The Klein Gordon equation (1926)

Scalar field (J=0) : $\phi(\mathrm{x})$

$$E^2 - \mathbf{p}^2 = m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

4 vector notation

$$A^\mu = (A^0, \underline{A}), \quad B^\mu = (B^0, \underline{B})$$ contravariant

$$A_\mu = (A^0, -\underline{A}), \quad B_\mu = (B^0, -\underline{B})$$ covariant

$$A_\mu = g_{\mu\nu} A^\nu \qquad A^\mu = g^{\mu\nu} A_\nu$$

$$A.B = A_\mu B^\mu = A^\mu B_\mu = A^0 B^0 - \underline{A}.\underline{B}$$

4 vectors

$$(ct, \underline{x}) \equiv x^\mu \qquad (\frac{E}{c}, \underline{p}) \equiv p^\mu$$

$$\partial^\mu = (\frac{\partial}{c\partial t}, -\underline{\nabla}) \qquad \partial_\mu = (\frac{\partial}{c\partial t}, \underline{\nabla})$$

$$p^\mu \rightarrow i\hbar\partial^\mu \qquad p_\mu p^\mu = E^2 - \underline{p}^2 \quad \rightarrow \quad -\Box^2 \equiv \partial_\mu \partial^\mu$$

Field theory of $\pi^{\pm}$

Scalar particle – satisfies KG equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

- Classical electrodynamics, motion of charge –e in EM potential $\mathrm{A}^\mu = (\mathrm{A}^0, \mathbf{A})$

is obtained by the substitution : $p^\mu \rightarrow p^\mu + eA^\mu$

- Quantum mechanics : $i\partial^\mu \rightarrow i\partial^\mu + eA^\mu$

The Klein Gordon equation becomes:

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

The smallness of the EM coupling, $\alpha_{em} = \frac{e^2}{4\pi} \approx \frac{1}{137}$, means that it is sensible to

Make a "perturbation" expansion of V in powers of α_{em}

$$V \approx -ie(\partial_\mu A^\mu + A^\mu \partial_\mu)$$

Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2m}\nabla^2\phi=0$$

$$i\phi^*(S.E.)-i\phi(S.E.)^* \Rightarrow \frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0 \text{ continuity eq.}$$

$$\rho=|\phi|^2$$

"probability density"

$$\mathbf{j}=-\frac{i}{2m}(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

"probability current"

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=2E|N|^2$$

Negative probability?

$$\rho=i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

Pauli and Weisskopf

$$j^\mu\to e(\rho,\mathbf{j})=j^\mu_{EM}$$

$$\frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0=\partial^\mu j_\mu$$

Want to solve : $(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$

Solution : $\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x)V(x')\psi(x')$

where $(\partial_\mu \partial^\mu + m^2)\phi = 0$

and $(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$

Feynman propagator (arrow to Δ_F) — Dirac Delta function (arrow to δ^4)

$$\int d^4x' \delta^4(x'-x) f(x') = f(x)$$

Want to solve : $$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution : $$\boxed{\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x')\psi(x')}$$

where $$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and $$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Feynman propagator Dirac Delta function

Simplest to solve for propagator in momentum space by taking Fourier transform

$$\frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} (\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) d^4(x'-x) = \frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} \delta^4(x'-x) d^4(x'-x)$$

$$\Rightarrow \quad (-p^2 + m^2)\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}$$

$$\boxed{\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}\frac{1}{-p^2+m^2+i\varepsilon}, \qquad \Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\frac{1}{p^2-m^2-i\varepsilon}}$$

