BFKL at NLL for diffraction and for jet production

Solving the BFKL Equation(s) by Iteration

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Outline

- The **High Energy Limit** of scattering processes and the **BFKL equation(s)**
- **Leading Logarithmic Accuracy**:
  - Analytic solution
    Problems for phenomenology intrinsic to this analytic approach
- **Next-to-Leading Logarithmic Accuracy**
  - Analytic problems
  - ...and the iterative solution
    Why one *has* to use a different approach at NLL
- **Non-forward** BFKL Equation at LLA and NLLA

Pulications

- J. R. Andersen and A. Sabio Vera, Phys. Lett. B **567**, 116
- J. R. Andersen and A. Sabio Vera, Nucl. Phys. B **679**, 345
- J. R. Andersen and A. Sabio Vera, Nucl. Phys. B **699**, 90
- J. R. Andersen and A. Sabio Vera, JHEP **0501**, 045

⋮

QCD

Well understood for scattering processes with one hard scale.
Total cross sections well described by:

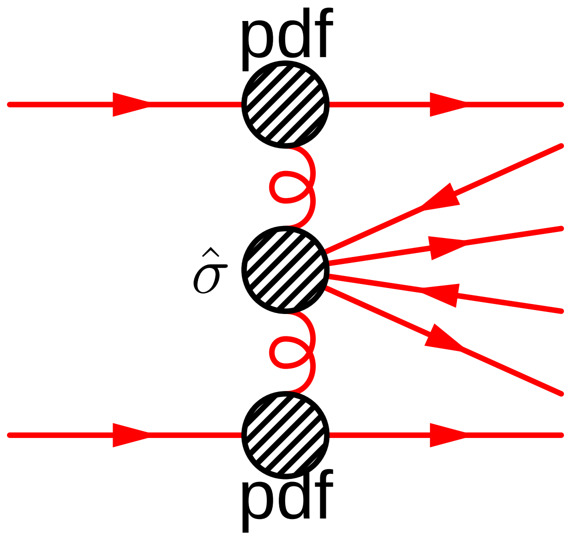
- Fixed (NL) order perturbative calculations
- DGLAP evolution of PDFs

Multi-Jet and Multi-Scale QCD: attempt to extend our understanding of perturbative QCD away from the study of total cross sections with one, super perturbative scale.

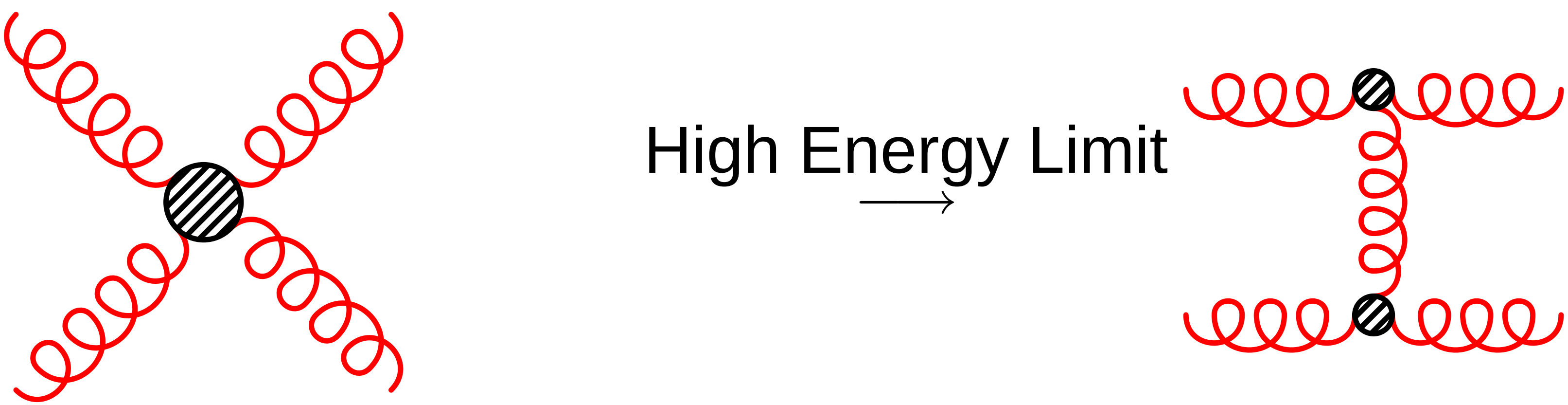
Excitement/Drawback: *not* 5-10% effects!

Surprises may lurk round the corner

High Energy Limit

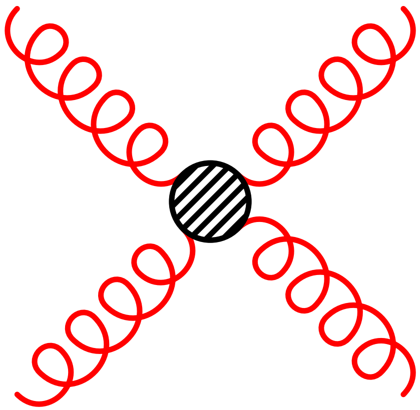


High energy limit:
$\frac{\hat{s}}{|\hat{t}|} \longrightarrow \infty$
$|\mathcal{M}|^2$ factorises.



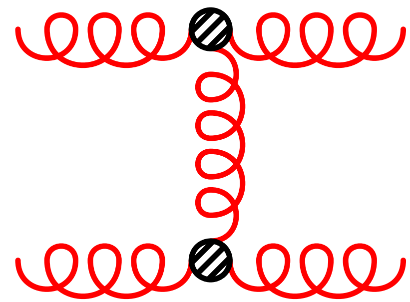
Diagrams with a t-channel gluon exchange dominate the cross section.

