2+1+1 Polyakov Linear-Sigma Model at finite Temperature and Density

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Contents

• A short introduction to sigma model

• Symmetries of $U(N_f)_r \times U(N_f)_\ell$ linear-sigma model

• $SU(2)$, $SU(3)$, and $SU(4)$:
  • Generation of mesons
  • Mean-field approximation
  • Pure meson potential

• Polyakov-loop potentials

• Results on $SU(4)$ Chiral Condensates and deconfinement order-parameters in finite Temperature and Densities

• Summary and Outlook

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Sigma Model

Sigma model was introduced by Gell-Mann and Levy in 1960.

The name \( \sigma \text{-model} \) comes from a field corresponding to the spinless meson scalar \( \sigma \) introduced earlier by Schwinger.

Sigma-Model is a physical system with the Lagrangian

\[
\mathcal{L}(\phi_1, \phi_2, \ldots, \phi_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} \, d\phi_i \wedge *d\phi_j
\]

fields \( \phi_i \) represent map from a base manifold space-time (worldsheet) to a target (Riemannian) manifold of the scalars linked together by internal symmetries, the scalars \( g_{ij} \) determines linear and non-linear properties.

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Importance of Linear-Sigma Model

• LSM is one of lattice QCD alternatives
• It doesn’t need supercomputers like lattice QCD, you can use suitable PC and easily computational techniques.
• Various symmetry-breaking scenarios can be investigated in a more easy way, for instance,

Various properties of strongly interacting matter can be studied
- QCD Equation of State
- Chiral phase structure of masses
- QGP transport properties
- ….etc.
**LSM (axial)vector Symmetries**

**Chiral symmetry** of vector field under unitary transformation

$$\theta^a$$ corresponding to rotational angle and $$T_{ij}^a$$ being a matrix generating transformation.

- **Vector transformation**

  $$\Psi \rightarrow e^{-i\frac{\gamma^5}{2}\theta} \Psi \approx (1 - i\frac{\gamma^5}{2}\theta) \Psi$$
  $$\bar{\Psi} \rightarrow e^{+i\frac{\gamma^5}{2}\theta} \bar{\Psi} \approx (1 + i\frac{\gamma^5}{2}\theta) \bar{\Psi}$$

For fermions: Dirac Lagrangian describing free Fermion particle of mass m is given by

$$L_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m^2)\Psi$$

Under **vector** transformation, the Lagrangian is invariant.

But for **axial-vector** transformation,

$$\Lambda_A : \quad m \bar{\Psi} \psi \rightarrow e^{-i\gamma^5 \frac{\gamma^5}{2}\theta} m \bar{\Psi} \psi \approx (1 - i\gamma^5 \frac{\gamma^5}{2}\theta) m \bar{\Psi} \psi,$$

$$= m \bar{\Psi} \psi - 2im\bar{\theta}(\bar{\Psi}\gamma^5 \frac{\gamma^5}{2}\psi)$$
**LSM chiral Symmetries**

Lagrangian of massless fermions is invariant under chiral transformations BUT massive one make an spontaneous symmetry breaking.

**Spontaneous symmetry breaking**

if Lagrangian symmetry is not realized in the ground state

Minimum energy configuration is given as shown in the potential energy

Minimum energy (density) is given by an any point on the circle (1,1)

Ground state is right in the middle (0,0) and the potential plus ground state are still invariant under rotations

Ground state is away from the center. The point at the center is a local maximum of the potential and thus unstable

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**LSM (pseudo)scalar Symmetries**

**(scalar Meson)**

*Sigma like state* \( J^p = 0^+ \)

\[
\sigma = \bar{\psi}\psi
\]

**(pseudoscalar Meson)**

*Pion like state* \( J^p = 0^- \)

\[
\pi = i\bar{\psi}\gamma_5\psi
\]

Gell-Mann & Levy obtained the invariant form if the squares of two states are summed

\[
\Lambda_V: \quad \pi^2 \rightarrow \pi^2 \\
\sigma^2 \rightarrow \sigma^2
\]

\[
\Lambda_A: \quad \pi^2 \rightarrow \pi^2 + 2\sigma\theta\pi \\
\sigma^2 \rightarrow \sigma^2 - 2\sigma\theta\pi
\]

\[
(L_V \cdot L_A) \frac{\Lambda_V \cdot \Lambda_A}{(\pi^2 + \sigma^2)} \quad L_V = i\bar{\psi}\gamma_\mu\partial_\mu\psi + \frac{1}{2}(\partial_\mu\pi\partial^\mu\pi + \partial_\mu\sigma\partial^\mu\sigma) - g_\pi \left[ (i\bar{\psi}\gamma_5\pi)\pi + (i\bar{\psi}\sigma)\sigma \right] - \frac{\lambda}{4} \left( (\pi^2 + \sigma^2) - f_\pi^2 \right)
\]

