Extension of the CPT Theorem to non-Hermitian Hamiltonians and Unstable States

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1 Antilinearity and the reality of eigenvalues

Beginning in 1998 with the work of Bender and collaborators it was rigorously established that the eigenvalues of the manifestly non-Hermitian Hamiltonian $H = p^2 + ix^3$ were all real. This reality was traced to the existence of an underlying antilinear PT symmetry that H possessed. (H is invariant under $p \to p$, $x \to -x$, $i \to -i$).

To see the implications of an antilinear symmetry such as PT, consider the eigenvector equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle = E|\psi(t)\rangle. \tag{1}$$

On replacing the parameter t by $-t$ and then multiplying by some general antilinear operator A (i.e. not necessarily PT itself), we obtain

$$i\frac{\partial}{\partial t}A|\psi(-t)\rangle = AHA^{-1}A|\psi(-t)\rangle = E^*A|\psi(-t)\rangle. \tag{2}$$

If H has an antilinear symmetry so that $AHA^{-1} = H$, then, as first noted by Wigner in his study of time reversal invariance, energies can either be real and have eigenfunctions that obey $A|\psi(-t)\rangle = |\psi(t)\rangle$, or can appear in complex conjugate pairs that have conjugate eigenfunctions ($|\psi(t)\rangle \sim \exp(-iEt)$ and $A|\psi(-t)\rangle \sim \exp(-iE^*t)$). Hermiticity is only **SUFFICIENT** to secure real eigenvalues.

As to the converse, suppose we are given that the energy eigenvalues are real or appear in complex conjugate pairs. In such a case not only would E be an eigenvalue but E^* would be too. Hence, we can set $HA|\psi(-t)\rangle = E^*A|\psi(-t)\rangle$ in (2), and obtain

$$(AHA^{-1} - H)A|\psi(-t)\rangle = 0. \tag{3}$$

Then if the eigenstates of H are complete, (3) must hold for every eigenstate, to yield $AHA^{-1} = H$ as an operator identity, with H thus having an antilinear symmetry. Antilinearity is thus the **NECESSARY** condition for the reality of energy eigenvalues.

2 A Simple Example

The matrix

$$M = \begin{pmatrix} 1+i & s \\ s & 1-i \end{pmatrix} \tag{4}$$

with real s is PT symmetric (set $P = \sigma_1$ and $T = K$ where K denotes complex conjugation).

Even though this M is not Hermitian, its eigenvalues are given by

$$E_{\pm} = 1 \pm (s^2 - 1)^{1/2}, \tag{5}$$

and both of these eigenvalues are real if s is greater than one. Moreover, these eigenvalues come in complex conjugate pairs if s is less than one. In addition if $s = 1$ M is a non-diagonalizable Jordan-block Hamiltonian with only one eigenvector despite having two solutions to $M - \lambda I = 0$ (both with $\lambda = 1$), and cannot be diagonalized by a similarity transformation:

$$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6}$$

As well as see the generic pattern of eigenvalues, we also see that by varying parameters we can continue from one realization of antilinear symmetry to another, crossing through, and in fact necessarily crossing through, the Jordan-block case on the way, as the transition from all real eigenvalues to complex pairs must be singular. For both $s > 1$ and $s < 1$ M has a complete set of eigenvectors and can be diagonalized, with its diagonal form being Hermitian when $s > 1$. When $s > 1$ M is thus "Hermitian in disguise", with the utility of antilinear symmetry being that without it, it is guaranteed that a Hamiltonian is not Hermitian in disguise.

The Jordan-block situation is a case where the Hamiltonian is manifestly non-diagonalizable and thus manifestly non-Hermitian and yet all eigenvalues are real. While Hermiticity implies reality of eigenvalues, reality of eigenvalues does not imply Hermiticity or even Hermiticity in disguise. The conformal gravity theory with action $I_{\rm W} = -\alpha_g \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$ where $C^{\lambda\mu\nu\kappa}$ is the Weyl conformal tensor also falls into the Jordan-block category (Bender and Mannheim 2008, Mannheim 2011, Mannheim 2012), and is able to be ghost free and unitary at the quantum level because of it, to thus provide a fully consistent quantum theory of gravity without any of the string theory need for supersymmetry or extra spacetime dimensions.

