# Treatment of correlated systematic errors

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Systematic differences combining ZEUS and H1 data

- In a QCD fit
- In a ‘theory free’ fit

# Treatment of correlated systematic errors

$$\chi 2 = \Sigma_i \left[ \frac{F_i^{QCD}(p) - F_i^{MEAS}}{(\sigma_i^{STAT})^2 + (\Delta_i^{SYS})^2} \right]^2$$

Quadratic combination: errors on the fit parameters, p, evaluated from $\Delta\chi 2 = 1$,

**THIS IS NOT GOOD ENOUGH** if experimental systematic errors are correlated between data points-

$$\chi^2 = \Sigma_i \Sigma_j [ F_i^{QCD}(p) - F_i^{MEAS}] V_{ij}^{-1} [ F_j^{QCD}(p) - F_j^{MEAS}]$$

$$V_{ij} = \delta_{ij}(\sigma_i^{STAT})^2 + \Sigma_\lambda \Delta_{i\lambda}^{SYS} \Delta_{j\lambda}^{SYS}$$

Where $\Delta_{i\lambda}^{SYS}$ is the correlated error on point **i** due to systematic error source $\lambda$

It can be established that this is equivalent to

$$\chi^2 = \Sigma_i \frac{[ F_i^{QCD}(p) - \Sigma_\lambda s_\lambda \Delta_{i\lambda}^{SYS} - F_i^{MEAS}]^2}{(\sigma_i^{STAT})^2} + \Sigma s_\lambda^2$$

Where $s_\lambda$ are systematic uncertainty fit parameters of zero mean and unit variance

**This has modified the fit prediction by each source of systematic uncertainty**

**CTEQ, ZEUS, H1, MRST/MSTW have all adopted this form of $\chi 2$ (MSTW still use some errors as quadratic) – but use it differently in the OFFSET and HESSIAN methods**

# How do experimentalists often proceed: OFFSET method

Perform fit without correlated errors ($s_\lambda = 0$) for central fit, and propagate statistical errors to the PDFs

$$< \sigma^2_q > = T \; \Sigma_j \Sigma_k \frac{\partial q}{\partial p_j} V_{jk} \frac{\partial q}{\partial p_k}$$

Where T is the χ2 tolerance, T = 1.

1. Shift measurement to upper limit of one **of its systematic uncertainties ($s_\lambda = +1$)**
2. **Redo fit, record differences of parameters from those of step 1**
3. **Go back to 2, shift measurement to lower limit ($s_\lambda = -1$)**
4. **Go back to 2, repeat 2-4 for next source of systematic uncertainty**
5. **Add all deviations from central fit in quadrature (positive and negative deviations added in quadrature separately)** A5
6. **This method does not assume that correlated systematic uncertainties are Gaussian distributed**

**Fortunately, there are smart ways to do this – see extras**

**(Pascaud and Zomer LAL-95-05, Botje hep-ph-0110123)**

**Slide 3**

**A5** Cooper-Sarkar, 3/15/2004

# HESSIAN method (covariance method)

Allow $s_\lambda$ parameters to vary for the central fit. The total covariance matrix is then the inverse of a single Hessian matrix expressing the variation of $\chi2$ wrt both theoretical and systematic uncertainty parameters.

If we believe the theory why not let it calibrate the detector(s)? Effectively the theoretical prediction is not fitted to the central values of published experimental data, but allows these data points to move collectively according to their correlated systematic uncertainties

The fit determines the optimal settings for correlated systematic shifts such that the most consistent fit to all data sets is obtained. In a global fit the systematic uncertainties of one experiment will correlate to those of another through the fit
The resulting estimate of PDF errors is much smaller than for the Offset method for $\Delta\chi2 = 1$

