EXCITATION OF PLASMA WAKEFIELDS

Basics: definition of laser peak intensity

- Consider a EM field as a plane wave

$$\mathbf{E_L}(z,t) = \frac{E_0}{2}\exp[-i(k_0 z - \omega_0 t)]\vec{\mathbf{e}}_\mathbf{x} + c.c.$$

- Peak intensity is defined as

$$I_0 = \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{c\varepsilon_0}{2} E_0^2$$

- For a Gaussian pulse (at focus)

$$I(r,t) = I_0 \exp\left[-2\frac{r^2}{w_0^2}\right]\exp\left[-4\ln 2\frac{t^2}{\tau_0^2}\right] \Rightarrow I_0 = \frac{2E}{\pi w_0^2 \tau_0}$$

- Example: E=1 J, w_0=20 µm, τ_0=30 fs
  $I_0 = 5\times10^{18}$ W/cm^2

a: the normalized vector potential

- Laser E field linked to potential vector a by $$E_L = -\frac{\partial A}{\partial t}$$
- Normalized vector potential $$a = \frac{eA}{m_e c}$$

$$a_0 = -\frac{eE_0}{m_e c \omega_0}$$

- In practical units $$a_0 = 8.5 \times 10^{-10} \lambda[\mu m] I_0^{1/2} [W/cm^2]$$
- Example: $I_0 = 2\times10^{18}$ W/cm^2 (λ=1µm)→ a=1.2

Relativistic regime of laser-plasma interaction

- Electron in laser field:

$$\frac{d\mathbf{p}}{dt} = -\frac{e}{m_e}(\mathbf{E_L} + \mathbf{v} \times \mathbf{B_L})$$

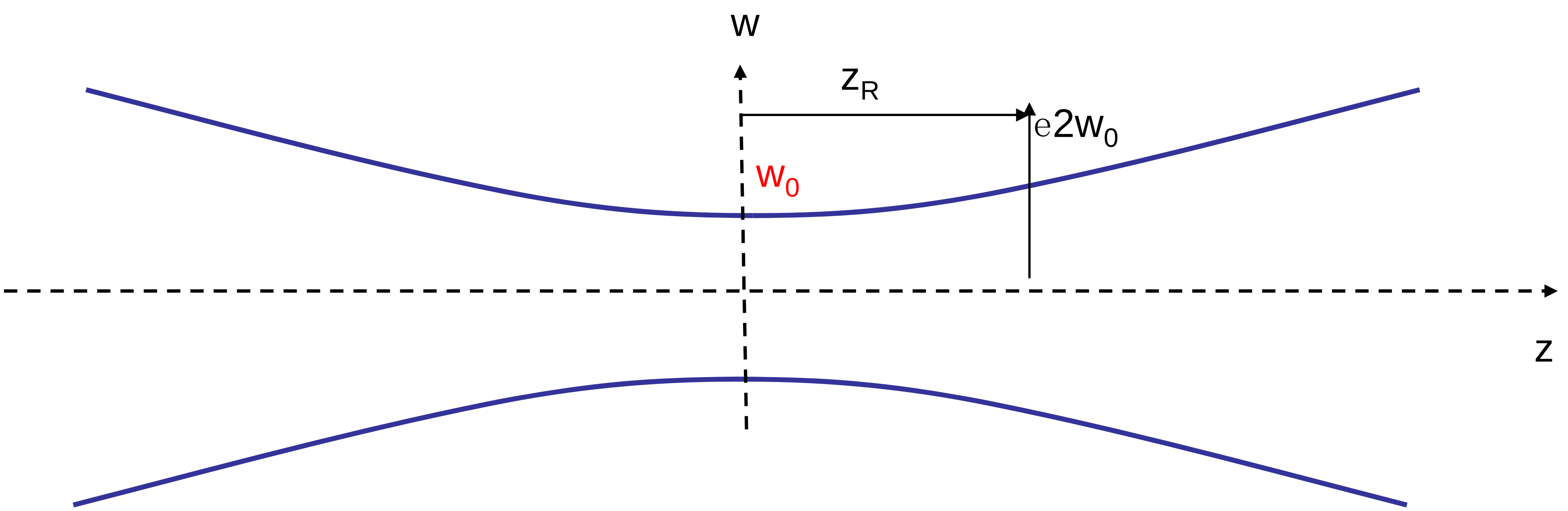
- Weakly relativistic case v/c <<1 (magnetic component is neglected)

$$\frac{dv_{osc}}{dt} = -e\frac{E_L}{m_e} = \frac{e}{m_e}\frac{\partial A}{\partial t}$$

$$\Rightarrow \quad \frac{v_{osc}}{c} = a$$

Relativistic regime is entered when a ~ 1 ($I_0 \sim 10^{18}$ W/cm^2)

Basics: gaussian beam propagation



$$w(z) = w_0\left(1 + z^2/z_R^2\right)^{1/2}$$

Rayleigh length: from I_0 to $I_0/2$

$$z_R = \pi w_0^2/\lambda_0$$

Example: λ=1 µm, w_0=20 µm → z_R=1.2 mm

Fluid model: hypothesis (1)