The Born series

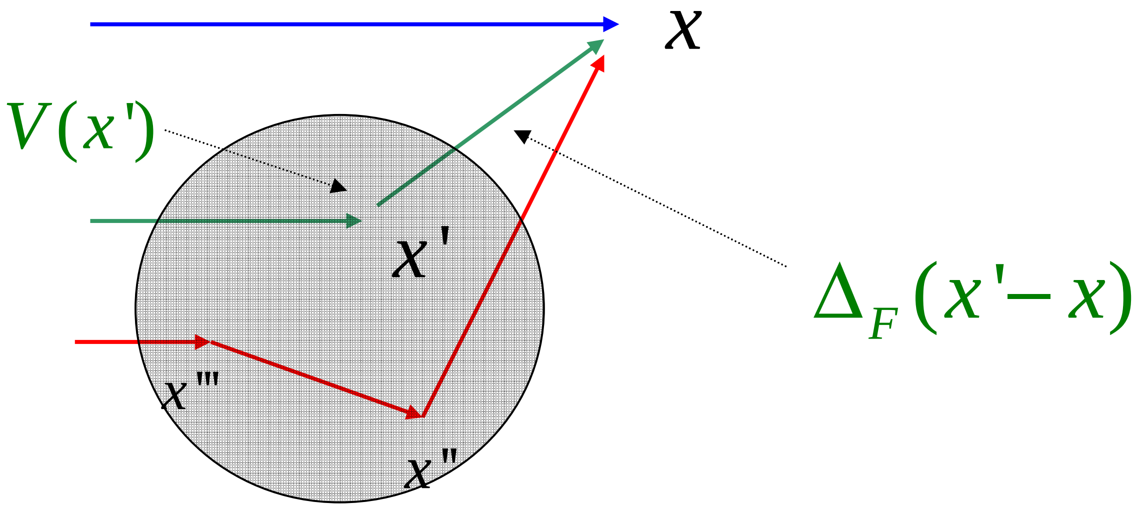
$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \psi(x')$$

Since V(x) is small can solve this equation iteratively :

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \phi(x')$$

$$+ \int d^4x'' \int d^4x''' \Delta_F(x''-x) V(x'') \Delta_F(x'''-x'') V(x''') \phi(x''') + \ldots$$

Interpretation :



But energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

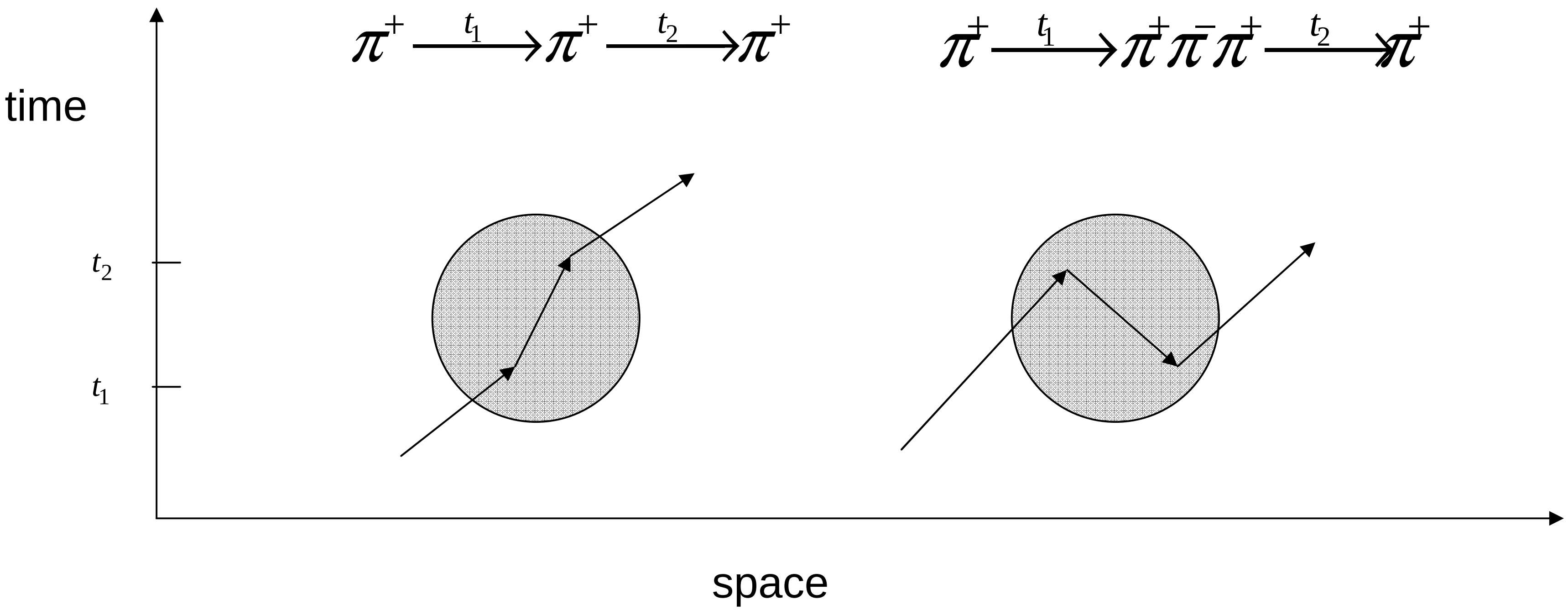
⇨ Feynman – Stuckelberg interpretation

$$\pi^+(E>0) \uparrow \quad \equiv \quad \pi^-(E<0) \downarrow$$

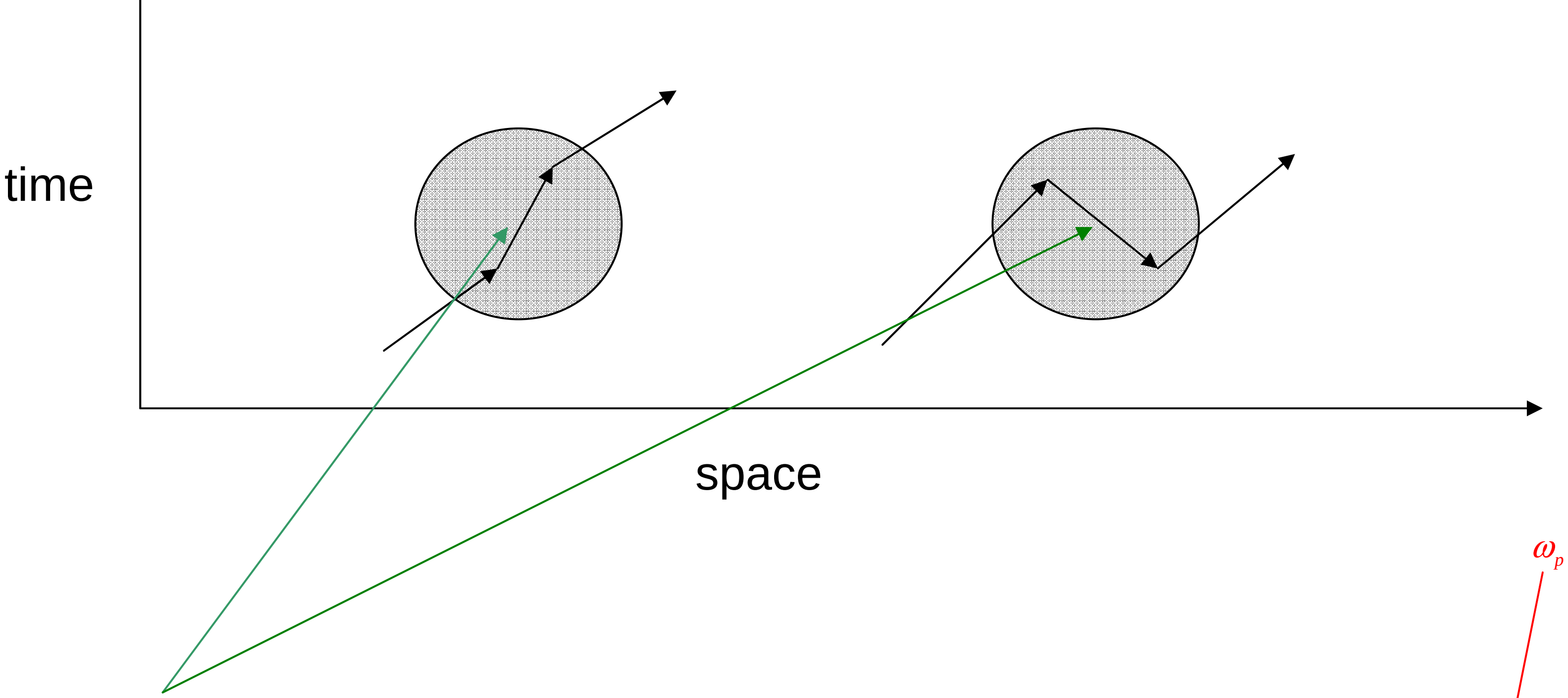
$$e^{-iEt} \qquad\qquad e^{-i(-E)(-t)}$$

