Fundamental principles of particle physics

Our description of the fundamental interactions and particles rests on two fundamental structures :

- Quantum Mechanics
- Symmetries

Symmetries

Central to our description of the fundamental forces :

Relativity - translations and Lorentz transformations

Lie symmetries - $SU(3) \otimes SU(2) \otimes U(1)$

Copernican principle : “Your system of co-ordinates and units is nothing special”

⇒ Physics independent of system choice

Fundamental principles of particle physics

Quantum Mechanics

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Relativity

} Quantum Field theory

Relativistic quantum field theory

Fundamental division of physicist's world :

speed

Action (S)	slow	fast
large	Classical Newton	Classical relativity
small	Classical Quantum mechanics	Quantum Field theory

$\hbar \quad (QM \text{ amplitude} \propto e^{i(\frac{S}{\hbar})})$

c

Quantum Mechanics : Quantization of dynamical system of particles

Quantum Field Theory : Application of QM to dynamical system of fields

Why fields?

- No right to assume that any relativistic process can be explained by single particle since $E=mc^2$ allows pair creation

- (Relativistic) QM has physical problems. For example it violates causality

Amplitude for free propagation from x_0 to x

$$U(t) = \langle x | e^{-i(p^2/2m)t} | x_0 \rangle = \int d^3p \langle x | e^{-i(p^2/2m)t} | p \rangle \langle p | x_0 \rangle$$

$$= \frac{1}{2\pi^3} \int d^3p \, e^{-i(p^2/2m)t} \, e^{ip(x-x_0)}$$

$$= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{im(x-x_0)^2/2t} \quad \ldots \text{ nonzero for all } x,\ t$$

Relativistic case : $\mathrm{U(t)} \propto \mathrm{e}^{-\mathrm{m}\sqrt{\mathrm{x}^2 - t^2}} \ldots$ nonzero for all $x,\ t$

- **Relativistic QM - The Klein Gordon equation (1926)**

Scalar particle (field) (J=0) : $\phi(\mathrm{x})$

$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

Energy eigenvalues $\underline{E = \pm(\mathbf{p}^2 + m^2)^{1/2}}$???

1927 Dirac tried to eliminate negative solutions by writing a relativistic equation linear in E (a theory of fermions)

1934 Pauli and Weisskopf revived KG equation with E<0 solutions as E>0 solutions for particles of opposite charge (antiparticles). Unlike Dirac's hole theory this interpretation is applicable to bosons (integer spin) as well as to fermions (half integer spin).

As we shall see the antiparticle states make the field theory causal

Special relativity

- Space time point $a^{\mu} = (ct, x, y, z)$ not invariant under translations
- Space-time vector $(a+\Delta a)^{\mu} - a^{\mu} = \Delta a^{\mu} = (c\Delta t, \Delta x, \Delta y, \Delta z)$

  Invariant under translations …but not invariant under rotations or boosts

- Einstein postulate : the real invariant distance is

$$\left(\Delta a^{0}\right)^{2} - \left(\Delta a^{1}\right)^{2} - \left(\Delta a^{2}\right)^{2} - \left(\Delta a^{3}\right)^{2} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \Delta a^{\mu} \Delta a^{\nu} = \Delta a^{\mu} \Delta a_{\mu} = \left(\Delta a\right)^{2}$$

$$g_{\mu\nu} = diag(+1, -1, -1, -1)$$

- Physics invariant under all transformations that leave all such distances invariant :

  Translations and Lorentz transformations

Lorentz transformations :

$$x^{\mu} \to \Lambda^{\mu}_{\nu} x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} = \tilde{x}^{\mu} \quad \Rightarrow \quad g_{\mu\nu} \tilde{x}^{\mu} \tilde{x}^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu} \quad \Rightarrow \quad g_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = g_{\alpha\beta}$$

(Summation assumed)

Solutions :

- 3 rotations R

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- 3 boosts B

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Space reflection – parity P

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Time reflection, time reversal T

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **The Lorentz transformations form a group, G** $(g_1 g_2 \in G \quad if \quad g_1, g_2 \in G)$

Rotations

$$R(\theta) = e^{-i\mathbf{J}.\theta/\hbar}, \quad J_z = i\hbar\left(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\right)..$$