CTEQ have used this method with $\Delta\chi2 \sim 100$ for 90%CL limits

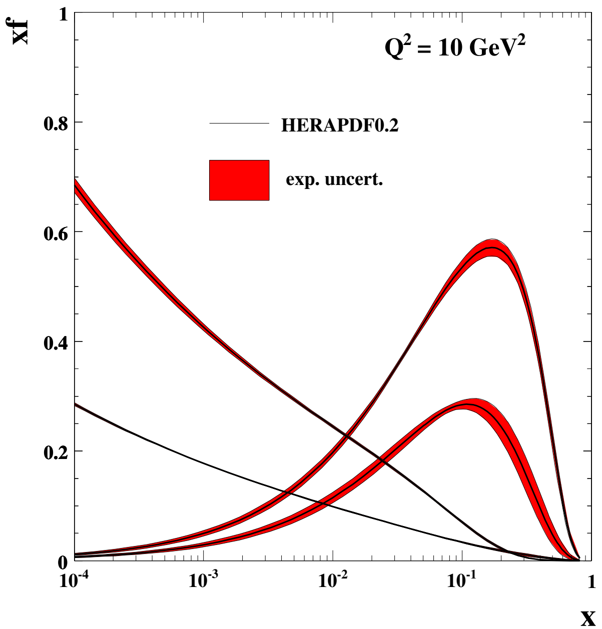
MRST have used $\Delta\chi2 \sim 50$

H1, Alekhin have used $\Delta\chi2 = 1$

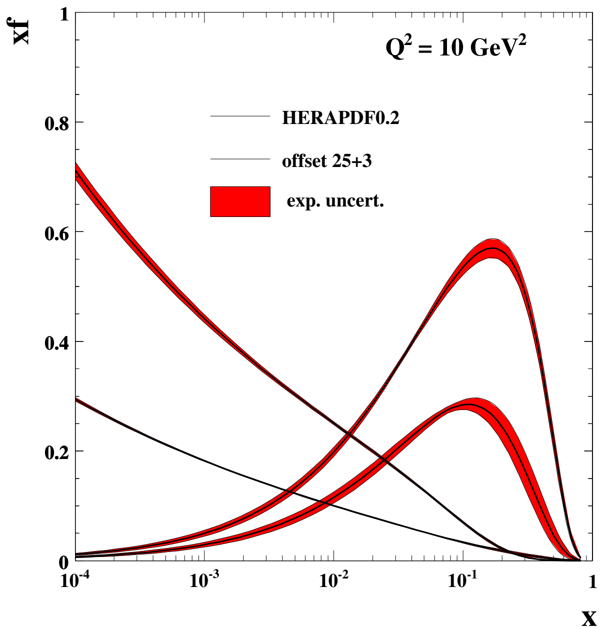
Tolerances are now dynamic according to the eigenvector

It is not my purpose to justify these methods Just to compare them.

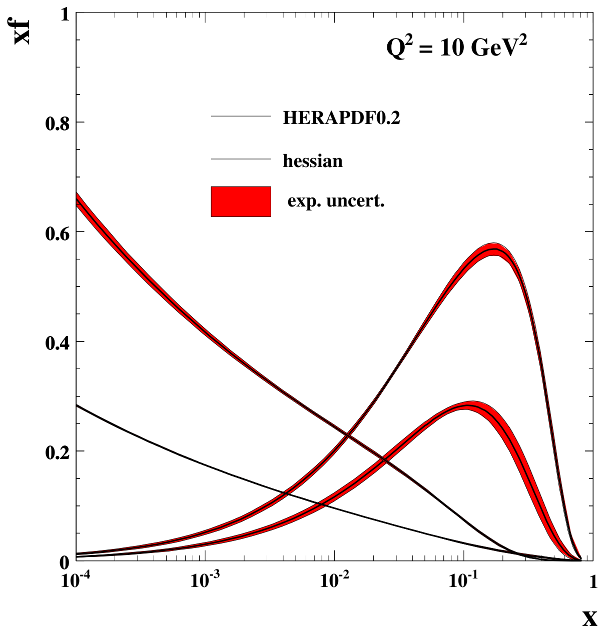
First for the HERAPDF to the combined HERA data for which correlated systematic errors are SMALL relative to the statistical and uncorrelated errors




