A}}_4^0(s_{12}) - \mathcal{A}_3^0(s_{12})\,\mathcal{A}_3^0(s_{12})\right)\Bigg]$$

$$\mathcal{X}^{VS,tree}_{q\bar{q}g,NNLO,N} = N\left[\frac{1}{2}\,\mathcal{D}_3^0(s_{13}) + \frac{1}{2}\,\mathcal{D}_3^0(s_{23})\right]$$

$$\mathcal{X}^{VS,\beta}_{q\bar{q}g,NNLO,N} = N\left[\frac{1}{2}\,\mathcal{D}_3^0(s_{13})\,\left[(s_{13})^{-\epsilon} - (s_{123})^{-\epsilon}\right] + (s_{13} \leftrightarrow s_{23})\right]$$

$$\mathcal{X}^{VS}_{q\bar{q}g,NNLO,N^2} = N^2\left[\frac{1}{2}\,\mathcal{D}_3^1(s_{13}) + \frac{1}{2}\,\mathcal{D}_3^1(s_{23}) - \tilde{\mathcal{A}}_3^1(s_{12})\right]$$

Summary

Status of $e^+e^- \to 3j$ at NNLO

- developed antenna subtraction formalism for infrared singular real radiation at NNLO
- constructed and implemented the 3, 4 and 5 parton contributions for $1/N^2$, N_F^2, $N_{F,\gamma}$ colour factors
- showed $\mathcal{P}oles(\text{three-parton}) = 0$ in all colour factors
- in progress: all colour factors in 3-jet rate