# Convexity, gauge-dependence, and tunneling rates

Carlos Tamarit, Durham IP[3]

In collaboration with Alexis Plascencia

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# The effective action

The effective action $\Gamma$ encodes the quantum dynamics of expectation values of scalar fields in the presence of sources.

$$\bar{\phi} \equiv \langle \phi \rangle_J, \quad \Gamma[\bar{\phi}] = \int d^D x \left[ \mathcal{O}(\partial \bar{\phi}) - V_{\text{eff}}(\bar{\phi}) \right]$$
$$= \sum (\text{1PI diagrams}),$$
$$J = \frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}}.$$

Its zero-momentum part, the effective potential, determines the vacuum structure of the theory.

$$J = 0, \partial \bar{\phi} = 0 \Rightarrow \frac{\partial V_{\text{eff}}(\bar{\phi})}{\partial \bar{\phi}} = 0.$$

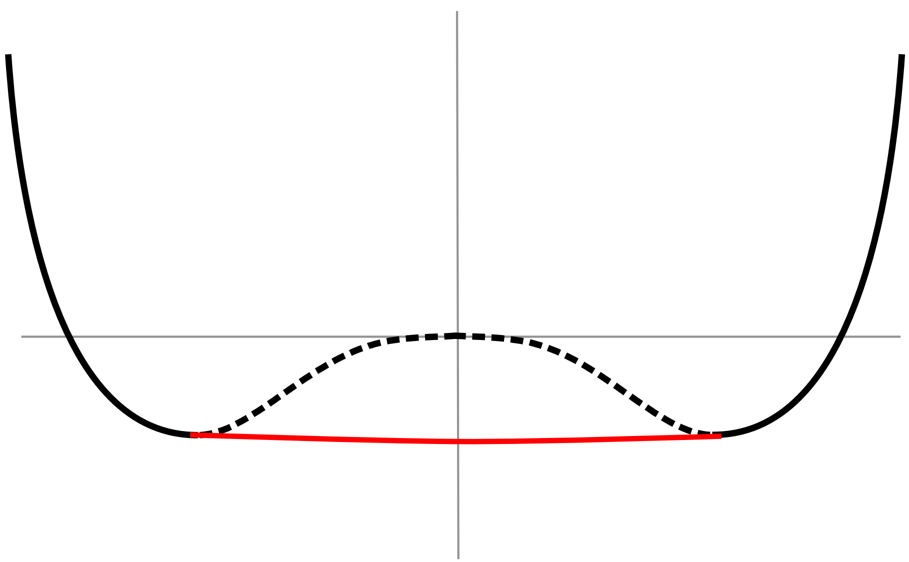
# Troubling issues with the effective action

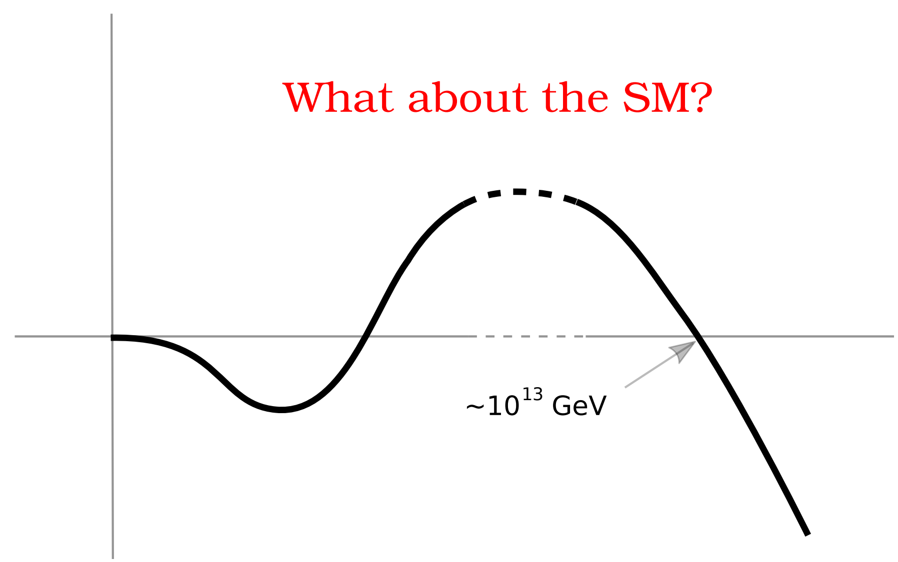
It is gauge-dependent! [Jackiw]

How can one extract gauge-independent information?