**Quadratic**



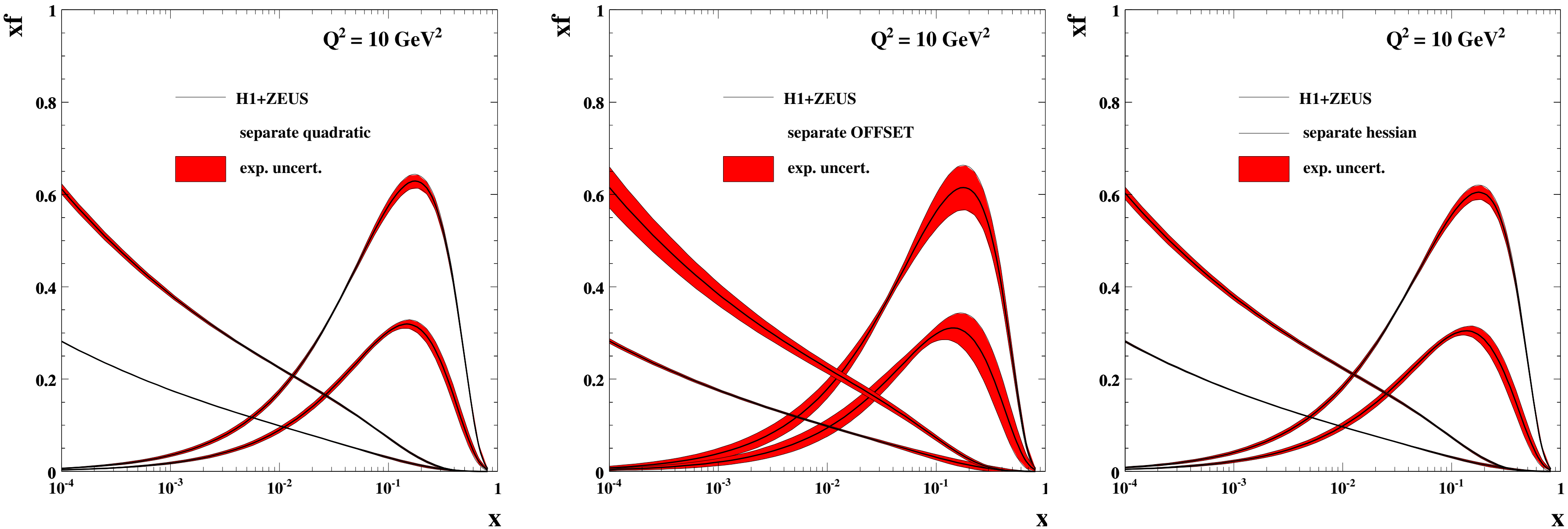

**OFFSET**




**HESSIAN**

For the HERAPDF with its small correlated systematic errors the treatment of these errors in the PDF fit gives only small differences in PDF central values and PDF uncertainty estimates

Secondly for the separate ZEUS and H1 data for which correlated systematic errors are LARGE relative to the statistical and uncorrelated errors for Q2 < 150