**K. E** Of nucleons

**K. E** Of Mesons

**Interaction term between nucleons and the mesons**

**Pion nucleon Potential**

**Nucleon mass term**

Chiral Lagrangian of $U(N_f)^{r} \times U(N_f)^{\ell}$ linear-sigma model for any $N_f$ flavors

Fermionic Part

$$\mathcal{L}_q = \sum_f \overline{\psi}_f (i \gamma^\mu D_\mu - g T_a (\sigma_a + i \gamma_5 \pi_a)) \psi_f$$

Mesonic Part

$$\mathcal{L}_m = \text{Tr} (\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi) - \lambda_1 [\text{Tr} (\Phi^\dagger \Phi)]^2 + \lambda_2 \text{Tr} (\Phi^\dagger \Phi)^2 + c [\text{Det} (\Phi) + \text{Det} (\Phi^\dagger)]$$

$$+ \text{Tr} [H (\Phi + \Phi^\dagger)] ,$$

where $\sigma$ is a complex $N_f \times N_f$ matrix parameterizing the scalar and pseudoscalar mesons,

$$\sigma = \overline{\psi} \psi$$

where $\sigma_a$ are the scalar ($J^P = 0^+$) fields and $\pi_a$ are the pseudoscalar ($J^P = 0^-$) fields.

The $N_f \times N_f$ matrix $H$ breaks the symmetry, explicitly

$$H = T_a h_a$$

where $h_a$ are nine external fields and $T_a = \hat{\lambda}_a / 2$ are generators of $U(3)$ with $\hat{\lambda}_a$ are Gell-Mann matrices.

The $T_a$ are normalized such that $\text{Tr}(T_a T_b) = \delta_{ab}/2$ and obey the $U(3)$

$$[T_a, T_b] = i f_{abc} T_c ,$$

$$\{T_a, T_b\} = d_{abc} T_c ,$$

where $f_{abc}$ and $d_{abc}$ are the standard antisymmetric and symmetric structure constants of $SU(N_f)$, respectively, and

$$a, b, c = 1, \ldots, N_f^2 - 1$$
Physical scalar and pseudoscalar meson fields

The complex $N_f \times N_f$ matrix reads

$$\Phi = T_a \phi_a = T_a (\sigma_a + i\pi_0 a)$$

By using Pauli and Gell-mann Matrices, we find

For $N_f = 2$,

$$T_a \sigma_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \sigma_0 + \frac{1}{\sqrt{2}} a_0^0 & a_0^+ \\ -\frac{1}{\sqrt{2}} \sigma_0 - \frac{1}{\sqrt{2}} a_0^0 & \frac{1}{\sqrt{2}} a_0^+ \end{pmatrix}, \quad T_a \pi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \pi_0 + \frac{1}{\sqrt{2}} \pi^0 & \pi^+ \\ -\frac{1}{\sqrt{2}} \pi_0 - \frac{1}{\sqrt{2}} \pi^0 & \frac{1}{\sqrt{2}} \pi^+ \end{pmatrix}.$$

For $N_f = 3$,

$$T_a \sigma_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} a_0^0 + \frac{1}{\sqrt{6}} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 & a_0^+ \\ -\frac{1}{\sqrt{2}} a_0^0 + \frac{1}{\sqrt{6}} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 & \frac{1}{2} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 \\ -\frac{\sqrt{2}}{\sqrt{3}} a_0^0 + \frac{1}{2} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 & \frac{1}{2} \sigma_8 + \frac{1}{\sqrt{3}} \sigma_0 \end{pmatrix}, \quad T_a \pi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \pi_0^0 + \frac{1}{\sqrt{6}} \pi_8 + \frac{1}{\sqrt{3}} \pi_0 & \pi^+ \\ -\frac{1}{\sqrt{2}} \pi_0^0 + \frac{1}{\sqrt{6}} \pi_8 + \frac{1}{\sqrt{3}} \pi_0 & \frac{1}{2} \pi_8 + \frac{1}{\sqrt{3}} \pi_0 \end{pmatrix}.$$  

For $N_f = 4$,

$$T_a \sigma_a = \frac{1}{\sqrt{2}} \begin{pmatrix} a_0^0 + \frac{1}{2} \sigma_8 + \frac{1}{2} \sigma_15 + \frac{\sigma_{30}}{2\sqrt{6}} & a_0^+ \\ a_0^- & \frac{1}{2} \sigma_8 - \frac{1}{2} \sigma_15 + \frac{\sigma_{30}}{2\sqrt{6}} \end{pmatrix}, \quad T_a \pi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} a_0^0 + \frac{1}{2} \sigma_8 + \frac{1}{2} \sigma_15 + \frac{\sigma_{30}}{2\sqrt{6}} & \frac{1}{2} \sigma_8 - \frac{1}{2} \sigma_15 + \frac{\sigma_{30}}{2\sqrt{6}} \\ a_0^- & \frac{1}{2} \sigma_8 - \frac{1}{2} \sigma_15 - \frac{\sigma_{30}}{2\sqrt{6}} \end{pmatrix}.$$  

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To calculate the grand potential in mean-field approximation, we start from the partition function.