HE limit and Dijet Production



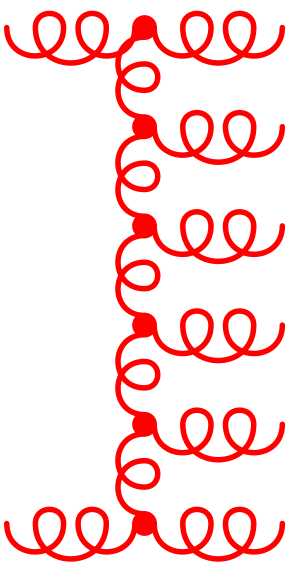
High Energy Limit $\longrightarrow$

$\hat{s}/|\hat{t}| \to \infty$



Large logarithm in multi-gluon production

$P_{Ta}, \Delta y$



$P_{Tb}, 0$

$$p_{Ti}^2 \sim p_{Ta}^2 \sim p_{Tb}^2$$

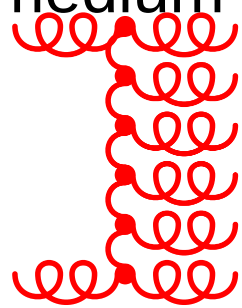
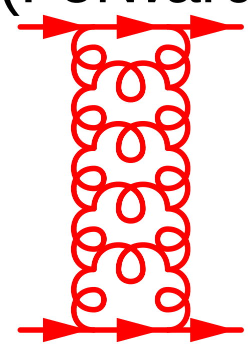
$$\hat{s}_{ij} \gg p_{Ti}^2$$

BFKL resums to all orders terms in the perturbative expansion of the form

$$\left(\alpha_s \ln \frac{\hat{s}_{ij}}{|\hat{t}_i|}\right) \sim (\alpha_s \Delta y)$$

BFKL formalism

- **BFKL** (**B**alitskii, **F**adin, **K**uraev, **L**ipatov): resummation of **large logarithms** in the perturbation series for QCD processes with **two large** (perturbative) and **disparate** energy scales $\hat{s} \gg |\hat{t}|$ ($\hat{s}$: E^2, $|\hat{t}|$: $p_\perp^2$)

Structure Functions	Forward Physics @ Hadron Colliders (Colour Octet Exchange)	Diffraction (Colour Singlet Exchange)
Small x	Large Rapidity (Forward) medium x 	Large Rapidity (Forward) 

BFKL formalism

- The cross section for the process $A + B \to A' + B'$ factorises as

$$\sigma(s) = \int \frac{d^2\mathbf{k}_a}{2\pi\mathbf{k}_a^2} \int \frac{d^2\mathbf{k}_b}{2\pi\mathbf{k}_b^2}\, \Phi_A(\mathbf{k}_a)\, f\left(\mathbf{k}_a, \mathbf{k}_b, \Delta = \ln\frac{s}{s_0}\right) \Phi_B(\mathbf{k}_b)$$

- $\Phi_A(\mathbf{k}_a), \Phi_B(\mathbf{k}_b)$ process dependent *impact factors* (calculated for many process at LL and for e.g. gg and (ongoing) $\gamma^*\gamma^*$ scattering at NLL)
- $f\left(\mathbf{k}_a, \mathbf{k}_b, \Delta\right)$ process independent *Gluon Green's function*

The BFKL Equation

The Gluon Green's function fulfil (to LLA and NLLA) the BFKL equation (in dim. regularisation $(D = 4 + 2\epsilon)$):

$$\omega f_\omega\left(\mathbf{k}_a, \mathbf{k}_b\right) = \delta^{(2+2\epsilon)}\left(\mathbf{k}_a - \mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}' \mathcal{K}_\epsilon\left(\mathbf{k}_a, \mathbf{k}'\right) f_\omega\left(\mathbf{k}', \mathbf{k}_b\right)$$

where the BFKL kernel $\mathcal{K}\left(\mathbf{k}_a, \mathbf{k}'\right)$ is calculated to LLA or NLLA respectively. At LL the kernel is **conformal invariant** (no running coupling) with **eigenfunctions** $\mathbf{k}^{2(\gamma-1)}$. Use (transverse) Mellin transform!

$$\int d^2\mathbf{k}'\ \mathcal{K}\left(\mathbf{k}, \mathbf{k}'\right)\ \mathbf{k}^{2(\gamma-1)} = \frac{N_c\alpha_s}{\pi}\xi^{\mathrm{LL}}(\gamma)\mathbf{k}^{2(\gamma-1)}$$

$$\omega(\gamma) = \langle\gamma|\mathcal{K}(k,k)|\gamma\rangle$$

$$f_\omega \sim \sum_\gamma \omega(\gamma)|\gamma\rangle$$

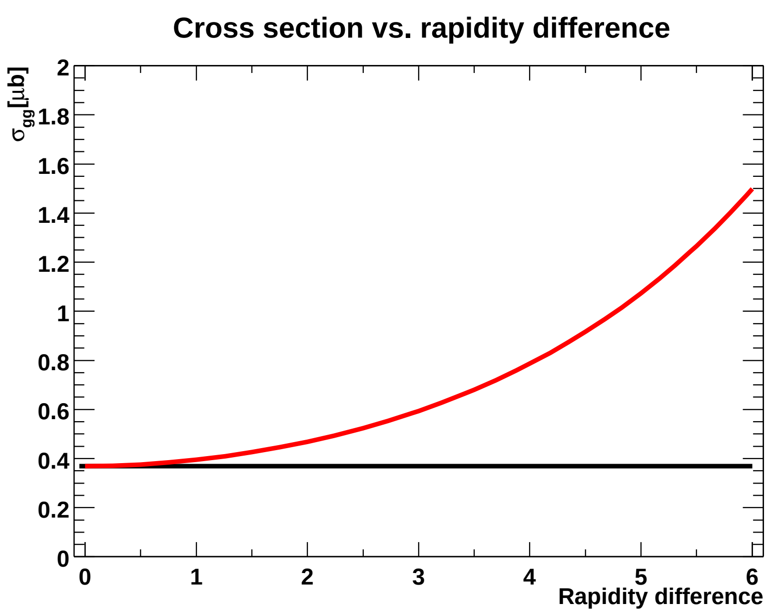
The BFKL Equation at LLA

Analytic solution for angular averaged gluon Green's function

$$\bar{f}\left(k_a, k_b, \Delta\right) = \frac{1}{k_a k_b} \int_0^\infty d\nu \left(\frac{k_a^2}{k_b^2}\right)^{i\nu} e^{\bar{\alpha}_s \Delta \chi_0(\nu)}$$

with the LL eigenvalue

$$\chi_0(\nu) = -2\,\mathrm{Re}\left\{\psi\left(\frac{1}{2} + i\nu\right) - \psi(1)\right\}.$$