3 Probability Conservation

Consider a right eigenstate of H in which H acts to the right as $i\partial_t|R(t)\rangle = H|R(t)\rangle$ with solution $|R(t)\rangle = \exp(-iHt)|R(0)\rangle$. The Dirac norm

$$\langle R(t)|R(t)\rangle = \langle R(0)|\exp(iH^\dagger t)\exp(-iHt)|R(0)\rangle \tag{7}$$

is not time independent if H is not Hermitian, and would not describe unitary time evolution. However, this only means that the Dirac norm is not unitary, not that no norm is unitary. Moreover, since $i\partial_t|R(t)\rangle = H|R(t)\rangle$ only involves ket vectors, there is some freedom in choosing bra vectors. So let us introduce a more general scalar product $\langle R(t)|V|R(t)\rangle$ with some as yet to be determined V, which we take to be time independent. We find

$$i\frac{\partial}{\partial t}\langle R_j(t)|V|R_i(t)\rangle = \langle R_j(t)|(VH - H^\dagger V)|R_i(t)\rangle. \tag{8}$$

Thus if we set $VH - H^\dagger V = 0$, then all scalar products will be time independent and probability is conserved (and V will indeed be time independent if H is). For the converse we note if we are given that all V scalar products are time independent, then if the set of all $|R_i(t)\rangle$ is complete we would obtain $VH - H^\dagger V = 0$ as an operator identity. The condition $VH - H^\dagger V = 0$ is thus both necessary and sufficient for the time independence of the V scalar products $\langle R(t)|V|R(t)\rangle$.

Since V obeys $VH - H^\dagger V = 0$, V depends on the particular Hamiltonian. Thus unlike the Dirac norm, now the theory dynamically determines its own norm each time. Just like general relativity ($g_{\mu\nu}$ metric determined dynamically) vis a vis special relativity (Minkowski $\eta_{\mu\nu}$ metric preassigned).

Now if $VH - H^\dagger V = 0$, we can set $VH|\psi\rangle = EV|\psi\rangle = H^\dagger V|\psi\rangle$. Consequently H and $H^\dagger$ have the same set of eigenvalues, i.e. for every E there is an E^*. (Also follows from $H^\dagger = VHV^{-1}$, an isospectral similarity transformation.) Energy eigenvalues are thus either real or in complex conjugate pairs. **Consequently, H must have an antilinear symmetry.**

For the two-dimensional matrix M above for instance, we have

$$M = \begin{pmatrix} 1+i & s \\ s & 1-i \end{pmatrix}, \qquad V = (s^2-1)^{-1/2}\begin{pmatrix} s & -i \\ i & s \end{pmatrix}, \qquad VMV^{-1} = M^{\dagger}.\ \text{V singular when s} = 1. \tag{9}$$

To reinforce the point we note that if $|R_i(t)\rangle$ is a right-eigenstate of H with energy eigenvalue $E_i = E_i^R + iE_i^I$, in general we can write

$$\langle R_j(t)|V|R_i(t)\rangle = \langle R_j(0)|V|R_i(0)\rangle e^{-i(E_i^R+iE_i^I)t+i(E_j^R-iE_j^I)t}. \tag{10}$$

Since V has been chosen so that the $\langle R_j(t)|V|R_i(t)\rangle$ scalar products are time independent, the only allowed non-zero norms are those that obey

$$E_i^R = E_j^R, \qquad E_i^I = -E_j^I, \tag{11}$$

with all other V-based scalar products having to obey $\langle R_j(0)|V|R_i(0)\rangle = 0$. We recognize (11) as being precisely none other than the requirement that eigenvalues be real or appear in complex conjugate pairs, with H thus possessing an antilinear symmetry.

