THE MYSTERY OF SPINORS

HAMILTON QUATERNIONS
(1843)

DIRAC EQUATION (1928)

PHYSICS

MATHMATICS

(2006)
Hamilton in 1842 as President of the Royal Irish Academy

(after The Dublin University Magazine, June 1842)
Brougham Bridge on the Royal Canal — Hamilton's place of enlightenment, with the modern plaque that commemorates his famous discovery of quaternions in 1843

(Drawing M. Caulfield; Plaque, Courtesy P.A. Wayman, Director, Dunsink Observatory)

there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between $i$, $j$, $k$, exactly such as I have used them ever since.

I pulled out, on the spot, a pocket-book which still exists and made an entry on which, at that very moment, I felt that it might be worth my while to expend the labour of at least ten (and it might be fifteen) years to come.²⁷

One month before his death, Hamilton again recorded the same event for the benefit of his son Archibald and likely future psychologists. The words now are different, but again the account is essentially similar:²⁸

Although (your mother) talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once an importance.

An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work . . .

Nor could I resist the impulse — unphilosophical as it may have been
Hamilton

Quaternions

Vector Analysis
Non-Commutative Algebra

Quantum Mechanics
Optics Mechanics

Cf. Einstein
COMPLEX NUMBERS

Long History ... Cauchy, Gauss

Gauss "The True Metaphysics of $\sqrt{-1}$ is Elusive"

$x + iy = r e^{i\theta}$

**Angular Phase**

**Algebra**

**Analysis, Geometry**

**2 Dim Physics**

Cauchy-Riemann Equations

Laplace Equation

**Abelian Gauge Theory**

**Abel, Riemann**

**Elliptic Functions**

**Algebraic Geometry**
Hamilton wanted to generalize to 3-dimensions for physical applications.

He found he needed 4 dimensions related to relativity (imaginary time).

Quaternions are spinors.

$\mathbb{S}^3 = U(1)$, unit quaternion 3-sphere or $SU(2)$ non-abelian phase (lie group)

Dirac operator

All key ideas of modern physics 150 years later, not disjoint from Hamiltonian theory.
SPINORS AND GEOMETRY

IN 4 DIMENSIONS

SPINORS = QUATERNIONS DIM 4

$S \otimes S = 1 + 4 + 6 + 4 + 1$
$(4 \times 4 = 16)$

DIFFERENTIAL FORMS OF ALL DEGREES

GEOMETRY LENGTH, AREA, VOLUME,

SPINORS = $\sqrt{6}GEOMETRY$

WHAT DOES THIS MEAN?

IF $R^4 = C^2$ SPINORS = $1 + 2 + 1$

COMPLEX GEOMETRY

(PENROSE Twistor Theory)
Hodge theory

Euclidean analogue of Maxwell's equations (in vacuo)

\[ d\omega = 0 \quad d^\ast \omega = 0 \quad \omega \text{ 2-form} \]

Hodge (1930-40) generalized to forms of all degrees \( p \) on Riemannian manifold \( M \)

For compact \( M \)

Solutions: harmonic \( p \)-forms

\[ = H^p(M,\mathbb{R}) \quad \text{cohomology} \]

Topological independent of metric

Interpreted by Witten (1984)

As super-symmetric quantum mechanics
DIRAC OPERATOR

ON COMPACT RIEMANNIAN MANIFOLD M

DIRAC OPERATOR (ACTING ON SPINOR FIELDS) IS ELLIPTIC AND EQUATION DS = 0 (HARMONIC SPINORS) HAS FINE DIMENSION

BUT VARIES WITH METRIC FOR \( \text{dim } M \geq 3 \) CAN BE ARBITRARY LARGE

ANY TOPOLOGICAL MEANING?

