Correspondence between Solutions of Scattering Equations and Scattering Amplitudes in Four Dimensions

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Outline

Backgrounds

Traditional Feynman diagram calculations on multi-particle scattering amplitudes are highly complicated because there are

- too many diagrams,
- correlation of color and kinematic factors,
- too many kinematic factors in each diagram.

New formalisms/tools/properties for scattering amplitudes are required.

Backgrounds

- In recent decades, many new compact expressions based on
  - spinor helicity formalism in 4D
  - color decomposition

  have been developed.

- In 4D, tree-level amplitudes with all same/all but one same helicity vanish. maximally-helicity-violation (MHV) configurations with two particles have different helicity from others are nontrivial and were shown to have unexpected compact simplicity:

Yang-Mills (YM)	Parke-Taylor
Gravity (GR)	Hodges
Einstein-Yang-Mills (EYM)	SBDW

All these new formulas/methods refreshed our knowledge on the hidden simplicity of scattering amplitudes.

Parke-Taylor formula

Color-ordered YM amplitudes at tree level in MHV configuration satisfy Parke-Taylor formula (Parke, Taylor, 1986)

$$A^{\mathsf{YM}}(1^+, \cdots, i^-, \cdots, j^-, \cdots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

i, j are two negative helicity gluons

Hodges formula

The gravity amplitude at tree level in MHV configuration has the form (Hodges 2012):

$$\mathcal{M}_n^{\text{GR}}(1^+,\cdots,i^-,\cdots,j^-,\cdots,n^+) = \langle ij\rangle^8 \bar{M}(12\ldots n),$$

where

- $\bar{M}$ is given by

$$\begin{aligned} \bar{M}(12\ldots n) &= (-1)^{n+1} c_{ijk} c^{pqr} \det(\phi^{ijk}_{pqr}), \\ c_{ijk} &= c^{ijk} = \frac{1}{\langle ij\rangle\langle jk\rangle\langle ki\rangle}. \end{aligned}$$

- The matrix ϕ (Hodges matrix) is given by:

$$\phi_{ab} = \begin{cases} \dfrac{\langle ab\rangle}{[ab]} & a \neq b \\ -\displaystyle\sum_{l\neq a} \dfrac{[al]\langle lm\rangle\langle ls\rangle}{\langle al\rangle\langle am\rangle\langle as\rangle} & a = b \end{cases}$$

The diagonal elements ϕ_{aa} do not depend on the choice of the spinors $|m\rangle$ and $|s\rangle$ (gauge invariance).

SBDW formula for MHV EYM amplitudes

Single-trace amplitudes for EYM in the MHV configurations $(g^- g^-)$ and $(g^- h^-)$ satisfy SBDW formula
(Selivanov 1997, 1998; Bern, De Freitas, Wong 1999)

$$A^{\mathsf{EYM}}(g_1, \cdots, g_r; h_1, \cdots, h_s) = \frac{\langle ij \rangle^4}{\langle g_1 g_2 \rangle \langle g_2 g_3 \rangle \ldots \langle g_r g_1 \rangle} S\left(i, j, \{h^+\}, \{g^+\}\right)$$

- i, j are negative helicity particles, S is a generating function:

$$S(\mathsf{h}_+) = \left(\prod_{m \in \mathsf{h}_+} \frac{\partial}{\partial a_m} \right) \exp\Bigg[\sum_{n_1 \in \mathsf{h}_+} a_{n_1} \sum_{l \in \mathsf{g}^+} \psi_{l n_1} \times \exp\Big[\sum_{\substack{n_2 \in \mathsf{h}_+ \\ n_2 \neq n_1}} a_{n_2} \psi_{n_1 n_2} \exp(\cdots) \Big] \Bigg] \Bigg|_{a_m = 0},$$

where

$$\psi_{ab} = \phi_{ab} \frac{\langle bi \rangle \langle bj \rangle}{\langle ai \rangle \langle aj \rangle}.$$

- $(h^- h^-)$ amplitude vanishes (conjecture).

Outline

Scattering equation (SE)

CHY formula (Cachazo, He, Yuan 2013) is based on scattering equations

$$\sum_{\substack{b=1\\b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad (a = 1, \cdots, n)$$

where k are on-shell momenta of external particles.

Properties of SE

- Only $(n-3)$ are independent:

$$f_a \equiv \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \qquad \Longrightarrow \qquad \sum_{a=1}^{n} \sigma_a^m f_a = 0 \text{ for } m = 0, 1, 2$$

- Möbius covariance: suppose $\{\sigma_a\}$ is a solution set, then

$$\xi_a = \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + \delta} \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

  is also a solution set.
- There are $(n-3)!$ solutions.