Two different time orderings giving same observable event :

$$\pi^+ \xrightarrow{t_1} \pi^+ \xrightarrow{t_2} \pi^+ \qquad\qquad \pi^+ \xrightarrow{t_1} \pi^+\pi^-\pi^+ \xrightarrow{t_2} \pi^+$$



$$\pi^+ \xrightarrow{t_1} \pi^+ \xrightarrow{t_2} \pi^+ \qquad\qquad \pi^+ \xrightarrow{t_1} \pi^+\pi^-\pi^+ \xrightarrow{t_2} \pi^+$$



$$\omega_p = \left(\underline{p}^2 + m^2\right)^{1/2}$$

$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.(x'-x)}\,\frac{1}{p^2-m^2-i\varepsilon} = -i\int \frac{d^3p}{(2\pi)^3\,2\omega_p}\, e^{-i\omega_p|t'-t|-ip.(x'-x)}$$

(p^0 integral most conveniently evaluated using contour integration via Cauchy's theorem)

$$\Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x} \frac{1}{p^2-m^2-i\varepsilon}$$

$$p^2+m^2-i\varepsilon \quad \Rightarrow \quad p_o^2 = \underline{p}^2+m^2-i\varepsilon \quad \Rightarrow \quad p_0 = \pm\left(\underline{p}^2+m^2\right)^{1/2} \mp i\delta = \pm\omega_p \mp i\delta$$

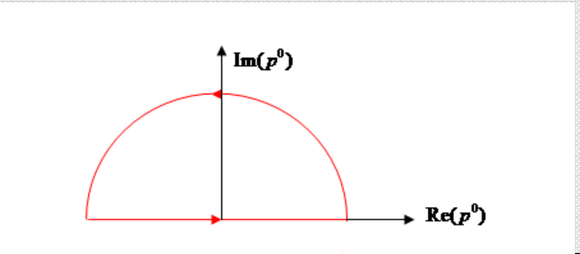
$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^3p\, e^{-i\underline{p}.(\underline{x}'-\underline{x})}\int dp_0 \underbrace{\frac{e^{-ip_0(t'-t)}}{\left(p_0-(\omega_p-i\delta)\right)\left(p_0-(-\omega_p+i\delta)\right)}}_{I}$$