Ordinarily in discussing decays one only keeps modes $e^{-i(E_R+iE_I)t}$ with negative imaginary part E_I. However now we keep both decaying and growing modes, with probability being conserved since the only transitions allowed by (11) are those in which the decaying mode couples to its growing partner, so that as the population of the decaying mode decreases, the population of the growing mode increases accordingly. Also with $U = e^{-iHt}$ obeying $U^{-1} = e^{iHt} = V^{-1}e^{iH^{\dagger}t}V = V^{-1}U^{\dagger}V$ unitarity is generalized to the non-Hermitian case.

With $VH - H^{\dagger}V = 0$ and $i\partial_t|R\rangle = H|R\rangle$, we obtain $-i\partial_t\langle R|V = \langle R|H^{\dagger}V = \langle R|VH$, and can thus identify left-eigenvectors $\langle L| = \langle R|V$, and can set $\langle R(t)|V|R(t)\rangle = \langle L(t)|R(t)\rangle = \langle L(0)|e^{-iHt}e^{iHt}|R(0)\rangle = \langle L(0)|R(0)\rangle$. Thus in the non-Hermitian case we should use the left-right norm, with this being the most general one possible if probability is to be conserved.

PROBALITY CONSERVATION IMPLIES ANTILINEARITY NOT HERMITICITY

4 Complex Lorentz Invariance

Is there any particular antilinear symmetry that might be preferred? If impose complex Lorentz invariance then CPT.
Can get real eigenvalues without assuming Hermiticity.
Can get probability conservation without assuming Hermiticity.
Can get CPT theorem without assuming Hermiticity.
Then if energies come in complex conjugate pairs, can describe decays and unstable states such as in K meson sector. In standard derivation of CPT theorem assume Hermiticity, so then no decays.

Lorentz transformations are of the form $\Lambda = e^{iw^{\mu\nu}M_{\mu\nu}}$ with six angles $w^{\mu\nu} = -w^{\nu\mu}$ and six Lorentz generators $M_{\mu\nu} = -M_{\nu\mu}$ that obey

$$[M_{\mu\nu}, M_{\rho\sigma}] \;=\; i(-\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} + \eta_{\nu\sigma}M_{\rho\mu}). \tag{12}$$

Under a Lorentz transformation the line element transforms as

$$x^\alpha \eta_{\alpha\beta} x^\beta \to x^\alpha \widetilde{\Lambda} \eta_{\alpha\beta} \Lambda x^\beta, \tag{13}$$

(tilde denotes transpose), with $\widetilde{\Lambda} = e^{iw^{\mu\nu}\widetilde{M_{\mu\nu}}}$. Given the Lorentz algebra one has $e^{iw^{\mu\nu}\widetilde{M_{\mu\nu}}}\eta_{\alpha\beta} = \eta_{\alpha\beta}e^{-iw^{\mu\nu}M_{\mu\nu}}$ (i.e. Minkowski metric orthogonal), with the line element thus being invariant. While this analysis familiarly holds for real $w^{\mu\nu}$, since $w^{\mu\nu}$ plays no explicit role in it, the analysis equally holds if $w^{\mu\nu}$ is **complex**.

For a general spin zero Lagrangian where $w^{\mu\nu}M_{\mu\nu}$ acts as $w^{\mu\nu}(x_\mu p_\nu - x_\nu p_\nu) = 2w^{\mu\nu}x_\mu p_\nu$, under an infinitesimal Lorentz transformation the action $I = \int d^4x L(x)$ transforms as

$$\delta I = 2w^{\mu\nu} \int d^4x x_\mu \partial_\nu L(x) = 2w^{\mu\nu} \int d^4x \partial_\nu [x_\mu L(x)], \tag{14}$$

to thus be a total derivative and thus be left invariant. However the change will be a total derivative even if $w^{\mu\nu}$ is complex. So again we see that we have invariance under **complex** Lorentz transformations.