- Plasma ions are supposed to be immobile. This is justified when the typical time for ion motion (ω_{pi}^{-1}) is large compared to the driver pulse duration ($\tau \ll \omega_{pi}^{-1}$).
- The plasma is represented by an electron fluid. This fluid is described by macroscopic quantities such as its density $n(\mathbf{r}, t)$, its velocity $\mathbf{v}(\mathbf{r}, t)$. Let us note that in such a model, kinetic effects (trapping, wavebreaking) are not taken into account.
- The electron fluid is cold. In the case of a laser driver this is justified when the quiver velocity of electrons in the laser field is orders of magnitude larger than the thermal velocity: $v_{osc} \simeq eE_{laser}/(m_e\omega_0) \gg v_{th} = (k_B T_e/m_e)^{1/2}$. In the case of a particle beam driver, this is true when the beam causes plasma electrons to move at velocities greater than v_{th}.

Fluid model: hypothesis (2)

- We consider the weakly relativistic case, also called the linear regime, in which plasma electrons are not relativistic. This implies that the laser intensity is sufficiently low $a_0^2 \ll 1$, that the beam density is sufficiently low $n_b \ll n_0$ and in consequence that the wakefield amplitude is low $\delta n/n_0 \ll 1$.
- The plasma is strongly underdense $\omega_p/\omega_0 \ll 1$. Here ω_0 is the laser frequency and ω_p is the plasma frequency.
- We will assume here that the problem has cylindrical symmetry.

Definition of drivers

- Laser driver

$$\mathbf{a} = \hat{a}(r, z, t)\cos(k_0 z - \omega_0 t)\mathbf{e_x}$$

  - envelope

$$\hat{a}^2(r, \zeta) = a_0^2 \exp(-\zeta^2/L_0^2)\exp(-r^2/\sigma^2)$$

$$\zeta = z - v_g t$$

- Electron beam driver

$$n_b(r, \zeta') = n_{b0} \exp(-\zeta'^2/L_b^2)\exp(-r^2/\sigma_b^2)$$

$$\zeta' = z - v_b t$$

Poisson equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$ with $$\rho = -e(n - n_0) + q n_b$$ (electron beam driver: q=-e)

In addition, the fields can be written in terms of potentials:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (5)$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad (6)$$

We now choose the Coulomb gauge, so that $\nabla \cdot \mathbf{A} = 0$. In this gauge, the Poisson equation becomes

$$\nabla^2\Phi = \frac{e}{\epsilon_0}(n + n_b - n_0)$$

Rewrite as: $$\nabla^2\Phi = \frac{e n_0}{\epsilon_0}\left(\frac{\delta n}{n_0} + \frac{n_b}{n_0}\right)$$

Physical meaning of potential

So that Φ here has a clear physical meaning: it represents the potential in the plasma due to charge separation. In this gauge, the vector potential A represents the laser field. In this case, the electric field $\mathbf{E}$ has two components: the laser component $\mathbf{E_L} = -\partial \mathbf{A}/\partial t$ which is a high frequency field ω_0 and the plasma field $\mathbf{E_p} = -\nabla \Phi$, which is at ω_p.

Fluid equations

We start with the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

We also have the equation of motion for fluid electrons

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{e}{m_e}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

which can be rewritten as:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{e}{m_e}(\mathbf{E_L} + \mathbf{v} \times \mathbf{B_L} - \nabla \Phi)$$

Ponderomotive force

We will neglect the plasma field for now in order to emphasize the ponderomotive force. So we will assume temporarily that the electrons only witness the laser field:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{e}{m_e}(\mathbf{E_L} + \mathbf{v} \times \mathbf{B_L}) \qquad (11)$$

Some algebra

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{e}{m_e}\mathbf{E_L} - c^2\nabla\frac{a^2}{2}$$