IF \( \text{dim } M = 2n \)
\( S = S^+ \oplus S^- \) (CHIRAL SPINORS)

HARMONIC SPINORS \( H = H^+ \oplus H^- \)

\( \text{dim } H^+ - \text{dim } H^- \) TOPOLOGICAL INVARIANT (INDEX D) OF M RELATED TO ANOMALIES IN QFT (1970's)
SPECTRAL FLOW

Dim M odd D self-adjoint

Spectrum D discrete unbounded

Above and below

"Dirac Sea"

Couple D with gauge field varying periodically in time

Family $D_t \ 0 \leq t \leq 1$ with $D_t$ gauge equivalent to $D_0$

Spectrum varies with $t$ but ends up same

But there can be a spectral flow or "shift"

Topological invariant of family $D_t$
IF \( \dim M = 3 \)

PHYSICAL INTERPRETATION

EXTERNAL FIELD HAS CREATED

(or annihilated particles)

IF WE FORM 4-MANIFOLD \( M \times S^1 \)

AND GLUE GAUGE-FIELD BY GIVEN EQUIVALENCE

WE GET ELLIPTIC DIRAC OPERATOR \( D \)

ON \( M \times S^1 \)

INDEX \( \Theta \) = SPECTRAL FLOW OF

\( 1 \)

FAMILY \( D_6 \)

IMAGINARY

TIME

PHYSICAL
Example - Dimension 1

\[ M = \text{circle} \]

\[ D_t = -i \frac{\partial}{\partial x} + t \quad x \mod 2\pi, \quad 0 \leq t \leq 1 \]

\[ D_1 = e^{-ix} D_0 e^{ix} \quad \text{Gauge Equivalence} \]

on \( M \times S^1 \) (torus)

\[ \Theta = \frac{\partial}{\partial t} + D_t \quad \text{acts on functions} \]

\[ f(x,t) \] periodic in \( x \) (period 2\pi)

and satisfy \( f(x, t+1) = e^{-it} f(x, t) \)

Theta-function

\[ f(x, t) = \sum_n \exp\left(-\frac{(n+\zeta)^2}{\epsilon} + i n x\right) \]

is unique solution of \( \Theta f = 0 \)

no solutions of \( \Theta^2 f = 0 \)

\[ \text{index } \Theta = 1 \]
Donaldson Theory

1983 opened up entirely new field in 4-dimensional geometry using ideas from physics.

Yang-Mills instantons

Solutions of self-duality equations $\ast F = F$

On compact Riemannian 4-dimensional manifold depends on finitely many continuous parameters (moduli).

Moduli space is manifold $M$

Depends on metric on $M$ but certain topological invariants of $M$ independent of metric.
DONALDSON INVARIANTS OF M

VERY UNEXPECTED

DEPEND ON SMOOTH STRUCTURE OF M

VERY POWERFUL

WHAT IS PHYSICAL SIGNIFICANCE?

WITTEN (1988)

EXPLAINED DONALDSON THEORY AS A PURELY TOPOLOGICAL QFT

N = 2 SUPER-SYMMETRIC (TWISTED) YANG-MILLS QFT IN 4 DIMENSIONS
Seiberg-Witten Theory is a top. QFT $U(1)$-Gauge Theory coupled to spinors equivalent (dual to) Donaldson Theory simpler - easier to use

Note: no spinors in Donaldson Theory

Mysterious
Role of Spinors?

- Describe Matter
- Explain Positivity
- Related to Mass
- Cosmological Constant?
Quaternions, according to Tait, freed the mathematical physicist from the artificial slavery of coordinates and allowed his thoughts to run in their most natural channels. In a marvelous display of Scottish industrialism and Victorian imperialism he compared coordinates to a steam-hammer, which an expert may employ on any destructive or constructive work of one general kind, say the cracking of an egg-shell, or the welding of an anchor. But you must have your expert to manage it, for without him it is useless. He has to toil amid the heat, smoke, grime, grease, and perpetual din of the suffocating engine-room. The work has to be brought to the hammer, for it cannot usually be taken to its work. . . . Quaternions, on the other hand, are like the elephant's trunk, ready at any moment for anything, be it to pick up a crumb or a field gun, to strangle a tiger or to uproot a tree. Portable in the extreme, applicable anywhere . . . directed by a little native who requires no special skill or training, and who can be transferred from one elephant to another without much hesitation. Surely this, which adapts itself to its work, is the grander instrument! But then, it is the natural, the other the artificial, one. 64

It can be assumed that Tait knew little about elephants or natives, but his feelings about the superiority of quaternions over coordinates are obvious enough.
Quaternions & Physics

Hamilton 1846

introduced the operator

\[ \nabla = \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \]

noted the identity

\[ -\nabla^2 = \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 + \left( \frac{\partial}{\partial z} \right)^2 \]

and remarked that

"Applications to Analytical Physics must be extensive to a high degree."