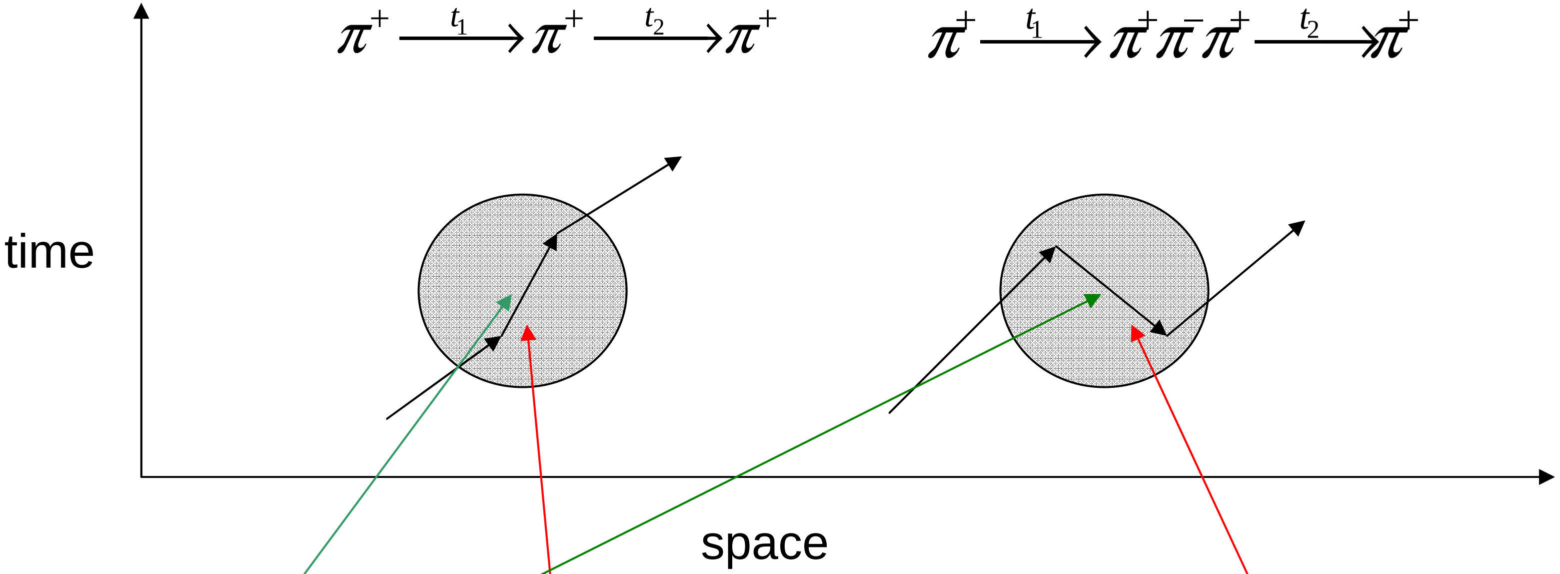
- If $t'-t>0$, choose contour such that $p_0=-ip_I$ $(p_I$ +ve$)\Rightarrow e^{-ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p}e^{-i\omega_p(t'-t)}\theta(t'-t)$$

- If $t'-t<0$, choose contour such that $p_0=+ip_I$ $(p_I$ +ve$)\Rightarrow e^{+ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p}e^{+i\omega_p(t'-t)}\theta(t-t')$$





$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.(x'-x)}\,\frac{1}{p^2-m^2-i\varepsilon} = -i\int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, e^{-i\omega_p|t'-t|-ip.(x'-x)}$$

$$\Delta_F(x-x') = -i\int d^3p\; f_p^{+}(x')\, f_p^{+*}(x)\,\theta(t'-t) - i\int d^3p\; f_p^{-}(x')\, f_p^{-*}(x)\,\theta(t-t')$$

where $f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$ are positive and negative energy solutions to free KG equation

Theory confronts experiment - Cross sections and decay rates

Scattering in Quantum Mechanics

- Prepare state at $t = -\infty$ $\qquad |\psi_{in}(t=-\infty)>=|i>$
- Time evolution (possibly scattering) $\qquad |\psi_{in}(t=+\infty)>=S|\psi_{in}(t=-\infty)>$
- Observe resulting system in state $\qquad |\psi_{out}(t=+\infty)>=|f>$