It is a convex functional, meaning the effective potential is concave (positive second derivative everywhere) [Iliopoulos, Itzykson, Martin,...].

How can it play a role regarding false vacua and tunnelling rates?





# Gauge-dependence: Nielsen identities

The gauge dependence of the effective action is encoded by Nielsen identities [Nielsen, Kugo, Fukuda, ...]

$$\xi \frac{\partial \Gamma}{\partial \xi}[\bar{\phi};\xi] + \Gamma_{,j}[\bar{\phi};\xi] K^{j}[\bar{\phi},\xi] = 0,$$

Gauge dependence ⟷ Nonlocal field redefinition

At extrema (e.g. vacua),

$$\Gamma_{,j} = 0 \to \xi \frac{\partial \Gamma}{\partial \xi}[\bar{\phi};\xi] = 0.$$

Energies and masses defined from $V_{\text{eff}}$ at the vacuum configurations remain independent of the choice of gauge.

What about tunneling rates?

# Tunneling rates

Tunneling à la Callan-Coleman

$$\gamma = \det(\text{Fluctuations}) e^{-S_E[\varphi_b]}.$$

$\varphi_b$ : Classical solution to Euclidean equations of motion, using $V_{class} \neq V_{eff}$, with boundary conditions set by false vacuum

What if false vacuum or instability is generated radiatively (e.g. SM)?

$V_{eff}$ suspected/assumed to play a role (e.g. SM)
But $V_{eff}$ should be convex (no false vacua!!)
How to avoid double-counting of fluctuations?

$\varphi_b$ depends on the potential between minima, known to be gauge-dependent. No formal proof of gauge-independence at all orders.
[Previous work by Metaxas and Weinberg, Garny and Konstandin,...]

# True-vacuum effective action

True-vacuum partition function Z[J]:

Identity insertions

$$Z[J] = \exp iW[J] = \lim_{T\to\infty} \langle 0|e^{-iHT}|0\rangle^J$$

$$= \lim_{T\to\infty} \int [dq][dq'] \langle 0|q\rangle^J \langle q|e^{-iHT}|q'\rangle^J \langle q'|0\rangle^J$$

$$= \int [dq][dq'] \psi_0^J(q')\psi_0^{J\star}(q) \int_{q'}^{q} [d\phi] \exp i\left[\tilde{S}_g[\phi;\xi] + J_j\phi^j\right]$$

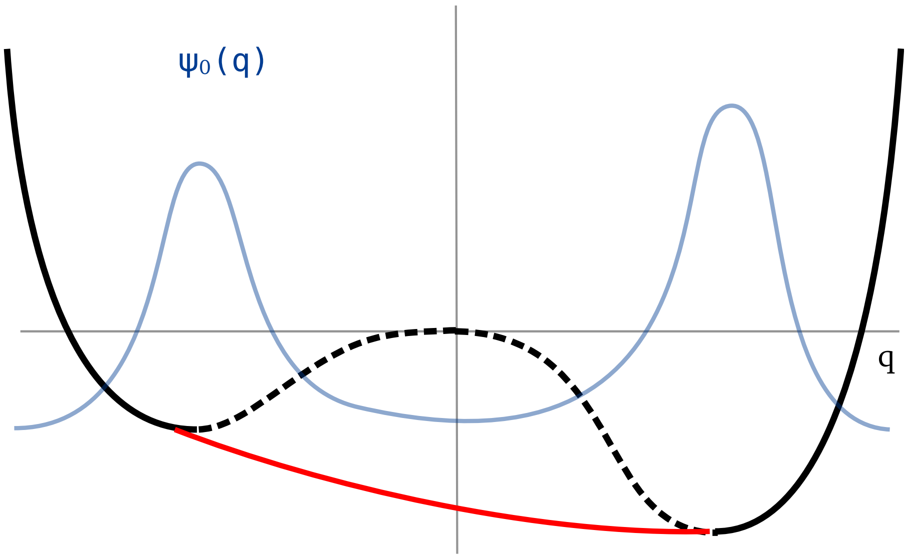
Usual single path integral

$$\Psi_0^J(q) \equiv \langle q|0\rangle^J$$

Vacuum wave function, usually ignored!