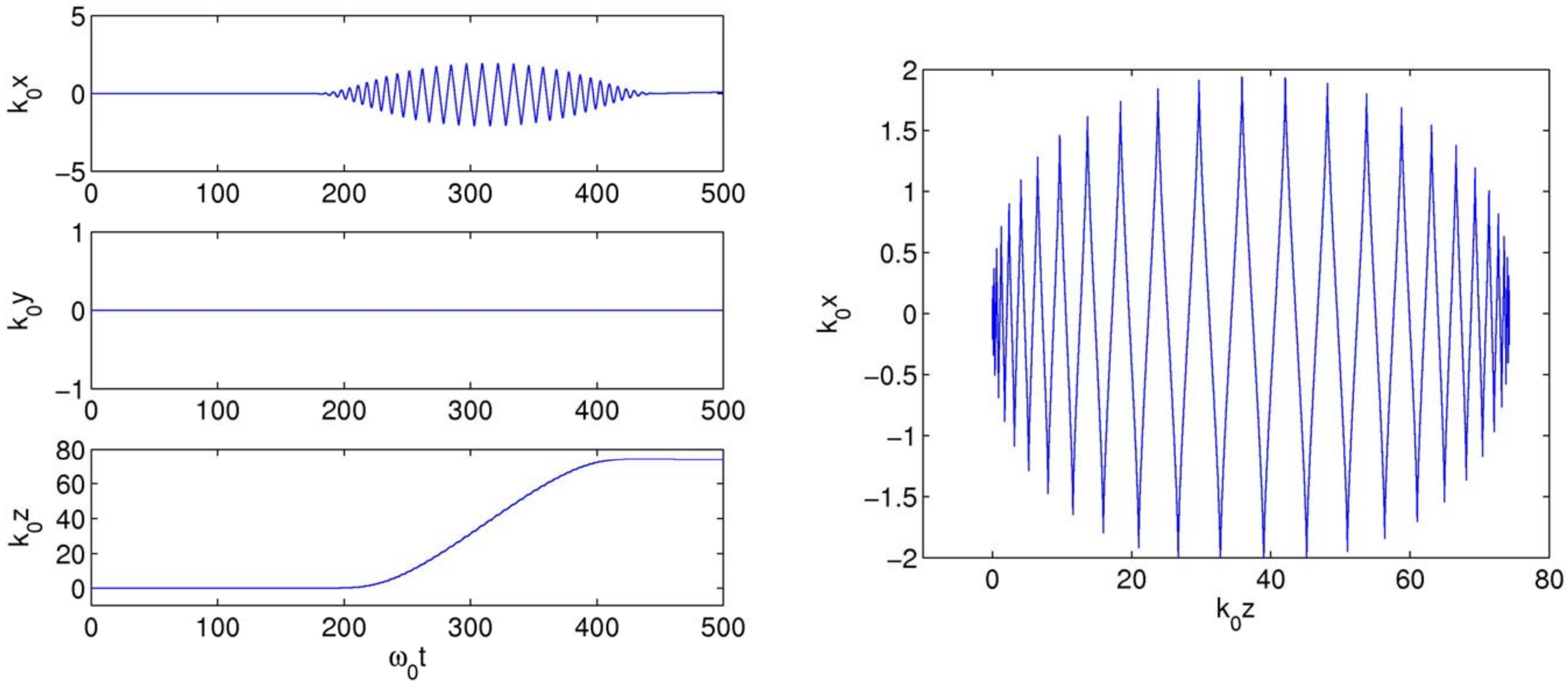
Motion of electrons in plasma after averaging over fast oscillations

$$\frac{\partial \mathbf{v}}{\partial t} = -c^2\nabla\frac{\hat{a}^2}{4} + \frac{e}{m_e}\nabla\Phi$$

Illustration of ponderomotive force

- Solving the motion of an electron in a laser field (propagating along z and polarized along x): $\frac{d\mathbf{p}}{dt} = -\frac{e}{m_e}(\mathbf{E_L} + \mathbf{v} \times \mathbf{B_L})$

a_0=2



Plasma wakefield equation

We now linearize the continuity equation (eq. 8) in order to get the following system of equations:

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \mathbf{v} = 0 \tag{14}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -c^2 \nabla \frac{\hat{a}^2}{4} + \frac{e}{m_e} \nabla \Phi \tag{15}$$

$$\nabla^2 \Phi = \frac{e n_0}{\epsilon_0} \left(\frac{\delta n}{n_0} + \frac{n_b}{n_0} \right) \tag{16}$$

⇩ Some algebra

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \frac{\delta n}{n_0} = c^2 \nabla^2 \frac{\hat{a}^2}{4} - \omega_p^2 \frac{n_b}{n_0}$$

Laser driver case

- Assume no electron beam $n_b=0$

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\frac{\delta n}{n_0}=c^2\nabla^2\frac{\hat{a}^2}{4}$$

- Write equation on potential using:

$$\phi = e\Phi/m_e c^2$$
$$\nabla^2\phi = k_p^2\delta n/n_0$$

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\phi=\omega_p^2\frac{\hat{a}^2}{4}$$

Moving window

The laser driver moves at the velocity of light (a is a function of $\zeta = z - v_g t$), so it is practical to change variables in order to follow the laser pulse: $\tau = t$ and $\zeta = z - v_g t$. For the new variables:

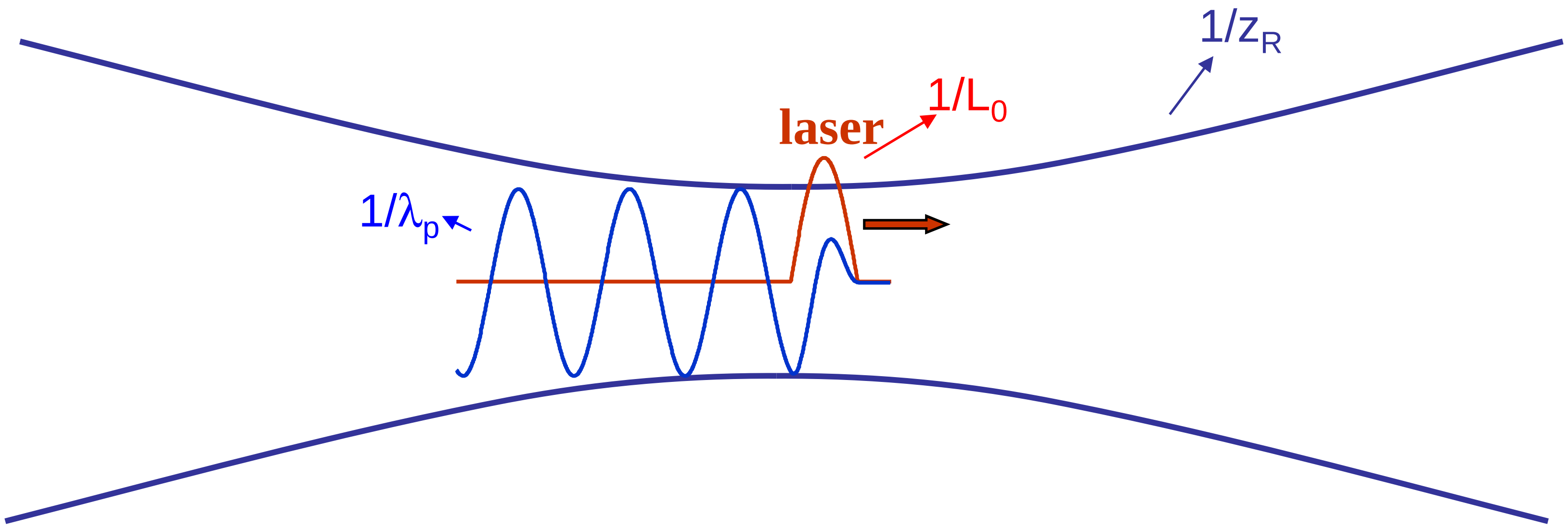
$$
\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau} - v_g \frac{\partial}{\partial \zeta} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial \zeta} \\
\frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial \tau^2} + v_g^2 \frac{\partial^2}{\partial \zeta^2} - 2 v_g \frac{\partial^2}{\partial \zeta \partial \tau} \\
\frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial \zeta^2}
\end{aligned}
$$

Considering laser driver only

$$
\left(\frac{\partial^2}{\partial \tau^2} + v_g^2 \frac{\partial^2}{\partial \zeta^2} - 2 v_g \frac{\partial^2}{\partial \tau \partial \zeta} + \omega_p^2 \right) \phi = \omega_p^2 \frac{\hat{a}^2}{4}
$$

Quasi-static approximation

- Neglect derivatives in τ compared to derivatives in ζ
- Physical meaning: the plasma responds adiabatically to slow changes of the driver



$$\left(\frac{\partial^2}{\partial\zeta^2} + k_p^2\right)\phi = k_p^2\frac{\hat{a}^2}{4}$$

Solutions

- Solution behind the pulse (gaussian shape)
  - Potential

$$\phi = -\sqrt{\pi}a_0^2\frac{k_pL_0}{4}e^{-k_p^2L_0^2/4}e^{-r^2/\sigma^2}\sin(k_p\zeta)$$

  - Electric field

$$\frac{\mathbf{E}}{E_0} = -\frac{1}{k_p}\nabla\phi = -\frac{1}{k_p}\left(\mathbf{e_z}\frac{\partial}{\partial\zeta} + \mathbf{e_r}\frac{\partial}{\partial r}\right)\phi$$

  - Longitudinal

$$\frac{E_z}{E_0} = \sqrt{\pi}a_0^2\frac{k_pL_0}{4}e^{-k_p^2L_0^2/4}e^{-r^2/\sigma^2}\cos(k_p\zeta)$$

  - Transverse

$$\frac{E_r}{E_0} = -\sqrt{\pi}\frac{a_0^2}{2}e^{-k_p^2L_0^2/4}\frac{L_0r}{\sigma^2}e^{-r^2/\sigma^2}\sin(k_p\zeta)$$

  - E_0 wavebreaking field

$$E_0 = \frac{m_e c\omega_p}{e} \propto n^{1/2}$$

Solutions

- Using Poisson equation, compute density perturbation
  - Longitudinal $$\frac{\delta n_z}{n_0} = \sqrt{\pi} a_0^2 \frac{k_p L_0}{4} e^{-k_p^2 L_0^2/4} e^{-r^2/\sigma^2} \sin(k_p \zeta)$$
  - Transverse $$\frac{\delta n_r}{n_e} = \frac{\delta n_z}{n_0} \frac{4}{\sigma^2 k_p^2} \left(1 - \frac{r^2}{\sigma^2}\right)$$