BFKL rise in cross section! Integrated over the **full k phase space** for gluon emission and allowing **any number** of gluons to radiate!!!

$$\hat{\sigma}_{gg} \to \frac{\pi C_A^2 \alpha_s^2}{2 P_{T,\min}^2} \frac{e^{\lambda \Delta y}}{\sqrt{\pi B \Delta y}}, B = 14\zeta(3)\bar{\alpha}_s, \quad \lambda = \frac{\alpha_s C_A}{\pi} 4 \ln 2 \approx 0.45$$

The NLL BFKL Story So Far

- BFKL equation at LL put forward and solved in 1978.
  - **non-forward** equation solved five years later by L. Lipatov
- 8-10 years effort to calculate the BFKL kernel at NLLA ended in 1998
  - Initial results were discouraging. NLL kernel applied to LL eigenfunctions lead to huge and unstable corrections.
  - We will see why this analysis is invalid.
- Calculation of the **non-forward** kernel finished Dec. 2004 by V. Fadin and collaborators.

BFKL at NLLA

- Two new effects appear:
  - Fermions
  - Running Coupling

Conformal invariance **broken** — Eigenfunctions **unknown.** Analyse what happens **if we pretend** the LL eigenfunctions are also eigenfunctions at NLL.

$$\omega^{\mathrm{NLL}}(\gamma) = \langle \gamma | \mathcal{K}^{\mathrm{NLL}}(k,k) | \gamma \rangle$$

$$f_\omega \sim \sum_\gamma \omega(\gamma) | \gamma \rangle$$

Leading Log tools at NLL

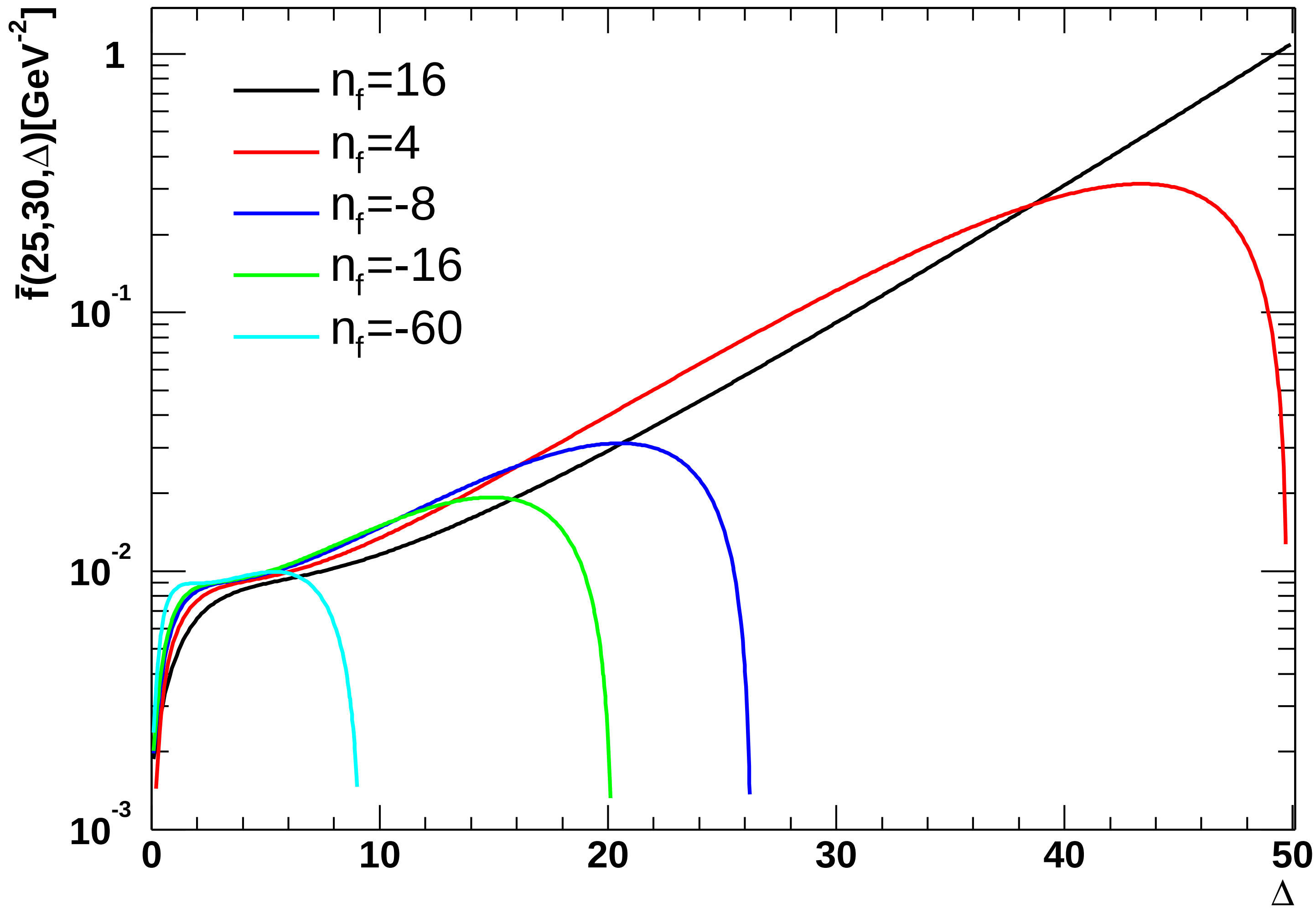
$$\omega^{\rm NLL}(\gamma) = \int \mathrm{d}^{D-2}\mathbf{k}\, \mathcal{K}^{\rm NLL}(\mathbf{k}_a, \mathbf{k}) \left(\frac{\mathbf{k}^2}{\mathbf{k}_a^2}\right)^{\gamma-1}$$
$$= \frac{\alpha_s(\mathbf{k}_a^2)N}{\pi} \left(\chi^{\rm LL}(\gamma) + \chi^{\rm NLL}(\gamma)\frac{\alpha_s(\mathbf{k}_a^2)N}{\pi}\right)$$