Special solutions in 4D

In four dimensions, there always exist two rational ones in terms of spinor variables: (Monteiro, O-Connell, 2013; Weinzierl, 2014)

$$\sigma_a^{(1)} = \frac{\langle a\theta\rangle\langle\eta\xi\rangle}{\langle a\xi\rangle\langle\eta\theta\rangle} \qquad \sigma_a^{(2)} = \frac{[a\theta][\eta\xi]}{[a\xi][\eta\theta]}$$

- The arbitrary spinors $|\theta\rangle$, $|\eta\rangle$ and $|\xi\rangle$ encode the whole $SL(2,\mathbb{C})$ freedom.
- They were conjectured to correspond to MHV and anti-MHV amplitudes before (Naculich, 2014), but there were no explicit proof.
- In addition, it was not clear what role other solutions play in the MHV and anti-MHV amplitudes.

Outline

CHY formula

CHY formula is a formula based on SE:

$$A_n = \oint_{\mathcal{O}} \frac{dz_1 \cdots dz_n}{\mathrm{Vol}[SL(2,\mathbb{C})]} \left[\prod_{a=1}^{n}{}' \delta(f_a) \right] \mathcal{I}(k,\epsilon,z)$$

Building blocks

- Integrand $\mathcal{I}(k,\epsilon,z)$ depends on theories, contains information of external particles.
- The integral measure:

$$\frac{dz_1 \cdots dz_n}{\mathrm{Vol}[SL(2,\mathbb{C})]} = \sigma_{pq}\sigma_{qr}\sigma_{rp} \prod_{c\neq p,q,r}^{n} dz_c$$

  The volume of $SL(2,\mathbb{C})$ are divided and $\sigma_{ab} \equiv \sigma_a - \sigma_b$.
- Support:

$$\prod_{a=1}^{n}{}' \delta(f_a) \equiv z_{ij} z_{jk} z_{ki} \prod_{a\neq i,j,k} \delta(f_a)$$

- $\oint_{\mathcal{O}}$ encloses all the solutions of the SE $f_a = 0$.

CHY integrands

CHY integrands $\mathcal{I}(k,\epsilon,z)$ for YM, GR, EYM are following.

- Color-ordered tree amplitude in YM:

$$\mathcal{I}^{\mathsf{YM}}(k,\epsilon,z) = \frac{1}{z_{12}z_{23}\cdots z_{n1}}\mathrm{Pf}'(\Psi)$$

- Tree amplitude in GR:

$$\mathcal{I}^{\mathsf{GR}}(k,\epsilon,z) = [\mathrm{Pf}'(\Psi)]^2$$

- Single-trace color-ordered tree amplitude in EYM:

$$\frac{1}{z_{g_1g_2}z_{g_2g_3}\cdots z_{g_rg_1}}\mathrm{Pf}(\Psi_{\mathrm{h}})\mathrm{Pf}'(\Psi_{\mathrm{h+g}})$$

Ψ matrix and reduced Pfaffian

Ψ is a $2n \times 2n$ skew-symmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

The $n \times n$ submatrices A, B and C are

	A_{ab}	B_{ab}	C_{ab}
$a \neq b$	$\frac{s_{ab}}{z_{ab}}$	$\frac{2\epsilon_a \cdot \epsilon_b}{z_{ab}}$	$\frac{2\epsilon_a \cdot k_b}{z_{ab}}$
$a = b$	0	0	$-\sum_{c \neq a} \frac{2\epsilon_a \cdot k_c}{z_{ac}}$

Reduced Pfaffian is define by

$$\mathrm{Pf}'(\Psi) = \frac{(-1)^{i+j}}{z_{ij}} \mathrm{Pf}(\Psi^{ij}_{ij})$$

Ψ^{ij}_{ij} means the i-th and j-th row and column of Ψ have been deleted ($i, j \in \{1, 2, \ldots, n\}$).

