Resonances in Coupled-Channel Scattering from Lattice QCD

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for the Hadron Spectrum Collaboration

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Extracting resonance properties

excited states seen as resonant enhancements in the scattering of lighter stable particles

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excited states seen as resonant enhancements in the scattering of lighter stable particles

Infinite volume

Bound states

Meson-meson continuum

Finite volume
Resonances in coupled-channel scattering

Infinite volume phase shifts from a finite volume

\[ \psi(0) = \psi(L), \quad \frac{\partial \psi}{\partial x} \bigg|_{x=0} = \frac{\partial \psi}{\partial x} \bigg|_{x=L} \]

\[ \sin \left( \frac{pL}{2} + \delta(p) \right) = 0 \]

\[ p = \frac{2\pi n}{L} - \frac{2}{L} \delta(p) \]

\[ m_\pi = 236 \text{ MeV} \]
ρ resonance

Phase shifts via the Lüscher method:

\[
\tan \delta_1 = \frac{\pi^{3/2} q}{Z_{00}(1; q^2)}
\]

\[
Z_{00}(1; q^2) = \sum_{n \in \mathbb{Z}^3} \frac{1}{|\vec{n}|^2 - q^2}
\]

\[m_\pi = 236 \text{ MeV}\]
$\rho$ resonance with moving frames

$P = [000]$
\[ m_\pi = 236 \text{ MeV} \]
Resonances in coupled-channel scattering

\[ t = \begin{pmatrix} \pi \pi \rightarrow \pi \pi & \pi \pi \rightarrow K \bar{K} \\ K \bar{K} \rightarrow \pi \pi & K \bar{K} \rightarrow K \bar{K} \end{pmatrix} \]

\[ t^{-1} = K^{-1} - i\rho \]

e.g.: \( K_{ij} = \frac{g_i g_j}{m^2 - s} + \gamma_{ij} \)

\[ m_\pi = 236 \text{ MeV} \]
ρ resonance into the coupled-channel region

PRD 92 094502, arXiv:1507.02599

$E_{\text{cm}}$/MeV

$m_\pi = 236$ MeV
An $a_0$ resonance

$\pi\eta$-$K\bar{K}$-$\pi\eta'$

$I = 1 \quad J = 0$

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**Figure 58.** Partial wave analysis of the $K^-\pi^+\rightarrow K^-\pi^+$ amplitudes deduced from the LASS results of Fig. 57, showing the magnitude and phase for the S, P and D-waves.

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**Figure 59.** Mass distribution for $\pi^0\eta$ from the GAMS experiment [11], where both $\pi^0$ and $\eta$ are detected in the $\gamma\gamma$ decay.

Partial wave analysis reveals that the $J = 0$ wave has two possible resonances $a_0(980)$ and $a_0(1430)$ in this mass region.
Resonances in coupled-channel scattering

\( a_0 \) resonance - two channel region

\[ \pi \eta - K \bar{K} \]

\[ K_{ij} = \frac{g_i f_j}{m_i^2 - s} + \gamma_{ij} \]

\( m = (1254 \pm 16) \cdot \text{GeV} \)

\( g_{\pi \eta} = (515 \pm 16) \cdot \text{GeV} \)

\( g_{K \bar{K}} = (-730 \pm 85) \cdot \text{GeV} \)

\( \gamma_{\pi \eta, \pi \eta} = -0.16 \pm 0.24 \)

\( \gamma_{\pi \eta, K \bar{K}} = -0.56 \pm 0.29 \)

\( \gamma_{K \bar{K}, K \bar{K}} = 0.12 \pm 0.38 \)

\[ \chi^2 / N_{\text{dof}} = \frac{58.0}{47 - 6} = 1.41 \]

\[ m_\pi = 391 \text{ MeV} \]
\( a_0 \) resonance - two channel region

S-wave \( \pi\eta-K\bar{K} \)

\[ m_\pi = 391 \text{ MeV} \]
a_0 resonance pole

\[ t_{ij} \sim \frac{c_i c_j}{s_0 - s} \]

\[
\sqrt{s_0} = \left( (1177 \pm 27) + \frac{i}{2} (49 \pm 33) \right) \text{ MeV}
\]

\[ |c_{\pi\eta}| = 652(130) \text{ MeV} \]
\[ |c_{K\bar{K}}| = 844(170) \text{ MeV} \]

\[ m_\pi = 391 \text{ MeV} \]
The $f_0(500)/\sigma$ resonance

elastic scattering with
vacuum quantum numbers
$\pi\pi$ in $I = 0, J = 0$

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The $f_0(500)/\sigma$ resonance

elastic scattering with vacuum quantum numbers $\pi\pi$ in $I = 0, J = 0$

$m_\pi = 236$ MeV

$m_\pi = 391$ MeV
The $f_0(500)/\sigma$ resonance
The $f_0(500)/\sigma$ resonance