QM : probability amplitude :

$$<\psi_{out}(t=+\infty)|\psi_{in}(t=+\infty)>=<\psi_{out}(t=+\infty)|S|\psi_{in}(t=-\infty)>$$
$$=<f|S|i>=S_{fi}$$

$$S_{fi}=\delta_{fi}+iT_{fi}$$

S matrix for Klein Gordon scattering

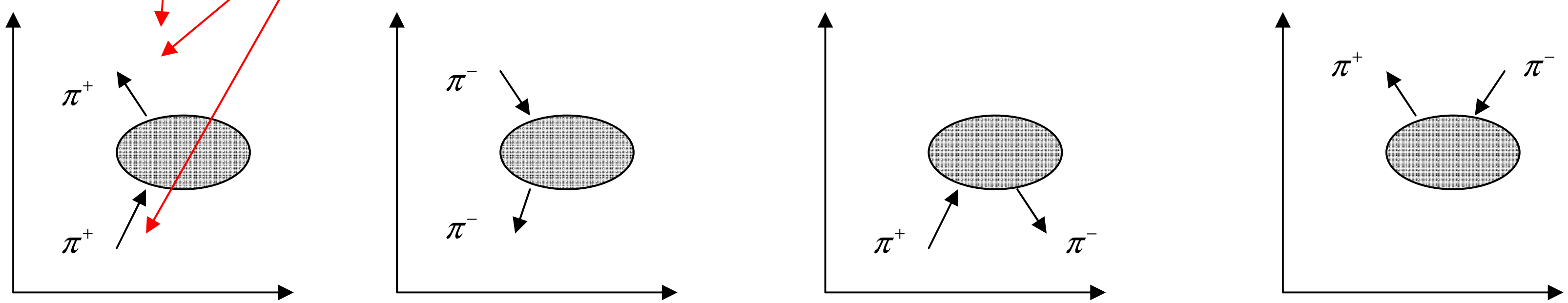
Relativistic probability density

$$S_{p'+,p+} = <\psi_{out}(t=+\infty)\,|\,\psi_{in}(t=+\infty)> \quad = \lim_{t\to\infty}\int d^3x\; f^{+*}_{p'+}\, i\partial_0 \psi(x)$$

$$\psi(x) = \phi(x) - \int d^4x'\,\Delta_F(x'-x)V(x')\phi(x') + \ldots$$

$$\Delta_F(x'-x) = -i\int d^3p\; f^{+}_{p}(x')\, f^{+*}_{p}(x)\theta(t'-t) - i\int d^3p\; f^{-}_{p}(x')\, f^{-*}_{p}(x)\theta(t-t')$$

$$S_{p'+,p+} = \delta^3(p'_+ - p_+) - i\int d^4x'\; f^{+*}_{p+}(x')\,V(x')\; f^{+}_{p+}(x') + \ldots$$



Feynman rules

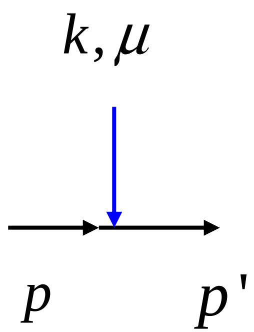
$$iT_{fi} = -i\int d^4y\; f^{+*}_{p+}(y)\, V(y)\; f^{+}_{p+}(y) = i\int d^4y\; f^{+*}_{p+}(y)\, ie(A^\mu\partial_\mu + \partial_\mu A^\mu)\; f^{+}_{p+}(y)$$

$$f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

$$= -i\int d^4y\; j^{fi}_\mu A^\mu$$

$$j^{fi}_\mu = -ie\Big(f^{+*}_{p+}(y)\big[\partial_\mu f^{+}_{p+}(y)\big] - \big[\partial_\mu f^{*+}_{p+}(y)\big]\, f^{+}_{p+}(y)\Big)$$

$$= -e(p_f + p_i)_\mu e^{i(p_f - p_i).x}$$



$$-ie(p_\mu + p'_\mu)$$

Feynman rule associated
with Feynman diagram