Expected to peak around local vacua in field-space

# True-vacuum effective action




Multi-peak structure of the wave function means that Z[J] can be approximated by a sum of path integrals

$$Z[J] \approx \sum_{m,n=1}^{N} Z^{m,n}[J], \quad Z^{m,n}[J] = \mathcal{N}[J]_{mn} \int_{q_0^{J,m}}^{q_0^{J,n}} [d\phi] \exp i \left[\tilde{S}_g[\phi;\xi] + J_j \phi^j\right]$$

Same for the true-vacuum effective action, explaining multi-path integral constructions of concave effective potentials! [Fujimoto et al, Bender et al,...]

$$\bar{\phi} = -\, i \frac{\delta \log Z[J]}{\delta J}, \quad \Gamma[\bar{\phi}] = -i \log Z[J] - J_j\, \bar{\phi}^j\,,$$

$$\exp i\Gamma[\bar{\phi};\xi] = \sum_{m,n=1}^{N} \mathcal{N}[\Gamma_{,j}]_{mn} \int_{q_0^{J,m}-\bar{\phi}_\infty}^{q_0^{J,n}-\bar{\phi}_\infty} [d\phi]\, \exp i \left[S_g[\bar{\phi},\phi;\xi] - \Gamma_{,j}[\bar{\phi};\xi]\phi^j\right]$$

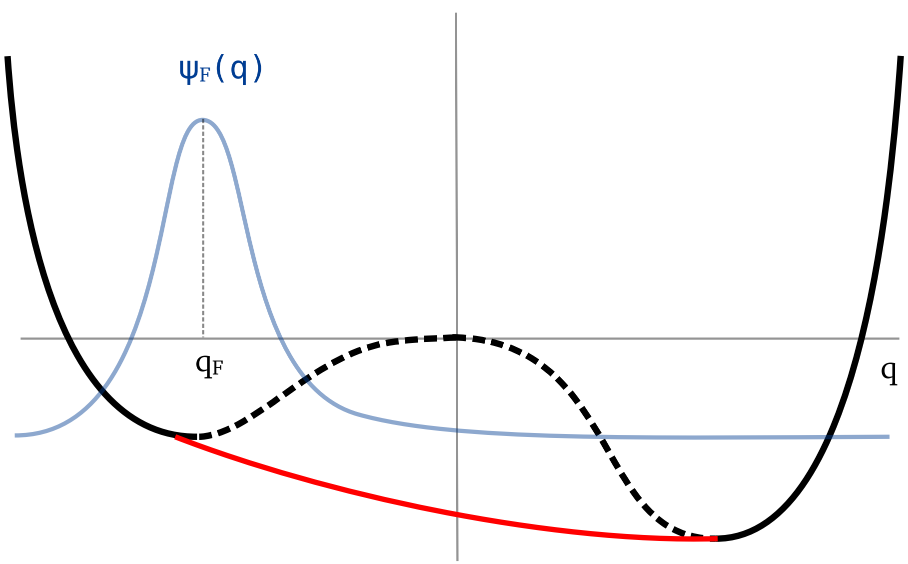
# False-vacuum effective action

By definition, a false vacuum state $|F\rangle$ has a localized field-space wave-function. Thus one can define a partition function which can be approximated by a single path integral.

$$Z_F^T[J] = \langle F|e^{-iHT}|F\rangle^J = e^{-i\epsilon VT}$$

$$= \int [dq][dq']\psi_F^J(q')\psi_F^{J\star}(q)\int_{q'}^{q}[d\phi]\exp i\left[\tilde{S}_g[\phi;\xi] + J_j\phi^j\right]$$

$$\approx \int_{q_F}^{q_F}[d\phi]\exp i\left[\tilde{S}_g[\phi;\xi] + J_j\phi^j\right]$$




The vacuum being unstable, $Z_F$ has an imaginary part related with the decay rate.

$$\gamma = -2\,\mathrm{Im}\epsilon = -\frac{2}{VT}\,\mathrm{Re}\left(\log Z_F^T[0]\right).$$

# False-vacuum effective action

One can construct a false vacuum effective action, which will be:

Approximated by a single path integral

Complex, not convex!

$$\bar\phi = -\,i\frac{\delta \log Z_F^T[J]}{\delta J},\quad \Gamma_F^T[\bar\phi] = \overbrace{-i\log Z_F^T[J]}^{-\epsilon VT,\ \text{complex!}} - J_j\,\bar\phi^j,$$

$$\exp i\Gamma_F^T[\bar\phi;\xi] = \int_0^0 [d\phi]\ \exp i\left[S_g[\bar\phi,\phi;\xi] - \Gamma^T_{F,j}[\bar\phi;\xi]\phi^j\right]$$

Can be understood as a realization of [Weinberg and Wu]'s "local" effective action. It is essentially the usual effective action used e.g. in the SM! [Einhorn, Jones]

The resulting effective potential is complex, not necessarily convex , explaining why one can see radiatively generated false-vacua or instabilities (e.g. SM).