$$\chi^{\rm NLL}(\gamma) = -\frac{1}{4}\Big[\left(\frac{11}{3} - \frac{2}{3}\frac{n_f}{N}\right)\frac{1}{2}\left(\chi^{\rm LL}(\gamma) - \psi'(\gamma) + \psi'(1-\gamma)\right)$$
$$- 6\zeta(3) + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \left(1 + \frac{n_f}{N^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$
$$- \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N}\right)\chi^{\rm LL}(\gamma) - \psi''(\gamma) - \psi''(1-\gamma) - \frac{\pi^3}{\sin(\pi\gamma)} + 4\phi(\gamma)\Big],$$

let us pretend:

$$\bar{f}(k_a, k_b, \Delta) = \frac{1}{k_a^2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathrm{d}\gamma}{2\pi i}\, e^{\Delta\omega^{\rm NLL}(\gamma)} \left(\frac{k_b^2}{k_a^2}\right)^{\gamma}$$

Leading Log tools at NLL



!!this would be a major catastrophe!!

Read the small print

Fadin and Lipatov say

Almost all the terms in the right hand side of eq. (12) except the contribution

$$\Delta(\gamma) = \frac{\alpha_s^2(\mu^2) N_c^2}{4\pi^2} \left(\frac{11}{3} - \frac{2n_f}{3N_c} \right) \frac{1}{2} \left(\psi'(\gamma) - \psi'(1-\gamma) \right)$$

are symmetric to the transformation $\gamma \leftrightarrow 1-\gamma$. Moreover, it is possible to cancel $\Delta(\gamma)$ if one would redefine the function $q^{2(\gamma-1)}$ by including in it the logarithmic factor $\left(\frac{\alpha_s(q^2)}{\alpha_s(\mu^2)} \right)^{-1/2}$.

This would **remove the imaginary part**, and therefore also **remove the oscillations**.

What to believe?

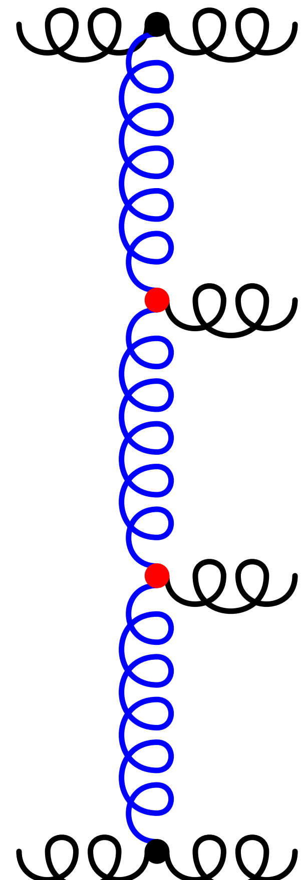
Iterative Solution at NLLA

We propose an iterative approach to the BFKL equation at NLLA that solves the equation with *no approximations*

- Directly in the physical rapidity and transverse momentum space
  (avoids the use of the troublesome Mellin transform completely)
- The right language for use of impact factors (physics predictions!)
- Hopeful in extending the approach to final state studies like at LL
- Expresses the solution in terms of effective vertices and no-emission probabilities (physical insight into the BFKL solution at NLLA!)