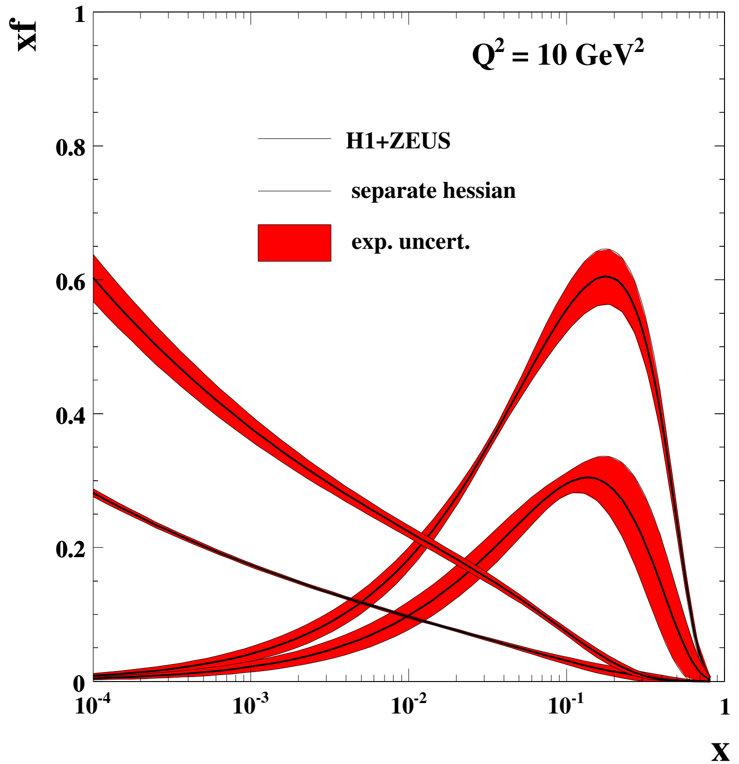


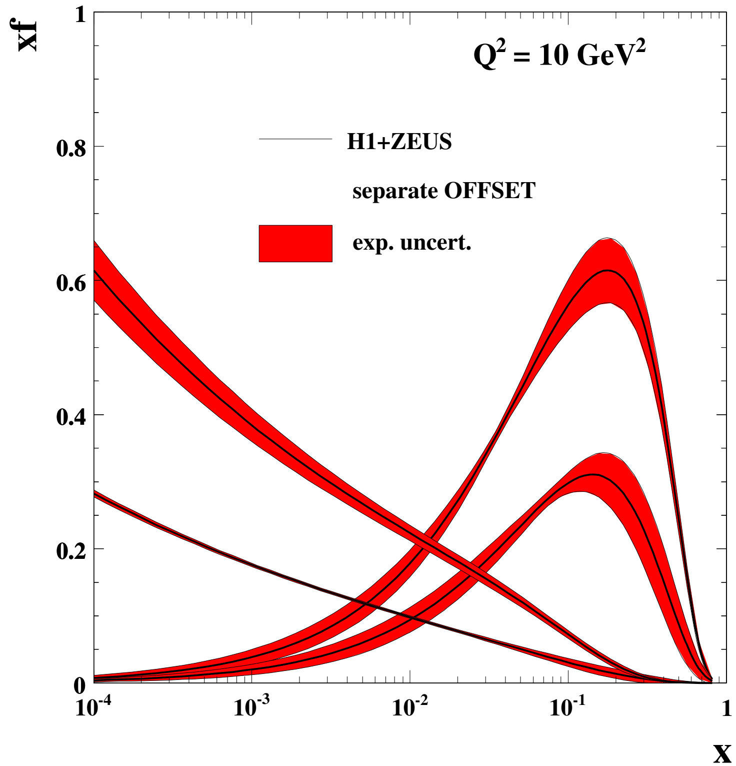

**Quadratic** **OFFSET** **HESSIAN**

When fitting to data with large correlated systematic errors the treatment of these errors in the PDF fit gives significant differences in PDF central values and PDF uncertainty estimates

NOTE the fit formalism, parametrization etc is the same as for the HERAPDF fit. The OFFSET method gives largest errors and covers the difference between the valence PDFs of the other two

For the separate ZEUS and H1 data for which correlated systematic errors are LARGE relative to the statistical and uncorrelated errors