In thermal equilibrium, the grand-canonical partition-function can be defined by using a path integral over the quark, antiquark and meson field.

\[
\mathcal{Z} = \text{Tr} \exp \left[ -\left( \hat{\mathcal{H}} - \sum_f \mu_f \hat{N}_f \right) / T \right] \\
= \int \prod_a D\sigma_a D\pi_a \int D\psi D\bar{\psi} \exp \left[ \int_x \left( \mathcal{L} + \sum_f \mu_f \bar{\psi}_f \gamma^0 \psi_f \right) \right],
\]

where \( \int_x \equiv i \int_0^{1/T} dt \int_V d^3x \) and \( \mu_f \) is the chemical potential.

For \( N_f = 2, f = u, d \), if \( N_f = 3, f = u, d \) and s and if \( N_f = 4, f = u, d, s \) and c.

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Mean-field approximation

Thermodynamic potential density

For $N_f = 2$, 
\[ \Omega = U(\phi, \phi^*) + U(\sigma_l) + \Omega_{\bar{q}q}(\phi, \phi^*, \sigma_l). \]

For $N_f = 3$, 
\[ \Omega = U(\phi, \phi^*) + U(\sigma_l, \sigma_s) + \Omega_{\bar{q}q}(\phi, \phi^*, \sigma_l, \sigma_s). \]

For $N_f = 4$, 
\[ \Omega = U(\phi, \phi^*) + U(\sigma_l, \sigma_s, \sigma_c) + \Omega_{\bar{q}q}(\phi, \phi^*, \sigma_l, \sigma_s, \sigma_c). \]

Polyakov-loop Potential

Pure mesonic potential

quarks and antiquarks potential

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Pure mesonic potential

For $N_f = 2$, 

$$U(\sigma) = \frac{\lambda}{4}(\sigma^2 - \nu^2)^2 - c\sigma.$$ 

where $g$, $\lambda$, $\nu$, and $c$, are model parameters.


For $N_f = 3$, 

$$U(\sigma_1, \sigma_s) = -h_l\sigma_l - h_s\sigma_s + \frac{m^2(\sigma_l^2 + \sigma_s^2)}{2} - \frac{c\sigma_l^2\sigma_s^2}{2\sqrt{2}} + \frac{\lambda_1\sigma_l^2\sigma_s^2}{2} + \frac{(2\lambda_1 + \lambda_2)\sigma_l^4}{8} + \frac{(\lambda_1 + \lambda_2)\sigma_s^4}{4}$$

where $m_2$, $h_l$, $h_s$, $\lambda_1$, $\lambda_2$, $c$, and $g$ are model parameters.


For $N_f = 4$, 

$$U(\sigma_1, \sigma_s, \sigma_c) = \frac{1}{8}
\left(-8\ h_c\ \sigma_c + 2\ \lambda_1\ \sigma_c^4 + 2\ \lambda_2\ \sigma_c^4 - 8\ h_1\ \sigma_1 + 4\ \lambda_1\ \sigma_c^2\ \sigma_1^2 + 2\ \lambda_1\ \sigma_1^4 + \lambda_2\ \sigma_1^4 -
2\ \left(4\ h_1 + c\ \sigma_c\ \sigma_1^2\right)\ \sigma_1^2 + 4\ \lambda_1\ \left(\sigma_c^2 + \sigma_1^2\right)\ \sigma_s^2 + 2\ \left(\lambda_1 + \lambda_2\right)\ \sigma_s^4 + 4\ m^2\ \left(\sigma_c^2 + \sigma_1^2 + \sigma_s^2\right)\right)$$

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Quarks-antiquarks potential

\[ \Omega_{\bar{q}q}(\phi, \phi^*, \sigma_f) = -2T \sum_f \int_0^\infty \frac{d^3 \vec{P}}{(2\pi)^3} \left\{ \ln \left[ 1 + 3 \left( \phi + \phi^* e^{-\frac{E_f - \mu_f}{T}} \right) \times e^{-\frac{E_f - \mu_f}{T}} + e^{-3\frac{E_f - \mu_f}{T}} \right] \right. \\
+ \ln \left[ 1 + 3 \left( \phi^* + \phi e^{-\frac{E_f + \mu_f}{T}} \right) \times e^{-\frac{E_f + \mu_f}{T}} + e^{-3\frac{E_f + \mu_f}{T}} \right] \right\} , \]

where the dispersion relation

\[ E_f = \sqrt{\vec{P}^2 + m_f^2} \]