ϕ, E_z/E_0, $\delta n_z/n_0$ are *normalized* quantities and have the same amplitude

$$\phi \simeq \sqrt{\pi} a_0^2 \frac{k_p L_0}{4} e^{-k_p^2 L_0^2/4}$$

Resonance condition

- Find optimimum pulse duration for plasma wave excitation

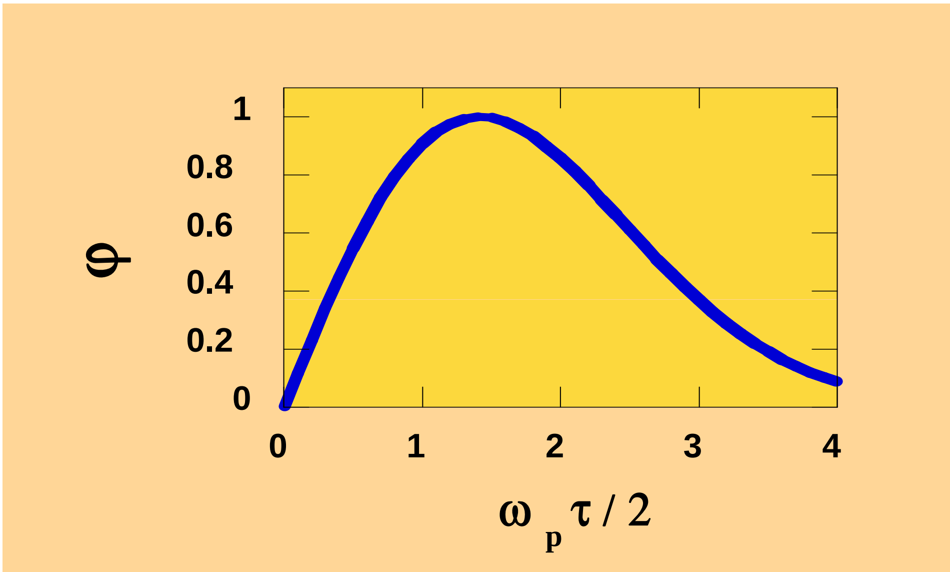
$$\partial\phi / \partial L_0 = 0$$

- Broad resonance condition:

$$k_p L_0 = \sqrt{2}$$

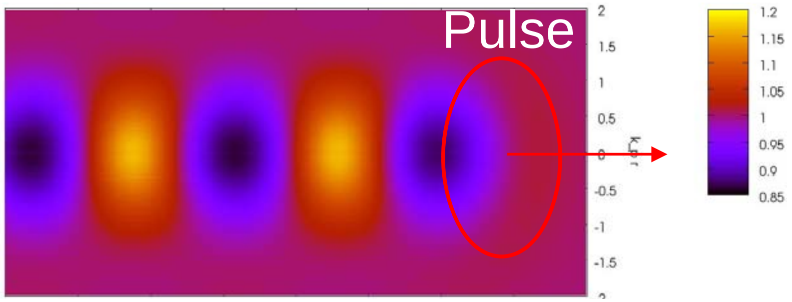
- Practical units

$$n_{er}(\mathrm{cm}^{-3}) = \frac{1.7 \times 10^{21}}{\tau_{fwhm}^2(\mathrm{fs})}$$

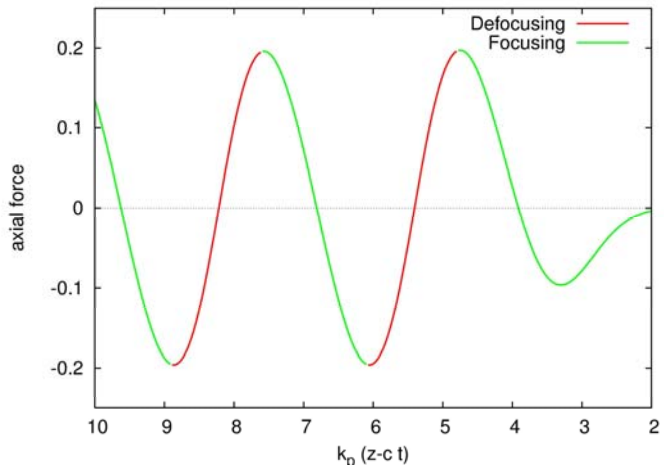


Accelerating and focusing fields

a=0.5



Electron density



1D nonlinear theory (fully relativistic)

- Basic equation: $$\frac{\partial^2\phi}{\partial\zeta^2} = \frac{k_p^2}{2}\left[\frac{1+a^2}{(1+\phi)^2} - 1\right]$$