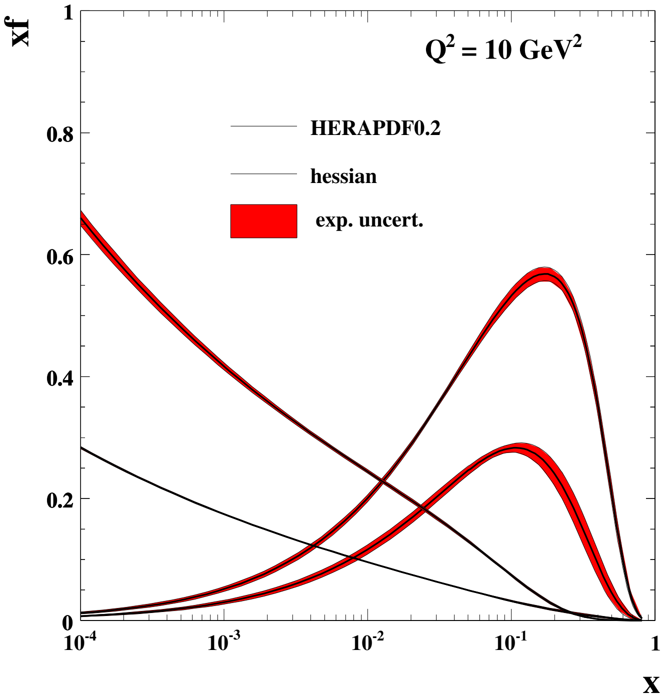




HESSIAN with increased χ2 tolerance

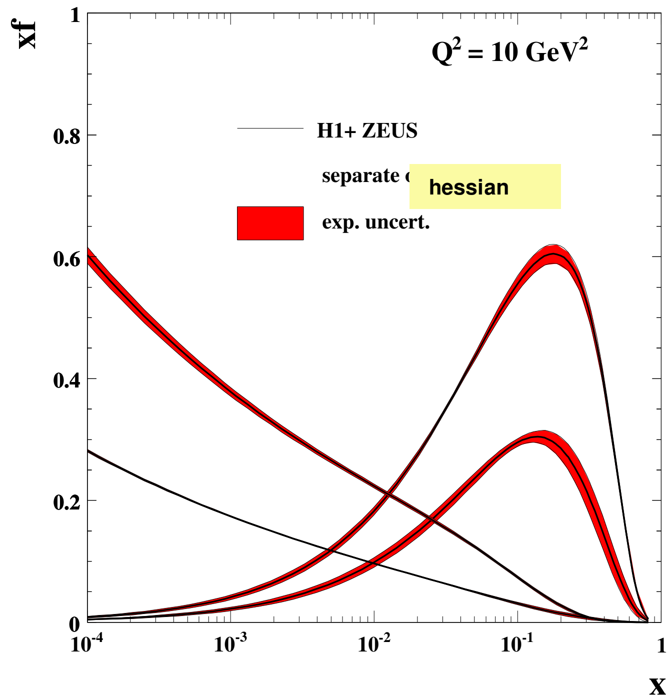
**OFFSET**

Applying an increased tolerance to the HESSIAN fit results in similar sized errors to those of the OFFSET method for Δχ2 ~ 10

Now compare Hessian fitting to the HERA combined data to Hessian fitting to separate H1 and ZEUS data sets.




**HESSIAN PDF fit to HERA combined data**




**HESSIAN PDF fit to H1 and ZEUS separately**

A comparable experimental error results BUT the shapes of the gluon and valence are significantly different. For the PDF fit to the HERA combined data the systematic error parameters of the separate data sets were already shifted in the fit which combined the data. The PDF fit does not make further significant shifts. For the separate H1 and ZEUS data the systematic error parameters of the separate data sets are shifted in the PDF fit itself.

When you put the separate data sets into a PDF fit it floats the systematic parameters of the data sets to different values than the 'theory free' combination fit. Representative examples below

| Fitted systematics | Combination fit | NLO QCD PDF fit |
|---|---|---|
| 20 zd1_e_eff | 0.2940 | 1.2284 |
| 21 zd2_e_theta_a | 0.6286 | -0.8520 |
| 22 zd3_e_theta_b | -0.0871 | -1.40265 |
| 23 zd4_e_escale | 0.4240 | -0.0090 |
| 24 zd5_had1 | 0.6210 | -0.9657 |
| 25 zd6_had2 | -0.1757 | -0.4113 |
| 26 zd7_had3 | -0.0167 | 0.6413 |
| 51 h1670e8 | 0.4860 | 0.5295 |
| 52 h1670e9 | -0.3290 | -0.0793 |
| 53 h1670e10 | 1.0718 | -0.7934 |
| 54 h1670e11 | 0.0833 | 0.8154 |
| 55 h1670e12 | -0.5428 | 1.2503 |
| 56 h1670e13 | 0.0820 | 0.0484 |
| 62 h195-00e10 | -1.1148 | 1.6024 |
| 63 h195-00e11 | -0.0917 | 0.3942 |
| 64 h195-00e12 | -0.5950 | 0.8569 |
| 65 h195-00e13 | 0.2882 | -0.2562 |
| 66 h195-00e14 | -0.1547 | -1.3632 |
| 67 h195-00e15 | -0.4395 | 0.4583 |
| 68 h195-00e16 | 0.1103 | -1.2396 |
| 69 h195-00e17 | 0.5173 | -1.8413 |