Integrated CHY formula

After integration, amplitude is expressed as a summation of contributions of each solution:

Color-ordered YM	$\sum_{\{\sigma_a\}\in\text{sol.}} \frac{\mathrm{Pf}'(\Psi)}{\det'(\Phi)\sigma_{12}\sigma_{23}\cdots\sigma_{n1}}$
GR	$\sum_{\{\sigma_a\}\in\text{sol.}} \frac{[\mathrm{Pf}'(\Psi)]^2}{\det'(\Phi)}$
Single-trace EYM	$\sum_{\{\sigma_a\}\in\text{sol.}} \frac{1}{\sigma_{g_1g_2}\sigma_{g_2g_3}\cdots\sigma_{g_rg_1}} \frac{\mathrm{Pf}(\Psi_{\mathrm{h}})\mathrm{Pf}'(\Psi_{\mathrm{h+g}})}{\det'(\Phi)}$

where

$$\det'(\Phi) = \frac{1}{\sigma_{ij}\sigma_{jk}\sigma_{ki}\sigma_{pq}\sigma_{qr}\sigma_{rp}} \det(\Phi^{ijk}_{pqr})$$

Φ^{ijk}_{pqr} is the Jacobian of the delta functions and $\det'(\Phi)$ is independent of the choice of (i,j,k) and (p,q,r).

Questions

- Can we directly evaluate helicity amplitudes and obtain simple compact formulas by using the CHY formalism?
- Is there any relation between solutions of SE and helicity configurations?

Outline

New identities Du, Teng, Wu, arxiv:1603.08158, JHEP05(2016)086

CHY building blocks satisfy following new identities under the special solution $\frac{\langle a\theta\rangle\langle\eta\xi\rangle}{\langle a\xi\rangle\langle\eta\theta\rangle}$

$$\mathrm{Pf}'(\Psi) = [F(\xi,\eta,\theta)]^n (P_\xi)^2 \langle xy\rangle^4 \bar{M}(12\ldots n)$$
$$\det{}'(\Phi) = [F(\xi,\eta,\theta)]^{2n} (P_\xi)^4 \bar{M}(12\ldots n)$$
$$\sigma_{12}\cdots\sigma_{n1} = \left[\frac{1}{F(\xi,\eta,\theta)}\right]^n \frac{\langle 12\rangle\cdots\langle n1\rangle}{(P_\xi)^2}$$

Möbius covariance are encoded in the factors

$$F(\xi,\eta,\theta) = \frac{\langle\theta\eta\rangle}{\langle\eta\xi\rangle\langle\theta\xi\rangle} \qquad P_\xi = \prod_{a=1}^{n}\langle a\xi\rangle,$$

Thus Möbius covariance is separated from gauge invariance.

Outline

The special solution and MHV in YM/GR

Putting these identities into the CHY formula for YM and GR, we have:

$$
\begin{aligned}
A^{\text{YM}} &= \frac{1}{\left[\frac{1}{F(\xi,\eta,\theta)}\right]^n \frac{\langle 12\rangle\cdots\langle n1\rangle}{(P_\xi)^2}} \frac{[F(\xi,\eta,\theta)]^n (P_\xi)^2 \langle xy\rangle^4 \bar{M}(12\ldots n)}{[F(\xi,\eta,\theta)]^{2n}(P_\xi)^4 \bar{M}(12\ldots n)} \\
&= \frac{\langle xy\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle} \qquad \text{Parke-Taylor formula} \\
\mathcal{M}^{\text{GR}} &= \frac{\left\{[F(\xi,\eta,\theta)]^n (P_\xi)^2 \langle xy\rangle^4 \bar{M}(12\ldots n)\right\}^2}{[F(\xi,\eta,\theta)]^{2n}(P_\xi)^4 \bar{M}(12\ldots n)} \\
&= \langle xy\rangle^4 \bar{M}(12\ldots n) \qquad \text{Hodges formula.}
\end{aligned}
$$

Other solutions do not contribute to MHV

For other solutions

- $\mathrm{Pf}'(\Psi) = 0$ for the other special rational solution $\frac{[a\theta][\eta\xi]}{[a\xi][\eta\theta]}$,
- All other solutions make $\mathrm{Pf}'(\Psi)$ vanish.
  (Du, Teng, Wu, appear soon)
- Similar result is true for $\sigma_a^{(2)}$ and anti-MHV amplitudes, if we exchange all $\langle \ldots \rangle$ with $[\ldots]$.

Outline

The special solution and MHV EYM amplitude

(Du, Teng, Wu, arxiv:1608.00883)

For single trace EYM amplitude and the special solution $\frac{\langle a\theta\rangle\langle\eta\xi\rangle}{\langle a\xi\rangle\langle\eta\theta\rangle}$, we prove

$$\frac{\mathrm{Pf}\left[\Psi_{\mathsf{h}}(\sigma)\right]}{\sigma_{g_1g_2}\sigma_{g_2g_3}\cdots\sigma_{g_rg_1}} \sim F^n\left(P_\xi\right)^2 \frac{\det(\phi_{\mathsf{h}})}{\langle g_1g_2\rangle\langle g_2g_3\rangle\cdots\langle g_rg_1\rangle},$$