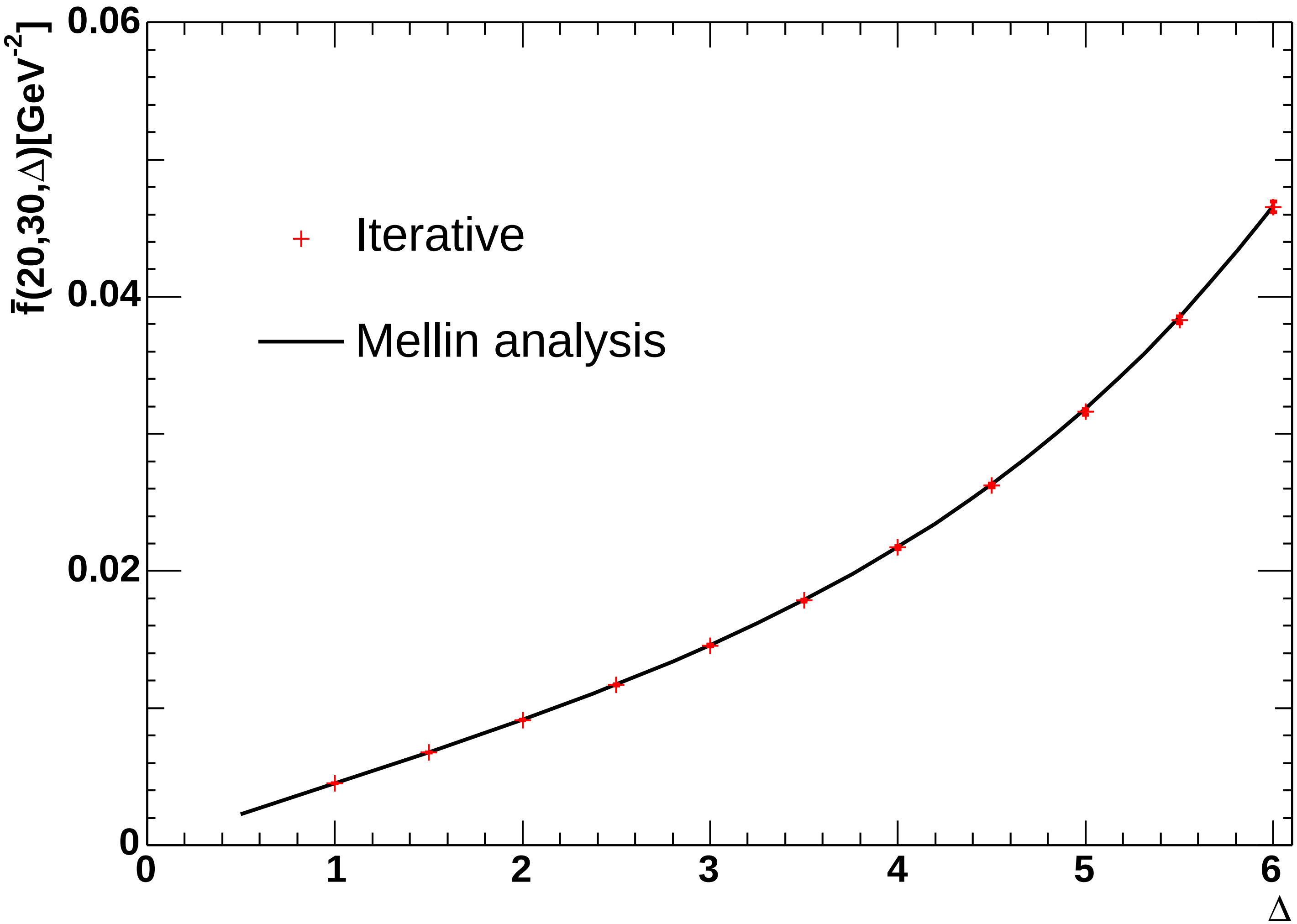
Enter Iteration at NLLA

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) \;&=\; \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
&+ \sum_{n=1}^{\infty}\prod_{i=1}^{n}\int d^2\mathbf{k}_i \int_0^{y_{i-1}} dy_i \left[V\left(\mathbf{k}_i, \mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mu\right)\right] \\
&\times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
&\qquad \times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right] \\
&\qquad \times \delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$



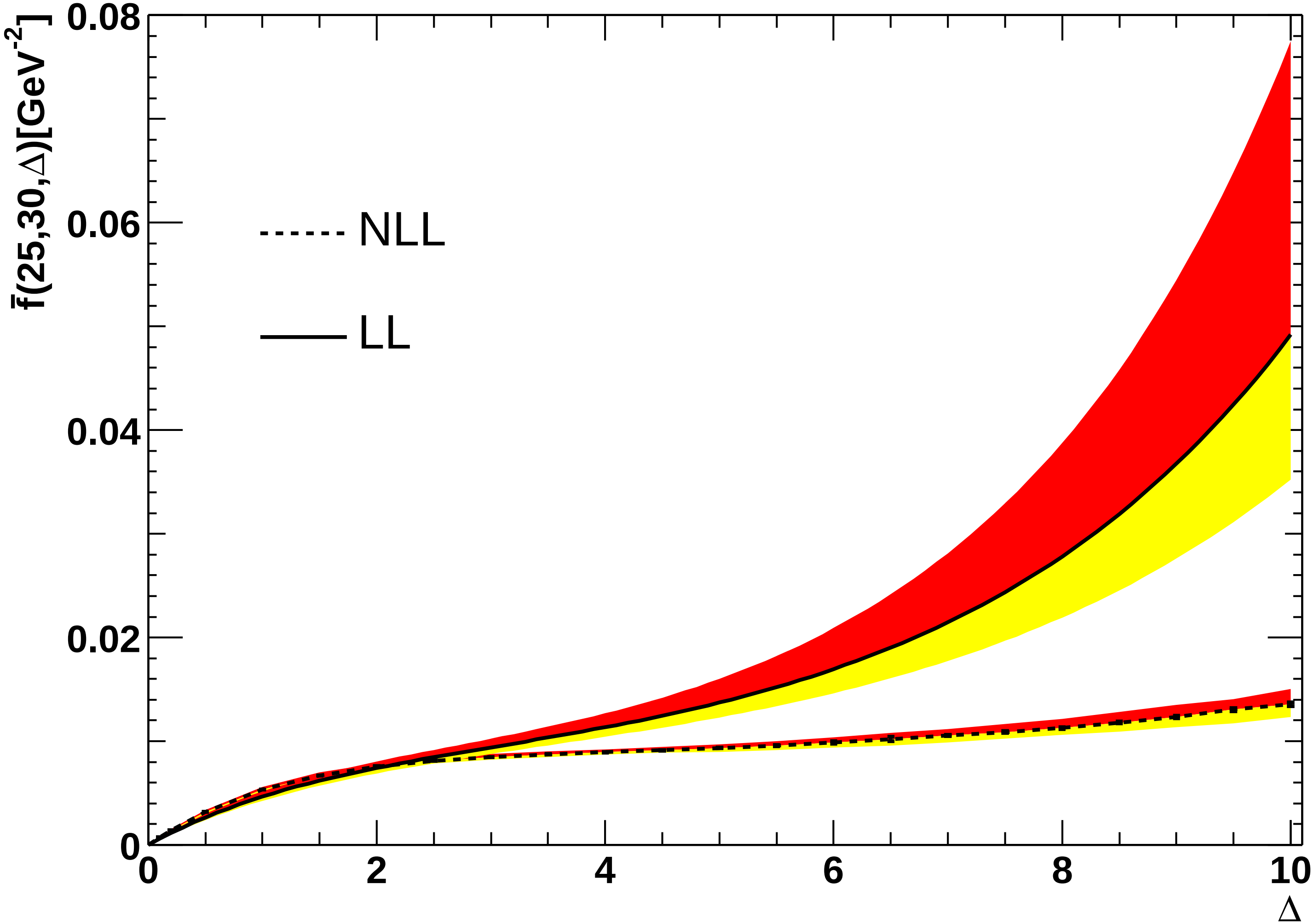
$\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM preserves conformal invariance at NLL