$E_\sigma$ / MeV

$\frac{1}{2} \Gamma_\sigma$ / MeV

$\pi\pi_{\text{phys.}}$ $\pi\pi_{\text{thr.}}$ $\pi\pi_{\text{thr.}}$ $\pi\pi_{\text{thr.}}$ $E_\sigma$ / MeV

$m_\pi = 391$ MeV

$m_\pi = 236$ MeV
Future directions

two-body coupled-channel

\[ f_0(980) \]
\[ D \bar{D} \]
\[ D \bar{D}^* \]
\[ N \pi \]
\[ \gamma a \rightarrow bc \]

further operator structures - glueball, tetraquark, ...

formalism for three-body and beyond
- needed for higher energies
- needed to get closer to the physical mass
$m_\pi = 391$ MeV

Backup
Coupled-channel scattering

\[ \text{a}_0(980), \text{f}_0(980) \]
\[ \text{a}_1(1260) \]
\[ \text{X}(3872), \text{and other XYZ states} \]
\[ \text{N}^*(1440), \Lambda(1405), \ldots \]

all decay into multiple final states
all are resonant enhancements in multiple channels
to understand these rigorously, we need coupled-channel analyses
Coupled-channel scattering

Direct extension of the elastic quantization condition derived by Lüscher

\[ \text{det} \left[ 1 + i \rho(E) \cdot t(E) \cdot (1 + i \mathcal{M}(E, L)) \right] = 0 \]

Many derivations, **all in agreement**:

- He, Feng, Liu 2005 - two channel QM, strong coupling
- Hansen & Sharpe 2012 - field theory, multiple two-body channels
- Briceño & Davoudi 2012 - strongly-coupled Bethe-Salpeter amplitudes
- Guo et al 2012 - Hamiltonian & Lippmann-Schwinger

Also derivations in specific channels, or for a specific parameterization of the interactions like NREFT, chiral PT, Finite Volume Hamiltonian, etc.

- Briceño 2014 - Generalised to scattering of particles with non-zero spin, and spin-$^{1/2}$.

Significant steps towards a general 3-body quantization condition have been made.
Amplitude parameterization

t = \begin{pmatrix} \pi\pi \to \pi\pi & \pi\pi \to \bar{K}K \\ K\bar{K} \to \pi\pi & K\bar{K} \to K\bar{K} \end{pmatrix}

\[ \det \left[ 1 + i\rho(E) \cdot t(E) \cdot (1 + iM(E, L)) \right] = 0 \]
determinant condition:
- several unknowns at each value of energy
- energy levels typically do not coincide
- underconstrained problem for a single energy

one solution: use energy dependent parameterizations
- Constrained problem when \(\#(\text{energy levels}) > \#(\text{parameters})\)
- Essential amplitudes respect unitarity of the S-matrix

\[ S^\dagger S = 1 \quad \Rightarrow \quad \text{Im} \; t^{-1} = -\rho \quad \rho_{ij} = \delta_{ij} \frac{2k_i}{E_{cm}} \]

K-matrix approach:
\[ t^{-1} = K^{-1} - i\rho \quad \text{e.g.:} \; K_{ij} = \frac{g_i g_j}{m^2 - s} + \gamma_{ij} \]
ρ resonance pole near a pole:

\[ t_{ij} \sim \frac{c_i c_j}{s_0 - s} \]

\[
\rho \text{ resonance pole}
\]
Other calculations

**Coupled $\pi K - \eta K$**

Combined S & P-wave analysis
80 energy levels from 3 volumes
arXiv:1406.4158, PRL 113 (2014) no.18, 182001

**Coupled $D\pi - D\eta - D_s \bar{K}$**

Combined S & P-wave analysis
3 coupled channels in S-wave
47 energy levels from 3 volumes
arXiv:1607.07093

$m_\pi = 391$ MeV
The $f_0(500)/\sigma$ resonance

$m_\pi = 236$ MeV

$m_\pi = 391$ MeV

$p^2 / \text{GeV}^2$
The $f_0(500)/\sigma$ resonance

from J. R. Pelaez, arXiv:1510.00653

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Resonances in coupled-channel scattering