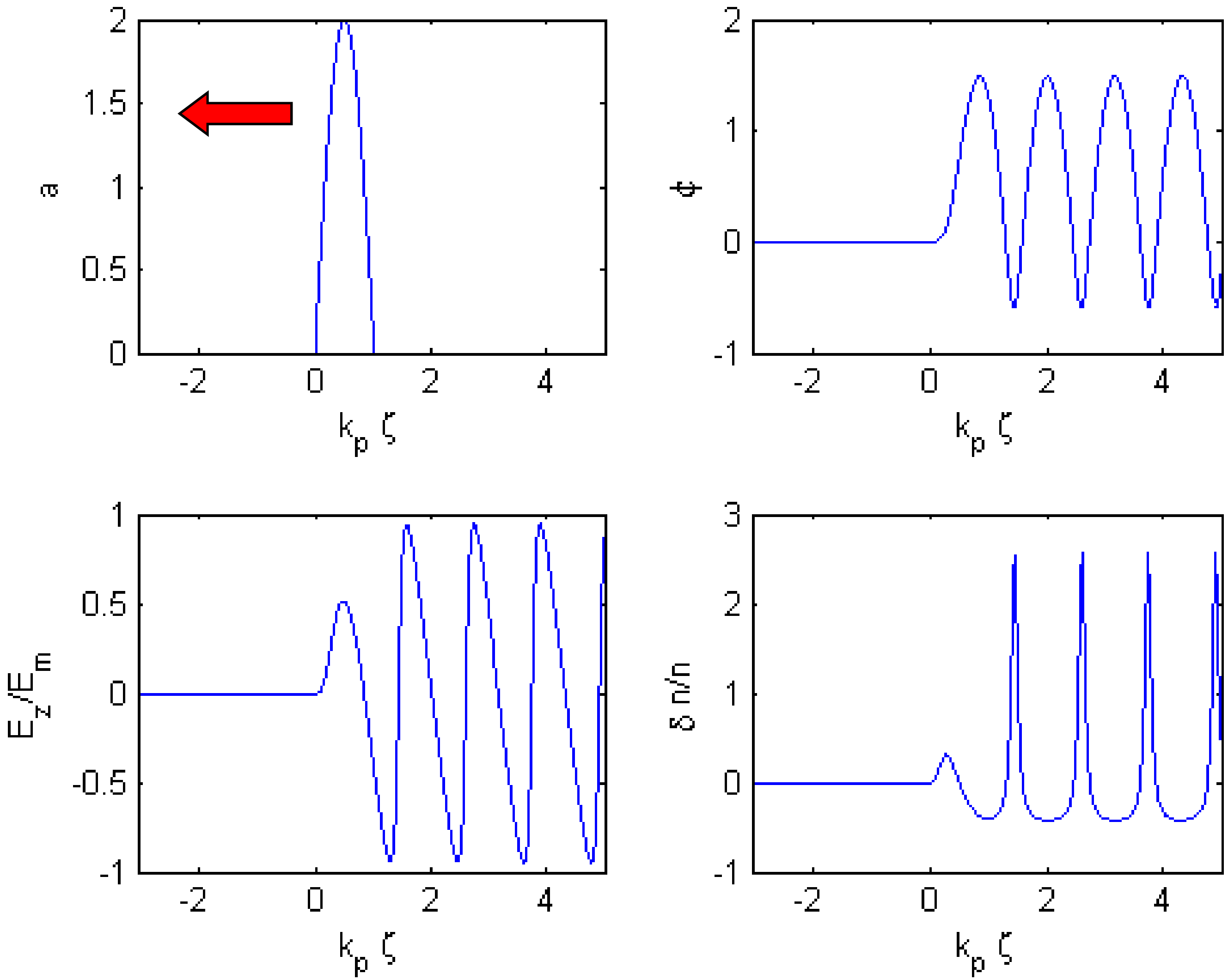
- No limit on a (until wavebreaking…)