**Agreement on Systematic shift**

**BAD**

**GOOD**

**Middling**

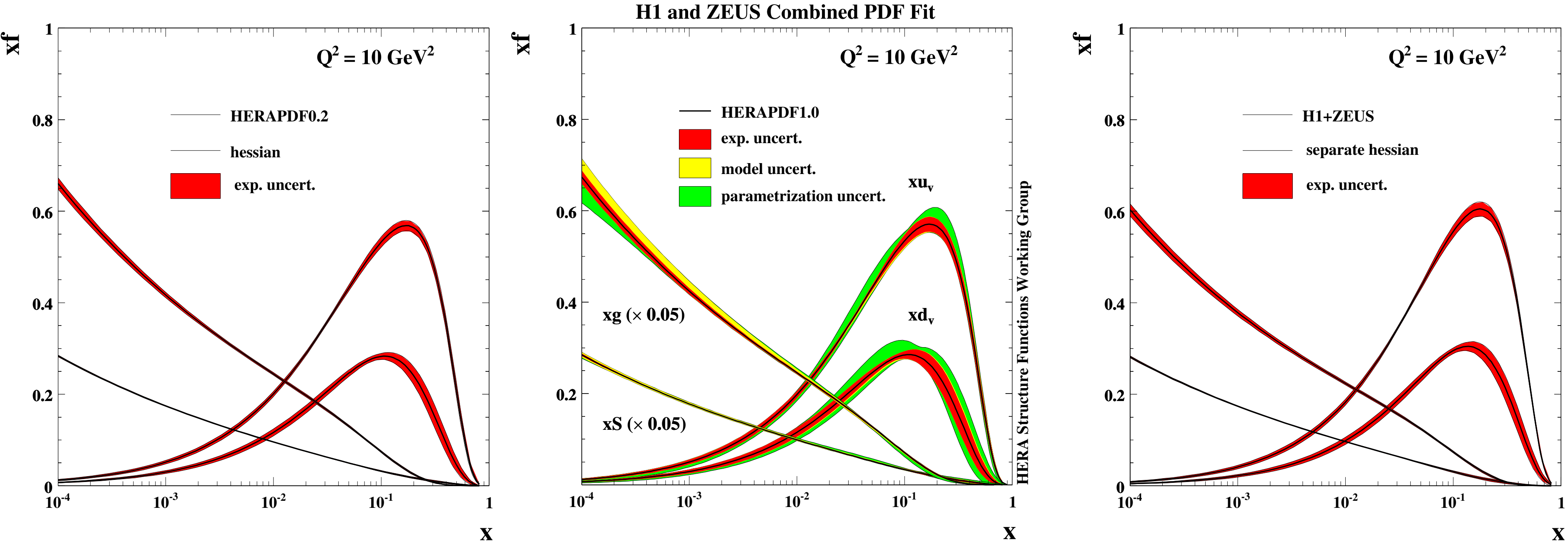
So whereas you might think that the QCD PDF fits are already doing the job of combining the HERA data- they are not getting the same answer as the HERA combination fit.

## Model dependence is also important

The statistical criterion for parameter error estimation within a particular hypothesis is $\Delta\chi2 = T^2 = 1$. But for judging the acceptability of an hypothesis the criterion is that $\chi2$ lie in the range $N \pm\sqrt{2N}$, where N is the number of degrees of freedom

There are many choices, such as the form of the parametrization at $Q^2_0$, the value of $Q_0^2$ itself, the flavour structure of the sea, etc., which might be considered as superficial changes of hypothesis, but the $\chi2$ change for these different hypotheses often exceeds $\Delta\chi2=1$, while remaining acceptably within the range $N \pm\sqrt{2N}$.

The model uncertainty on the PDFs generally exceeds the experimental uncertainty, if this has been evaluated using T=1, with the Hessian method.




Compare HESSIAN fitting to HERA combined data

To Hessian fitting to HERA combined data PLUS model and param. errors

To HESSIAN fitting to H1 and ZEUS separately

The differences between fitting the combination and fitting the separate data sets are covered by the model and param uncertainties- and would also be covered by a larger $\Delta\chi 2$ tolerance

BUT we are trying our best to get it right and to define clearly where the uncertainties come from

# extras

Fortunately, there are smart ways to do this (Pascaud and Zomer LAL-95-05)

Define matrices $M_{jk} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial p_j \partial p_k}$ $C_{j\lambda} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial p_j \partial s_\lambda}$

Then M expresses the variation of χ2 wrt the theoretical parameters, accounting for the statistical errors, and C expresses the variation of χ2 wrt theoretical parameters and systematic uncertainty parameters.

Then the covariance matrix accounting for statistical errors is $V^p = M^{-1}$ and the covariance matrix accounting for correlated systematic uncertainties is $V^{ps} = M^{-1} C C^T M^{-1}$. The total covariance matrix $V^{tot} = V^p + V^{ps}$ is used for the standard propagation of errors to any distribution F which is a function of the theoretical parameters

$$< \sigma^2_F > = T \sum_j \sum_k \frac{\partial F}{\partial p_j} V_{jk}^{tot} \frac{\partial F}{\partial p_k}$$

Where T is the χ2 tolerance, T = 1 for the OFFSET method.