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The $f_0(500)/\sigma$ resonance

from J. R. Pelaez, arXiv:1510.00653
An $a_0$ resonance

$\pi \eta - K \bar{K} - \pi \eta'$

$m_\pi = 391$ MeV
An $a_0$ resonance

$\pi\eta - K\bar{K} - \pi\eta'$

$m_\pi = 391$ MeV
An $a_0$ resonance - three channel region

$m_\pi = 391$ MeV
Poles

$m_\pi = 391$ MeV
Poles

$\pi^+\pi^- = 391$ MeV

$m_\pi = 391$ MeV
Extracting resonance properties

build a large basis of operators: $\mathcal{O}^\dagger \sim \bar{\psi} \Gamma \overleftrightarrow{D} ... \overleftrightarrow{D} \psi$

compute large correlation matrices: $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) | 0 \rangle$

solve GEVP: $C_{ij}(t)v_n^j = \lambda_n(t, t_0)C_{ij}(t_0)v_n^j$

$m_\pi = 236$ MeV
Extracting resonance properties

add in $\pi \pi$ operators using a variationally optimal pion $\pi^\dagger = \sum \nu^\pi_i O^\dagger_i$

combine in pairs $(\pi \pi)^\dagger = \sum_{\vec{p}_1 + \vec{p}_2 = \vec{P}} C(\vec{p}_1, \vec{p}_2) \pi^\dagger(\vec{p}_1) \pi^\dagger(\vec{p}_2)$

$m_\pi = 236$ MeV
Extracting resonance properties

essential to have operators that overlap onto “meson” and “meson-meson” contributions to the physical spectrum

\[ [000] T_{1}^{-} \]

\[ m_{\pi} = 236 \text{ MeV} \]
Determinant

\[ t = (\pi\pi \rightarrow \pi\pi) \]

\[
\det \left[ 1 + i \rho \cdot t \cdot (1 + iM(L)) \right]
\]

\[ a_t E_{cm} \]

\[ \delta /^{\circ} \]

\[ \delta_{\pi\pi} \]

Real

Imag
$t = (\pi\pi \rightarrow \pi\pi)$
Determinant

\[
\begin{pmatrix}
\pi\pi \rightarrow \pi\pi & 0 \\
0 & K\bar{K} \rightarrow K\bar{K}
\end{pmatrix}
\]

\[
\operatorname{det} \left[ 1 + i \rho \cdot t \cdot (1 + i M(L)) \right]
\]

\[
\delta^{\pi\pi}
\]

\[
\delta^{KK}
\]

\[
\eta
\]
\[ t = \begin{pmatrix} \pi\pi \rightarrow \pi\pi & \pi\pi \rightarrow K\bar{K} \\ K\bar{K} \rightarrow \pi\pi & K\bar{K} \rightarrow K\bar{K} \end{pmatrix} \]

\[ \det \left[ 1 + i \rho \cdot t \cdot (1 + i M(L)) \right] \]

\[ \delta_{\pi\pi} \]

\[ \delta_{K\bar{K}} \]

\[ \eta \]

\[ a_t E_{cm} \]
Determinant

\[
\begin{bmatrix}
\pi\pi \rightarrow \pi\pi & \pi\pi \rightarrow K\bar{K} \\
K\bar{K} \rightarrow \pi\pi & K\bar{K} \rightarrow K\bar{K}
\end{bmatrix}
\]

\[
\delta = \arctan \left( \frac{\text{Im}[t]}{\text{Re}[t]} \right)
\]

\[
\eta = \frac{1}{\sqrt{1 + \left( \frac{\text{Im}[t]}{\text{Re}[t]} \right)^2}}
\]
Determinant

\[ t = \begin{pmatrix} \pi \pi \to \pi \pi & \pi \pi \to K\bar{K} \\ K\bar{K} \to \pi \pi & K\bar{K} \to K\bar{K} \end{pmatrix} \]

\[ \det [1 + i \rho \cdot \hat{t} \cdot (1 + i \mathcal{M}(L))] \]

\[ \delta^\pi \pi \]

\[ \delta^K \bar{K} \]

\[ \eta \]

\[ a_t E_{cm} \]

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Resonances in coupled-channel scattering
Chiral perturbation theory $\rho$ extrapolation

$m_\rho = 754(3)(19)_{01}^{19}\text{MeV}$

$\Gamma_\rho = 129(3)(17)_{01}^{17}\text{MeV}$

$m_\pi = 140 \text{ MeV}$

$\delta_1 / \phi$

$E_{\pi\pi}^* [\text{MeV}]$

Lattice QCD + U$\chi$PT

Protopopescu et al.

Estabrooks and Martin

$E_{\pi\pi}^* / \text{MeV}$

$\delta_1 / \phi$

$m_\pi = 236 \text{ MeV}$, fit

$m_\pi = 140 \text{ MeV}$, postdiction

$m_\pi = 391 \text{ MeV}$, postdiction