for (g^-g^-) and

$$\frac{\mathrm{Pf}\left[\Psi_{\mathsf{h}}(\sigma)\right]}{\sigma_{g_1g_2}\sigma_{g_2g_3}\cdots\sigma_{g_rg_1}} \sim F^n\left(P_\xi\right)^2 \frac{\det[(\phi_{\mathsf{h}})^i_i]}{\langle g_1g_2\rangle\langle g_2g_3\rangle\cdots\langle g_rg_1\rangle},$$

for (h^-g^-). Putting these identities and the identities for $\mathrm{Pf}'(\Psi)$, $\det'(\Phi)$, we obtain a compact formula

$$A^{EYM}(\cdots h_i^- \text{ or } g_i^- \cdots g_j^- \cdots) \propto \frac{\langle ig_j\rangle^4}{\langle g_1g_2\rangle\langle g_2g_3\rangle\cdots\langle g_rg_1\rangle}\det(\phi_{\mathsf{h}+})$$

Other solutions, $(h^- h^-)$ configuration

- As in the YM/GR cases

  Other solutions make the reduced Pfaffian $\mathrm{Pf}'(\Psi)$ vanishing.

- $(h^- h^-)$ configuration:

  We prove that the special solution makes $\mathrm{Pf}\,[\Psi_\mathrm{h}]$ vanish at the $(h^- h^-)$ configuration.

Using the CHY formalism, we have proved analytically that the $(h^- h^-)$ MHV amplitudes in EYM are zero.

We can further prove that general EYM amplitudes with gluons having the same helicity also vanish.

EYM from CHY and SBDW

(Du, Teng, Wu, arxiv:1608.00883)

We derive that the SBDW generating function has a graph theory interpretation:

$$
\begin{aligned}
S(\mathsf{h}_+) &= \left(\prod_{m\in\mathsf{h}_+}\frac{\partial}{\partial a_m}\right)\exp\Bigg[\sum_{n_1\in\mathsf{h}_+} a_{n_1}\sum_{l\in\mathsf{g}^+}\psi_{ln_1} \\
&\quad \times\exp\Bigg[\sum_{\substack{n_2\in\mathsf{h}_+\\ n_2\neq n_1}} a_{n_2}\psi_{n_1n_2}\exp\left(\cdots\right)\Bigg]\Bigg]\Bigg|_{a_m=0} \\
&= \sum_{F\in\mathcal{F}_{\mathsf{g}^+}}\left(\prod_{v_av_b\in E(F)}\psi_{ab}\right)
\end{aligned}
$$

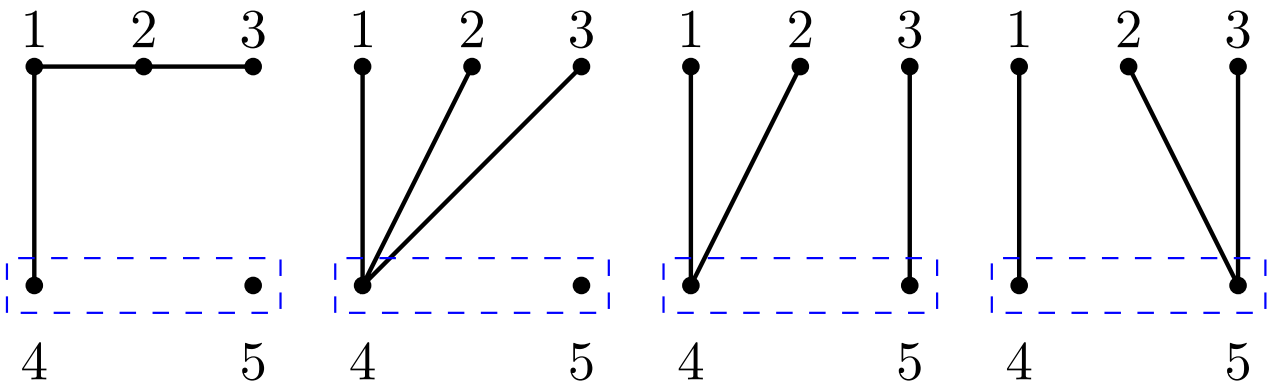
vertices $\to \mathsf{g}\bigcup\mathsf{h}$, roots $\to \mathsf{g}^+$

edge weight $\to \psi_{ab}$, $\mathcal{F}_{\mathsf{g}^+} \to$ Forests with roots in g^+

Example

A diagram for $A^{\text{EYM}}(h_1^+, h_2^+, h_3^+; g_1^+, g_2^+, g_3^-, g_4^-)$, where we define

$$h_1 \equiv 1\,, \quad h_2 \equiv 2\,, \quad h_3 \equiv 3\,, \quad g_1 \equiv 4\,, \quad g_2 \equiv 5\,, \quad g_3 \equiv 6\,, \quad g_4 \equiv 7\,.$$



Vertex 6 and 7 are disjoint to all the others due to our gauge choice.