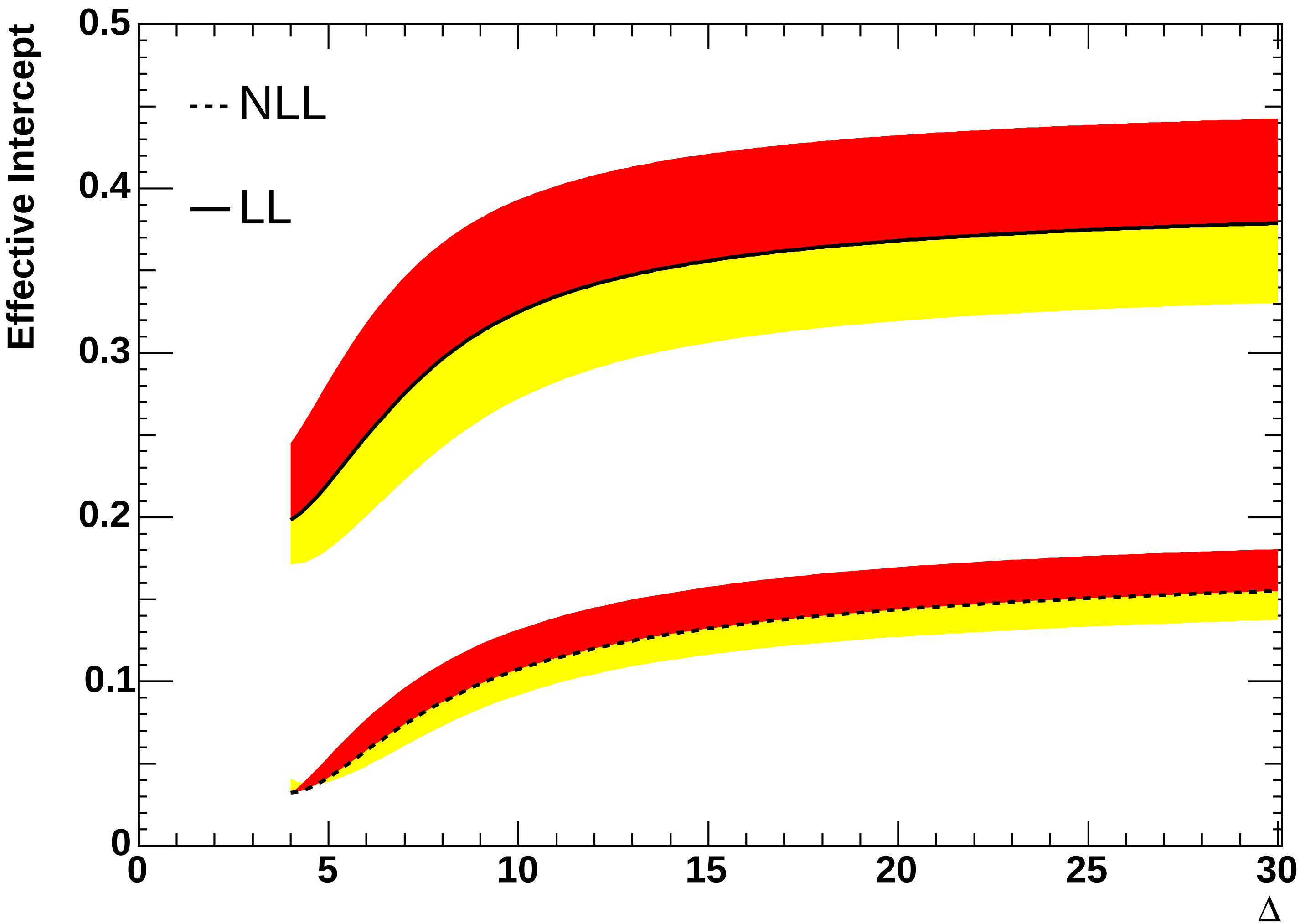
Dependence of f on Δ

QCD, $n_f = 4$, one loop running

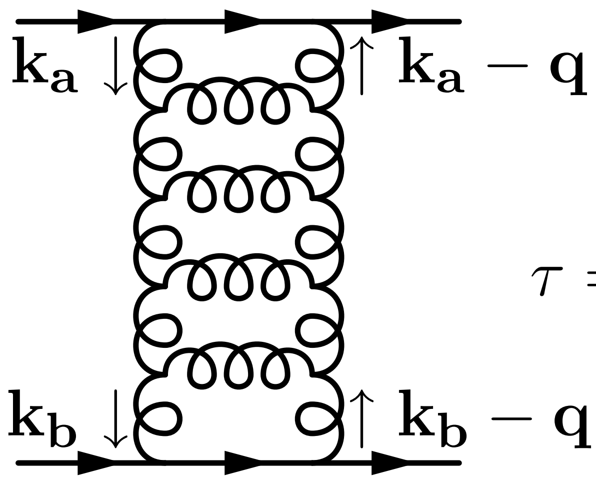


BFKL Intercept

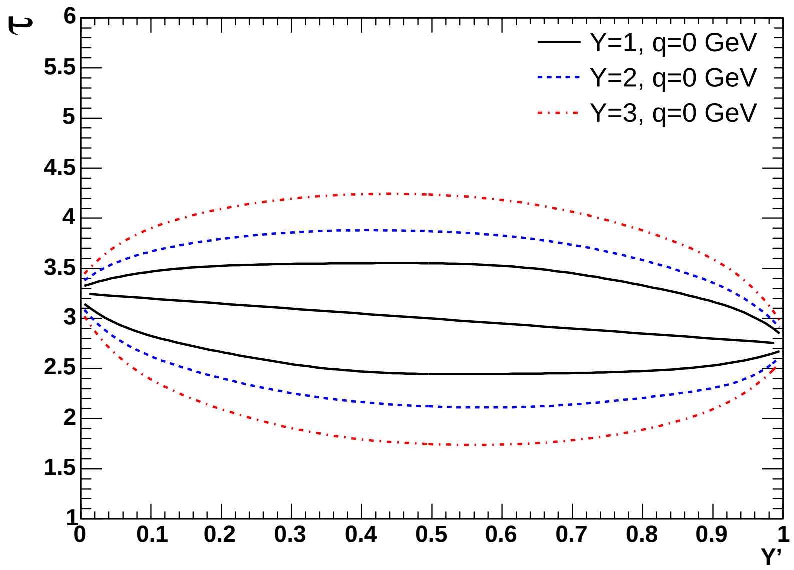
QCD, $n_f = 4$, one loop running

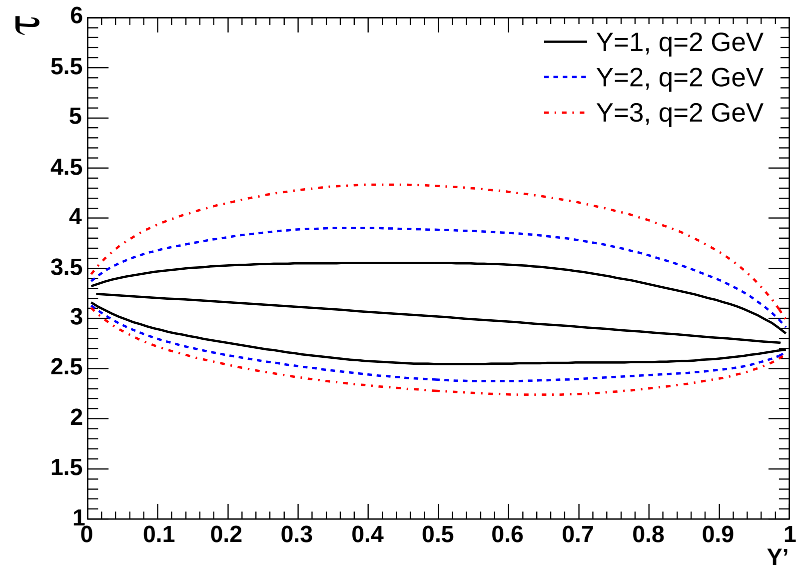


Diffusion and Diffraction at LLA



$$\tau = \ln\left(\left(k_a + \sum k_i\right)^2 / \mathrm{GeV}^2\right)$$



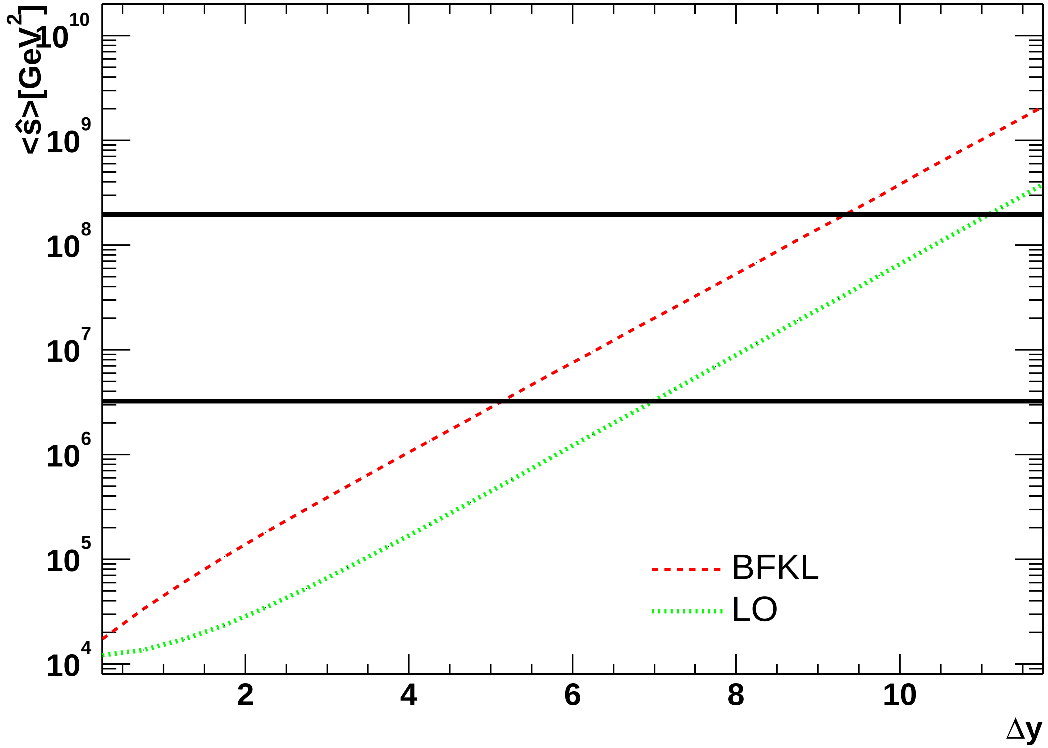


Conclusions

- We have solved the BFKL equation at full **Next–to-leading logarithmic accuracy** (No approximation: keeping all scale invariant and scale dependent terms, and full angular information.)
- . . . in a form that is directly suitable for calculation of cross sections (inclusion of impact factors)
- Explore **non-problem** of NLL BFKL
- Will extend study to **final states at NLL** - necessary for phenomenology at full NLL accuracy (resum **only available phase space**)
- Method also applicable to the non-forward (NLL) BFKL equation

Extra Slides

Energy Consumption of BFKL evolution



$$\hat{s}_{\rm BFKL} \propto \hat{s}_{\rm LO} \frac{1}{(1-e^{-\delta y})^2}$$

Large effects - Resum only the phase space accessible at a given collider!