- Wavebreaking field: $$E_{WB}^{1D} = \sqrt{2(\gamma_p - 1)}E_0$$

- Fields higher than E_0 are possible

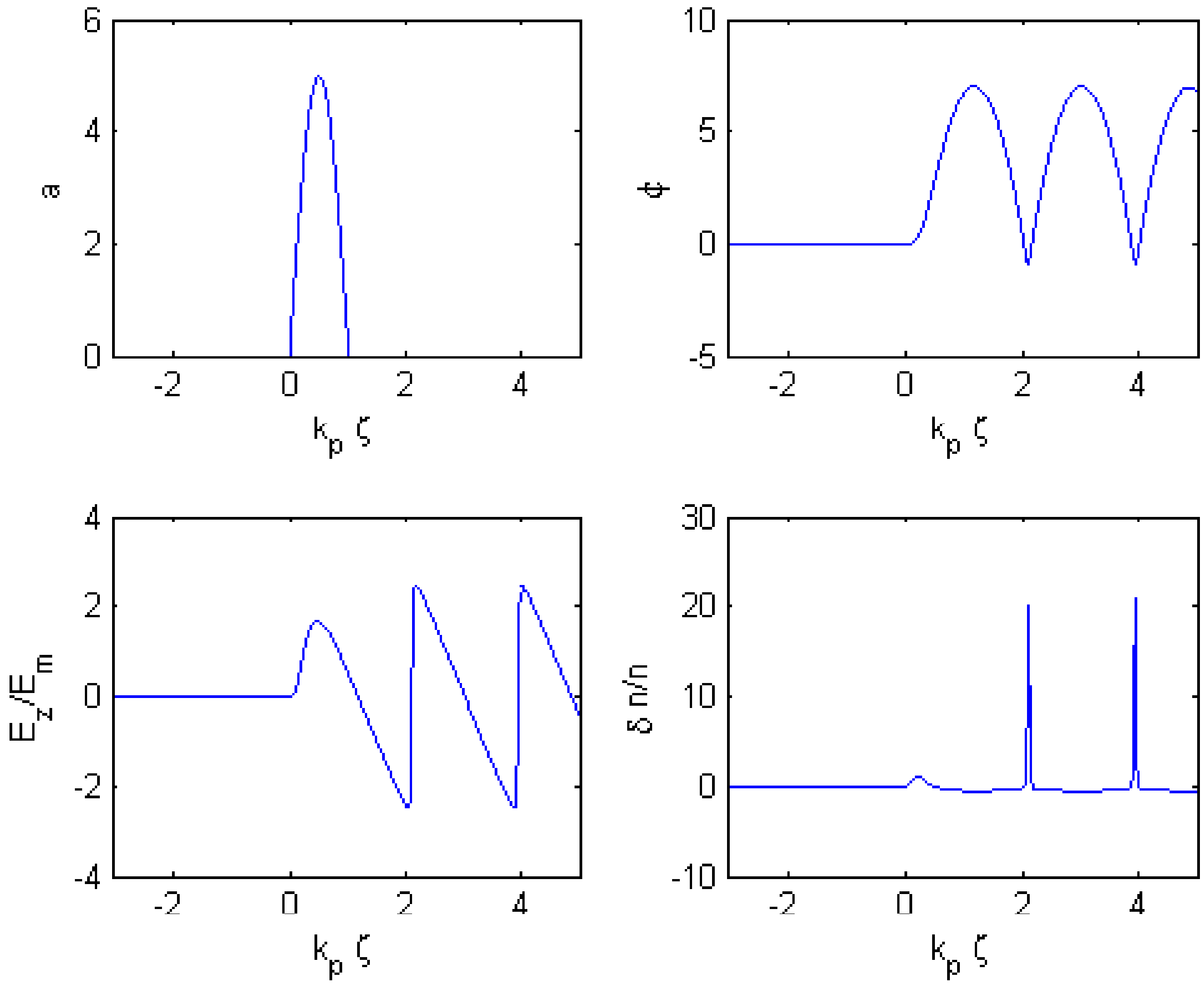
1D nonlinear plasma waves

30 fs pulse
a=2



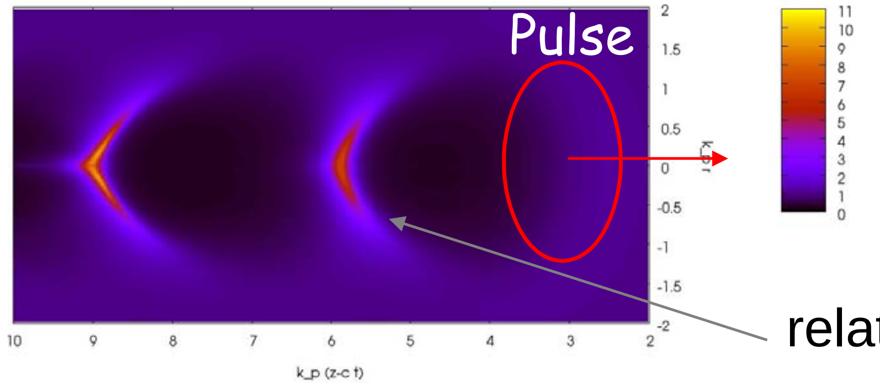
Limit of 1D nonlinear fluid theory: wave breaking