This is a conservative method which gives predictions as close as possible to the central values of the published data. It does not use the full statistical power of the fit to improve the estimates of $s_\lambda$, since it chooses to distrust that systematic uncertainties are Gaussian distributed.

**Luckily there are also smart ways to do perform the Hessian method**

**CTEQ have given an analytic method** CTEQ hep-ph/0101032,hep-ph/0201195

$$\chi 2 = \sum_i \frac{[F_i^{QCD}(p) - F_i^{MEAS}]^2}{(s_i^{STAT})^2} - B\, A^{-1} B$$

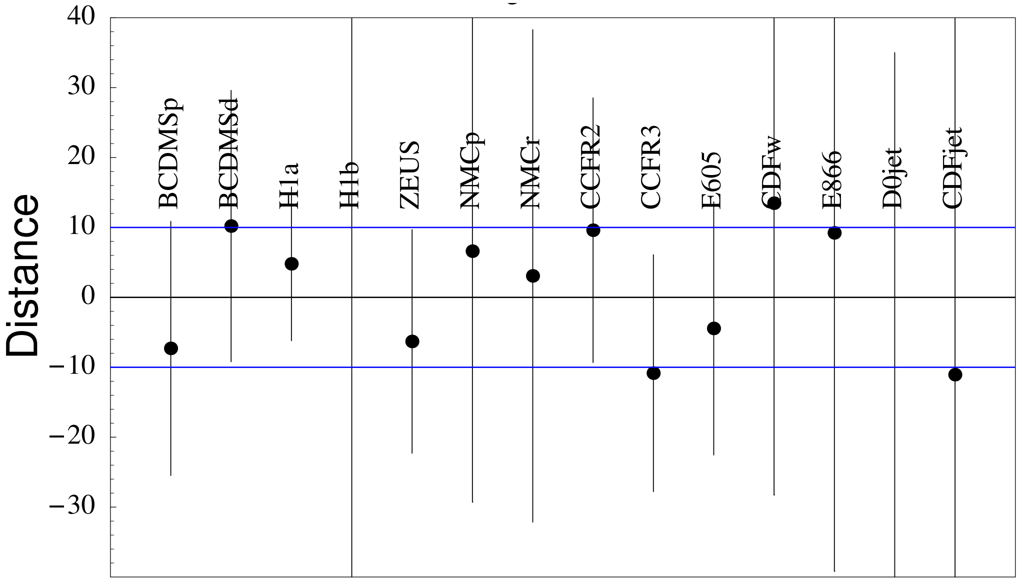
**where**

$$B_\lambda = \sum_i \Delta_{i\lambda}^{sys} \frac{[F_i^{QCD}(p) - F_i^{MEAS}]}{(s_i^{STAT})^2} \quad , A_{\lambda\mu} = \delta_{\lambda\mu} + \sum_i \frac{\Delta_{i\lambda}^{sys} \Delta_{i\mu}^{sys}}{(s_i^{STAT})^2}$$

**such that the contributions to $\chi 2$ from statistical and correlated sources can be evaluated separately.**

## WHY change the χ2 tolerance?

illustration for eigenvector-4




CTEQ6 look at eigenvector combinations of their parameters rather than the parameters themselves. They determine the 90% C.L. bounds on the distance from the global minimum from

$\int P(\chi_e^2, N_e)\, d\chi_e^2 = 0.9$

for each experiment

**This leads them to suggest a modification of the χ2 tolerance, Δχ2 = 1, with which errors are evaluated such that $\Delta\chi^2 = T^2$, T = 10.**

**Why? Pragmatism.** The size of the tolerance T is set by considering the distances from the χ2 minima of individual data sets from the global minimum for all the eigenvector combinations of the parameters of the fit.

**All of the world's data sets must be considered acceptable and compatible at some level, even if strict statistical criteria are not met, since the conditions for the application of strict statistical criteria, namely Gaussian error distributions are also not met.**

**One does not wish to lose constraints on the PDFs by dropping data sets, but the level of inconsistency between data sets must be reflected in the uncertainties on the PDFs.**