Mass of the quark flavors couples to the sigma field via Yukawa coupling $g$

For $N_f = 2$, $f = l = u, d$

\[ m_l = g\sigma_l \]

For $N_f = 3$, $f = u, d$ and $s$

\[ m_l = g\sigma_l/2 \text{ and } m_s = g\sigma_s/\sqrt{2} \]

For $N_f = 4$, $f = u, d, s, c$

\[ m_l = g\sigma_l/2, \ m_s = g\sigma_s/\sqrt{2} \text{ and } m_c = g\sigma_c/\sqrt{2} \]

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Polyakov-loop potential introduces the gluons degrees-of-freedom and the dynamics of the quark-gluon interactions to the QCD matter.

**Polynomial-logarithmic parameterization** of the Polyakov-loop potential, they even included higher-order terms,

\[
\frac{U_{\text{PolyLog}}(\phi, \phi^*, T)}{T^4} = \frac{-a(T)}{2} \phi^* \phi + b(T) \ln \left[ 1 - 6 \phi^* \phi + 4 (\phi^* \phi^3 + \phi^3) - 3 (\phi^* \phi)^2 \right] \\
+ \frac{c(T)}{2} (\phi^* \phi^3 + \phi^3) + d(T) (\phi^* \phi)^2.
\]

\[
x(T) = \frac{x_0 + x_1 (T_0/T) + x_2 (T_0/T)^2}{1 + x_3 (T_0/T) + x_4 (T_0/T)^2},
\]

\[
b(T) = b_0 \left( \frac{T_0}{T} \right)^{b_1} \left( 1 - e^{b_2 (T_0/T)^{b_3}} \right),
\]

where the coefficients \( x = a, c \) and \( d \)

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The thermodynamic potential

$$\Omega = U(\phi, \phi^*) + U(\sigma_l, \sigma_s, \sigma_c) + \Omega_{qq}(\phi, \phi^*, \sigma_l, \sigma_s, \sigma_c).$$

has the parameters

- chiral condensates
- deconfinement order-parameters

The analysis of the order parameters is given by minimizing the real part of the thermodynamic potential \(\text{Re} \ \Omega\).

The solutions of these equations can be determined by minimizing the real potential at a saddle point,

$$\left. \frac{\partial \Omega}{\partial \sigma_l} = \frac{\partial \Omega}{\partial \sigma_s} = \frac{\partial \Omega}{\partial \sigma_c} = \frac{\partial \Omega}{\partial \phi} = \frac{\partial \Omega}{\partial \phi^*} \right|_{\text{min}} = 0.$$

The behavior of the chiral condensate \(\sigma_l, \sigma_s, \sigma_c\) and the Polyakov-loop expectation values \(\phi\) and \(\phi^*\) as functions of \(T\) and \(\mu\) shall be presented.

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results

Normalized chiral condensates, $\sigma_l/\sigma_{l0}$ and $\sigma_s/\sigma_{s0}$ which are correspondent to light and strange quarks, respectively are given as functions of temperature.

At $\mu = 0$, the (thermal) expectation values of Polyakov-loops are identical, i.e. $\phi = \phi^*$

**Constituent quark mass**

$m_l \sim 300$ MeV,
$m_s \sim 433$ MeV, and
$m_c \sim 1750$ MeV.

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Results of $l$ and $s$ condensates at finite $\mu$

\[ \frac{\sigma_l}{\sigma_{l_0}} \quad \text{and} \quad \frac{\sigma_s}{\sigma_{s_0}} \]

- $\mu = 0.0 \text{ MeV}$
- $\mu = 100.0 \text{ MeV}$
- $\mu = 200.0 \text{ MeV}$

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Results of Polyakov-loop variables at finite $\mu$
Results of $c$ condensates and finite $\mu$

\[
\begin{align*}
\mu &= 0.0 \text{ MeV} \\
\mu &= 200.0 \text{ MeV} \\
\mu &= 400.0 \text{ MeV} \\
\mu &= 600.0 \text{ MeV} \\
\mu &= 800.0 \text{ MeV}
\end{align*}
\]
Summary and Outlook

- Constructing PLSM Lagrangian in SU(2), SU(3) and SU(4)
- Introducing Ployakov-Loop Corrections to LSM
- Constructing SU(4) PLSM formalism at finite T and mu
- Estimating T- and mu-dependence of chiral and deconfinement order-parameters

- Estimate T- and mu-inmedium effects on hadrons with c-quarks
- Comparing with lattice QCD simulations
- Electromagnetic effects at finite T and mu

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Thanks for your attention!