Angular momentum operator

$(c.f.\ \mathbf{J} = \mathbf{r} \times \mathbf{p})$

$$\boxed{[J_i, J_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} J_k}$$

ε_{ijk} totally antisymmetric Levi-Civita symbol,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1; \quad \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$$

Demonstration that $R_Z(\theta) = e^{-iJ_z\theta}$

$$R_Z(\theta)\psi(x,y) \equiv R_Z(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \psi(x',y')$$

For small $\theta = \varepsilon$, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x+\varepsilon y \\ y-\varepsilon x \end{pmatrix}$

$$R_Z(\varepsilon)\psi(x,y) = \psi(x+\varepsilon y, y-\varepsilon x) \approx \psi(x,y) + \varepsilon\left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right)$$

$$= (1 - i\varepsilon(xp_y - yp_x))\psi(x,y)$$

$$\text{i.e. } R_Z(\varepsilon) = (1 - i\varepsilon(xp_y - yp_x)) = 1 - i\varepsilon J_z$$

Hence $\boxed{R_Z(\theta = n\varepsilon) \approx (1 - i\varepsilon J_Z)^n \xrightarrow[n\to\infty]{} e^{-iJ_z\theta}}$

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Derivation of the commutation relations of SO(3) (SU(2))

$$R_x(\varepsilon)R_y(\eta)R_x(-\varepsilon)R_y(-\eta) \qquad \varepsilon, \eta \textit{ small } \text{(infinitesimal)}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \eta \\ 0 & 1 & 0 \\ -\eta & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \eta \\ 0 & 1 & 0 \\ -\eta & 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & \varepsilon\eta & 0 \\ -\varepsilon\eta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_z(\varepsilon\eta) \approx (1 - i\varepsilon\eta J_z)$$

$$R_x(\varepsilon)R_y(\eta)R_x(-\varepsilon)R_y(-\eta)$$

$$= (1 - i\varepsilon J_x)(1 - i\varepsilon J_y)(1 + i\varepsilon J_x)(1 + i\varepsilon J_y) = -\varepsilon\eta(J_x J_y - J_y J_x)$$

Equating the two equations implies

$$\boxed{[J_x, J_y] = iJ_z}$$

QED

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Representations $\quad |\mathbf{J}| = 0, \tfrac{1}{2}, 1, \ldots$

Pauli spin matrices

$e.g. \quad J = 1/2 \quad \mathbf{J}_i = \sigma_i / 2$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \text{Spin} \quad \psi_{J_z=-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ldots$$

(SU(2))

The Klein Gordon equation (1926)

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Energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t} + \frac{1}{2m}\nabla^2\phi = 0$$

$$i\phi^*(S.E.) - i\phi(S.E.)^* \;\Rightarrow\; \frac{\partial\rho}{\partial t} + \nabla.\mathbf{j} = 0 \text{ continuity eq.}$$

$$\boxed{\rho = |\phi|^2}$$

$$\mathbf{j} = -\frac{i}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

"probability density" *"probability current"*

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi$$

$$\boxed{\rho = i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)}$$

$$\mathbf{j} = -i(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

$$j^\mu = (\rho, \mathbf{j})$$

$$\phi = Ne^{-ip.x}, \quad \rho = 2E|N|^2$$

$$\int_V \rho\, dV = \int \rho\, d^3x = 2E$$

$$f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

Physical interpretation of Quantum Mechanics

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$$\boxed{\rho=|\phi|^2}$$

$$\mathbf{j}=-\frac{i}{2m}(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

"probability density"

"probability current"

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\boxed{\rho=i(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t})}$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

$$\phi=Ne^{-ip.x}, \quad \rho=2E|N|^2$$

Negative probability?

$$f_p^\pm=e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

Field theory of $\pi^{\pm}$

Scalar particle – satisfies KG equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0 \qquad \partial^\mu = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, -\nabla), \quad \partial_\mu = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, \nabla)$$

- Classical electrodynamics, motion of charge –e in EM potential $\mathrm{A}^\mu = (\mathrm{A}^0, \mathbf{A})$

is obtained by the substitution : $p^\mu \to p^\mu + eA^\mu$

- Quantum mechanics : $i\partial^\mu \to i\partial^\mu + eA^\mu$

The Klein Gordon equation becomes:

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

The smallness of the EM coupling, $\alpha_{em} = \frac{e^2}{4\pi} \approx \frac{1}{137}$, means that it is sensible to

Make a “perturbation” expansion of V in powers of α_{em}