30 fs pulse
a=5



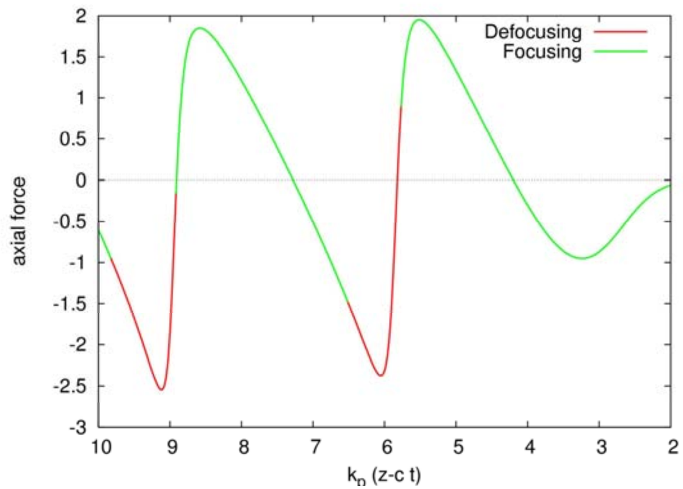
3D nonlinear wakefields

$a_0=2$



Electron density

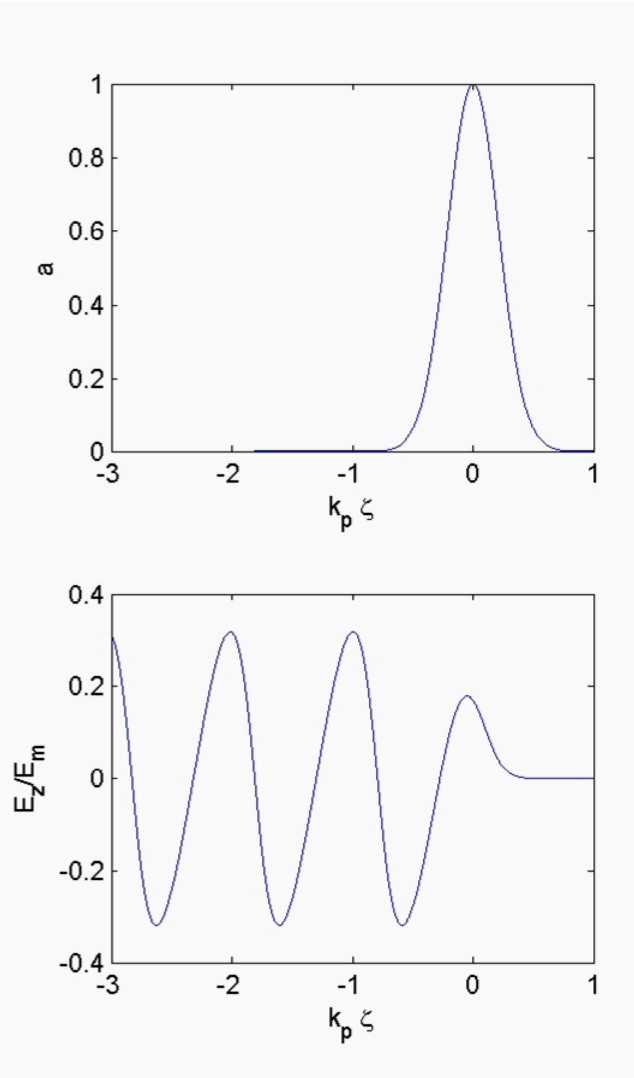
relativistic shift of ω_p



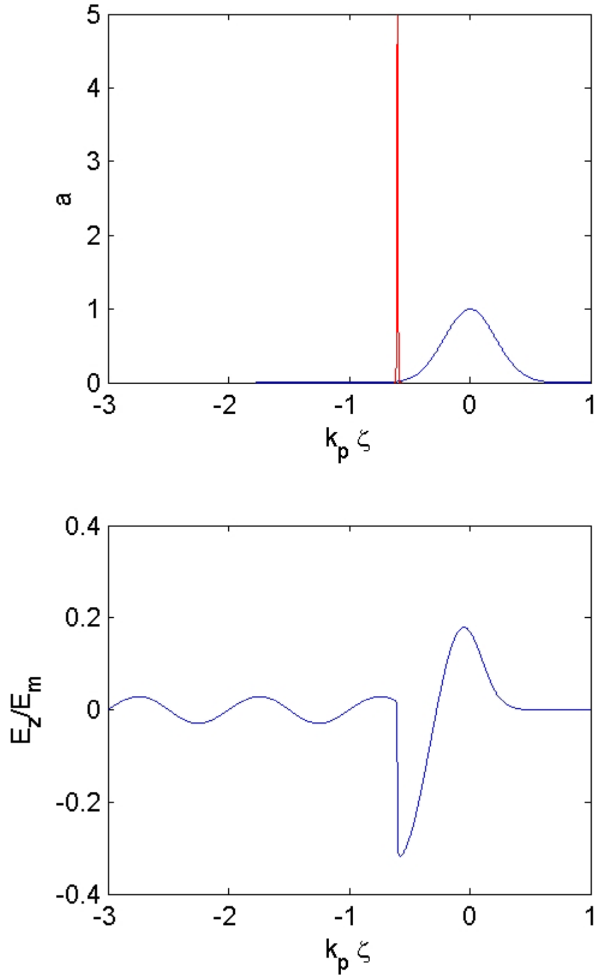
➡ Requires more complex models (fluid or kinetic)
+ computer simulations

Beam loading considerations

No particle beam



with particle beam



Charge limit

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\frac{\delta n}{n_0}=c^2\nabla^2\frac{\hat{a}^2}{4}-\omega_p^2\frac{n_b}{n_0}$$

- Limit on beam density: $n_b \sim n_0 \hat{a}_2$
- Limit on charge: $N_{part} = V_{beam} \times n_0 \hat{a}^2$
- Typical example: bunch 5 µm × 5 µm × 10 µm, â=1, $n_0=10^{19}$ cm^{-3}

  → Q=400 pC

Summary

- Intensity and a:

$$a_0 = 8.5\times 10^{-10}\,\lambda[\mu m] I_0^{1/2}[W/cm^2]$$

- Wakefield amplitude: *proportional to laser intensity*

$$\phi \simeq \sqrt{\pi} a_0^2 \frac{k_p L_0}{4} e^{-k_p^2 L_0^2/4}$$

- Wakefield max at resonance

$$n_{er}(\mathrm{cm}^{-3}) = \frac{1.7\times 10^{21}}{\tau_{fwhm}^2(\mathrm{fs})}$$

- Linear regime: sinusoidal field + radial fields
- Nonlinear regime: focusing phase is longer
- Charge limited by beam loading: $N_{part} = V_{beam} \times n_0 \hat{a}^2$
- Scaling law with particle beam, replace $\hat{a}^2/4$ by n_b/n_0