Iteration at NLL

Start from the BFKL equation

$$\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) = \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}'\,\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}'\right) f_\omega\left(\mathbf{k}',\mathbf{k}_b\right)$$

$$\mathcal{K}\left(\mathbf{k}_a,\mathbf{k}\right) = 2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\,\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right) + \mathcal{K}_r\left(\mathbf{k}_a,\mathbf{k}\right)$$

Need all terms (IR) finite to be able to iterate: split the kernel $\mathcal{K}_r$ into two parts: a ϵ–dependent, $\mathcal{K}_r^{(\epsilon)}$, and a ϵ–independent, $\widetilde{\mathcal{K}}_r$

$$\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) = \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right) f_\omega\left(\mathbf{k},\mathbf{k}_b\right)$$
$$+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right) f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right) + \int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right) f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right).$$

Iteration at NLL, 2

Introduce a slice in the phase space (no approximation)

$$\begin{aligned}\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) &= \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)+\int d^{2+2\epsilon}\mathbf{k}\,2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)\delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}\right)f_\omega\left(\mathbf{k},\mathbf{k}_b\right)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\left(\theta\left(\mathbf{k}^2-\lambda^2\right)+\theta\left(\lambda^2-\mathbf{k}^2\right)\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\end{aligned}$$

approximate $f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right)\simeq f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)$ for $|\mathbf{k}|<\lambda$

$$\begin{aligned}\omega f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right) &= \delta^{(2+2\epsilon)}\left(\mathbf{k}_a-\mathbf{k}_b\right)\\ &+\left\{2\,\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\theta\left(\lambda^2-\mathbf{k}^2\right)\right\}f_\omega\left(\mathbf{k}_a,\mathbf{k}_b\right)\\ &+\int d^{2+2\epsilon}\mathbf{k}\,\left\{\mathcal{K}_r^{(\epsilon)}\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\theta\left(\mathbf{k}^2-\lambda^2\right)+\widetilde{\mathcal{K}}_r\left(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}\right)\right\}f_\omega\left(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b\right).\end{aligned}$$

($\lambda\to 0$ limit can be obtained)

Iteration at NLL, 3

$$\left(\omega - \omega_0\left(\mathbf{k}_a^2, \lambda^2\right)\right) f_\omega\left(\mathbf{k}_a, \mathbf{k}_b\right) = \delta^{(2)}\left(\mathbf{k}_a - \mathbf{k}_b\right) + \int d^2\mathbf{k}\left(\frac{1}{\pi\mathbf{k}^2}\xi\left(\mathbf{k}^2\right)\theta\left(\mathbf{k}^2 - \lambda^2\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}\right)\right) f_\omega\left(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b\right)$$

$$\omega_0\left(\mathbf{q}^2, \lambda^2\right) \equiv -\xi\left(|\mathbf{q}|\,\lambda\right)\ln\frac{\mathbf{q}^2}{\lambda^2} + \eta$$

$$\xi\left(\mathrm{X}\right) \equiv \bar{\alpha}_s + \frac{\bar{\alpha}_s^2}{4}\left[\frac{4}{3} - \frac{\pi^2}{3} + \frac{5}{3}\frac{\beta_0}{N_c} - \frac{\beta_0}{N_c}\ln\frac{\mathrm{X}}{\mu^2}\right]$$

$$\eta \equiv \bar{\alpha}_s^2\frac{3}{2}\zeta(3).$$

$$\widetilde{\mathcal{K}}_r\left(\mathbf{q}, \mathbf{q}'\right) = \frac{\bar{\alpha}_s^2}{4\pi}\left\{\text{6 lines of equations}\ldots\right\}.$$

Iteration at NLL, 4

Iterate and take the inverse Mellin transform to find

$$
\begin{aligned}
f(\mathbf{k}_a, \mathbf{k}_b, \Delta) \quad &= \quad \exp\left(\omega_0\left(\mathbf{k}_a^2, \lambda^2, \mu\right)\Delta\right)\delta^{(2)}(\mathbf{k}_a - \mathbf{k}_b) \\
+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int d^2\mathbf{k}_i & \left[\frac{\theta\left(\mathbf{k}_i^2 - \lambda^2\right)}{\pi \mathbf{k}_i^2}\xi\left(\mathbf{k}_i^2, \mu\right) + \widetilde{\mathcal{K}}_r\left(\mathbf{k}_a + \sum_{l=0}^{i-1}\mathbf{k}_l, \mathbf{k}_a + \sum_{l=1}^{i}\mathbf{k}_l, \mu\right)\right] \\
\times \int_0^{y_{i-1}} dy_i \ \exp & \left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{i-1}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_{i-1} - y_i)\right] \\
& \times \exp\left[\omega_0\left(\left(\mathbf{k}_a + \sum_{l=1}^{n}\mathbf{k}_l\right)^2, \lambda^2, \mu\right)(y_n - 0)\right]\delta^{(2)}\left(\sum_{l=1}^{n}\mathbf{k}_l + \mathbf{k}_a - \mathbf{k}_b\right)
\end{aligned}
$$

JRA and A. Sabio Vera

Convergence

$$\bar{f}\left(k_a, k_b, \Delta\right) \;=\; \int_0^{2\pi} d\theta\, f\left(k_a, k_b, \theta, \Delta\right),$$

$k_a = 25$ GeV, $k_b = 30$ GeV, $\lambda = 1$ GeV