# Tunneling and gauge-independence

The gauge dependence of $\Gamma_F$ is encoded by its Nielsen identities. In particular, $\Gamma_F[\varphi]$ is gauge-independent if $\varphi$ solves the quantum equations of motion

$$\Gamma_{F,i}[\varphi(\xi);\xi] = 0 = -J_i \Rightarrow \xi \frac{\partial \Gamma_F}{\partial \xi}[\varphi(\xi);\xi] = 0.$$

The tunneling rate is related to the effective action evaluated at a solution to the quantum equations of motion! Gauge-independence follows immediately from the Nielsen identities.

$$\left.\begin{array}{l} \gamma = -\frac{2}{VT}\,\mathrm{Re}\left(\log Z_F^T[0]\right) \\ \Gamma_F[\bar{\phi}] = -i\log Z_F^T[J] - J_i\bar{\phi}^i \end{array}\right\} \quad J_i = 0 \Rightarrow \; \Gamma_F^T[\varphi] = -i\log Z_F^T[0]$$

$$\gamma = \frac{2}{VT}\,\mathrm{Im}\,\Gamma_F^T[\varphi_F(\xi);\xi].$$

# Which solution exactly?

From the definition of $\gamma$ in terms of $Z_F[0]$, being careful with the boundary conditions, we get a result that generalizes Callan and Coleman's sum over multiple bounce solutions. After rotating to Euclidean space:

$$\gamma = \frac{2}{VT^E}\,\mathrm{Im}\ e^{-\Gamma_F^E[\varphi^{1,E}(\xi);\xi]} = 2\mathcal{J}\,\mathrm{Im}\ e^{-\Gamma_F'^E[\varphi^{1,E}(\xi);\xi]}.$$

[see also Garbrecht, Millington in theories without gauge fields]

Single quantum bounce solution, with boundary conditions fixed by the false vacuum.

The exponential of the classical Euclidean action recovered in the semiclassical limit, $\Gamma_F'^E = S^E + \mathcal{O}(\hbar)$

Generalized Jacobian $\quad \mathcal{J} \sim (2\pi V''_{\mathrm{eff}}(q_F))^{-2}(\Gamma_F'^E)^2$

The formula includes all quantum corrections. Clarifies how to compute tunneling rates with radiatively generated vacua or instabilities.

# Summary

We clarified issues of convexity of the effective action and gauge-independence of tunneling rates.

We introduced the notion of false-vacuum effective action. In contrast to the true-vacuum effective potential, the false-vacuum one is neither real nor convex, and can be captured with a single path integral.

False vacua and tunneling rates can only be defined and found with the false-vacuum effective action.

Tunneling rates are related to the exponential of the false-vacuum effective action evaluated at a generalized bounce solution. This encodes all quantum corrections and clarifies how to compute tunneling rates with radiatively generated vacua/instabilities (e.g. SM).

Our result stresses the role of the imaginary part of $\Gamma_F$ (and $V_{Feff}$).

Formal proof of gauge-independence of tunneling rates.

# Connection with Andreassen et al's formalism

arXiv:1602.01102

$$\gamma = \frac{2}{V}\frac{\mathrm{Im}\int[d\phi]e^{-S^E}\delta(\tau_\Sigma[\phi])}{\int[d\phi]e^{-S^E}}.$$

Denominator argued to be real. In our formalism $\int[d\phi]e^{-S^E} = \exp\left(-\Gamma_F^E[\varphi]\right)$

for an extremal φ. For a real contribution it can only be the constant configuration at the false vacuum, and in our normalization $V_{eff}[q_F]=0$

$$\exp\left(-\Gamma_E^F[q_F]\right) = \exp\left(-VTV_{\text{eff}}[q_F]\right) = 1$$

Numerator: Using invariance under time translations

$$\int[d\phi]e^{-S^E}\delta(\tau_\Sigma[\phi]) = \frac{1}{T}\int[d\phi]d\tau e^{-S^E}\delta(\tau_\Sigma[\phi]) \sim \frac{1}{T}\int[d\phi]e^{-S^E},$$

and our formula is recovered:

$$\gamma = \frac{2}{VT}\,\mathrm{Im}\;e^{-\Gamma_F^E[\varphi]}$$

for a quantum bounce φ such that $\Gamma_F[\varphi]$ has an imaginary part.