EYM from CHY and SBDW

Our new formula from CHY:

$$A^{EYM}(\cdots h_i^- \text{ or } g_i^- \cdots g_j^- \cdots) \propto \frac{\langle i g_j \rangle^4}{\langle g_1 g_2 \rangle \langle g_2 g_3 \rangle \ldots \langle g_r g_1 \rangle} \det(\phi_{\mathsf{h}+})$$

is equivalent to SBDW since $\det(\phi_{\mathsf{h}+})$ has exactly the same graph theory interpretation. (Feng, He, 2012)

- This formula is much simpler and easier to compute compared with SBDW.
- Same method can prove analytically other important results (for example, gluon-same-helicity amplitudes).

Summary and Outlook

In this talk

- We directly derived the Parke-Talyor/Hodges formulas in YM/GR from the CHY formula.
- We derived a much more compact formula for single-trace EYM MHV amplitudes and proved the equivalence with SBDW formula.
- We proved the conjecture: single-trace EYM amplitudes where all gluons have same helicity have to vanish.
- We find further correspondence between other solutions and configurations beyond MHV recently. (Du, Teng, Wu, work to appear)

Thank you for your attention!

Spinor-helicity formalism (Xu, Zhang, Chang 1987)

In 4D, one can use tow copy of Weyl spinors to express vectors:

- On-shell momentum k of a massless particle has the form:

$$k_{\alpha\dot{\alpha}} = (k \cdot \sigma)_{\alpha\dot{\alpha}} = |k]_\alpha \langle k|_{\dot{\alpha}}$$

- Polarization vectors $\epsilon^{\pm}(k)$ of gluons have the form:

$$\epsilon^{+}_{\alpha\dot{\alpha}} \sim \frac{|q]_\alpha \langle k|_{\dot{\alpha}}}{\sqrt{2}[qk]} \quad \epsilon^{-}_{\alpha\dot{\alpha}} = \frac{|k]_\alpha \langle \tilde{q}|_{\dot{\alpha}}}{\sqrt{2}\langle \tilde{q}k\rangle}, |q], \langle \tilde{q}| : \text{reference spinors}$$

  For gravitons $\epsilon^{\pm}_{GR} \sim \epsilon^{\pm}(k)\tilde{\epsilon}^{\pm}(k)$.

- Index manipulation: $(\varepsilon^{12} = -\varepsilon^{21} = 1,\ \varepsilon^{11} = \varepsilon^{22} = 0)$

$$[k|^\alpha = \varepsilon^{\alpha\beta}|k]_\beta \qquad |k\rangle^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\langle k|_{\dot{\beta}}$$

- Spinor inner product:

$$[kp] \equiv [k|^\alpha |p]_\alpha \qquad \langle kp\rangle \equiv \langle k|_{\dot{\alpha}}|p\rangle^{\dot{\alpha}}$$

- The transverse condition and gauge invariance of physical amplitudes are preserved.

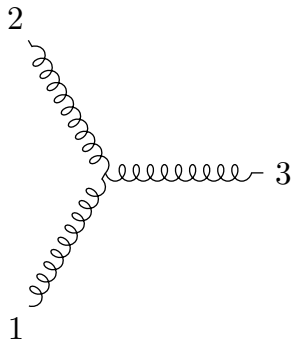
Color-decomposition

Color-dressed YM tree amplitude can be decomposed as

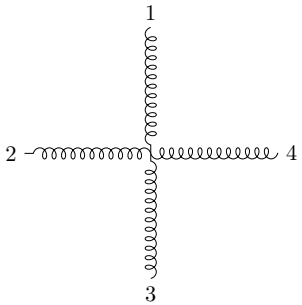
$$\mathcal{M}_n^{\mathsf{YM}} = \sum_{\sigma} \mathrm{Tr}\left(T^{a_1} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}\right) A_n^{\mathsf{YM}}\left[1, \sigma(2) \cdots \sigma(n)\right]$$

$\mathcal{M}_n^{\mathsf{YM}}$: color-dressed amplitude $\quad A_n^{\mathsf{YM}}$: color-ordered amplitude

Vertices in color-ordered Feynman rules (Feynman gauge)



$$= \frac{i}{\sqrt{2}}\Big[\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2}\Big]$$



$$= i\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \frac{i}{2}\left[\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}\right]$$