For spinors ψ we cannot use $\psi^\dagger$ since we would then have to use $\Lambda^\dagger = e^{-i[w^{\mu\nu}]^* M^\dagger_{\mu\nu}}$. So instead use Majorana spinors in the Majorana basis of the Dirac gamma matrices, since in that basis they are self conjugate. In Grassmann space one has line element $\widetilde{\psi} C\psi$ where C effects $C\gamma^\mu C^{-1} = -\widetilde{\gamma^\mu}$ and thus $CM_{\mu\nu}C^{-1} = -\widetilde{M_{\mu\nu}}$. (In Majorana basis $C = \gamma^0$.) Thus under a Lorentz transformation we have

$$\widetilde{\psi} C\psi \to \tilde{\psi} e^{iw^{\mu\nu}\widetilde{M}_{\mu\nu}} C e^{iw^{\mu\nu}M_{\mu\nu}}\psi = \tilde{\psi} C e^{-iw^{\mu\nu}M_{\mu\nu}} e^{iw^{\mu\nu}M_{\mu\nu}}\psi = \widetilde{\psi} C\psi. \tag{15}$$

So once again we see that we have invariance under **complex** Lorentz transforms and not just under real ones.

Thus for a general Dirac spinor set $\psi = \psi_1 + i\psi_2$ with $\psi_1 = \psi_1^\dagger$, $\psi_2 = \psi_2^\dagger$. Then under charge conjugation we obtain $\hat{C}\psi_1\hat{C}^{-1} = \psi_1$, $\hat{C}\psi_2\hat{C}^{-1} = -\psi_2$. Then set $\psi^\dagger\gamma^0\psi = (\tilde{\psi}_1 - i\tilde{\psi}_2)C(\psi_1 + i\psi_2)$ and afterwards apply Lorentz transformation, to thus remain invariant under complex Lorentz transformations. In Majorana basis of Dirac gamma matrices $\hat{P}$, $\hat{T}$, and $\hat{C}\hat{P}\hat{T}$ implement

$$\begin{aligned} &\hat{P}\psi(\vec{x},t)\hat{P}^{-1} = \gamma^0\psi(-\vec{x},t), \qquad \hat{T}\psi(\vec{x},t)\hat{T}^{-1} = \gamma^1\gamma^2\gamma^3\psi(\vec{x},-t), \\ &\hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)]. \end{aligned} \tag{16}$$

a relation that will prove central to the discussion.

If $\hat{M}_{\mu\nu} = \hat{M}^\dagger_{\mu\nu}$ and $w^{\mu\nu}$ is complex, then $\Lambda^\dagger = e^{-i[w^{\mu\nu}]^* M_{\mu\nu}} \neq \hat{\Lambda}^{-1} = e^{-iw^{\mu\nu}M_{\mu\nu}}$, and Dirac norm $\langle R|R\rangle \to \langle R|\Lambda^\dagger\Lambda|R\rangle$ is not invariant under a complex Lorentz transform. Thus need some additional operator that will convert $\exp(-i[w^{\mu\nu}]^*\hat{M}_{\mu\nu})$ into $\exp(-iw^{\mu\nu}\hat{M}_{\mu\nu})$. It would have to be antilinear in order to complex conjugate $[w^{\mu\nu}]^*$. But since it would also conjugate the i factor it would at the same time have to convert $\hat{M}_{\mu\nu}$ into $-\hat{M}_{\mu\nu}$. Because of the factor i in the Lorentz algebra where $[\hat{M},\hat{M}] = i\hat{M}$, the operator that does this is precisely $\hat{C}\hat{P}\hat{T}$, with $\hat{C}\hat{P}\hat{T}\hat{M}_{\mu\nu}[\hat{C}\hat{P}\hat{T}]^{-1} = -\hat{M}_{\mu\nu}$. Consequently $\langle R|\hat{C}\hat{P}\hat{T}|R\rangle$ is invariant under complex Lorentz transformations when $\hat{M}_{\mu\nu}$ is Hermitian, while $\langle R|V\hat{C}\hat{P}\hat{T}|R\rangle$ is invariant when $V\hat{M}_{\mu\nu} = \hat{M}^\dagger_{\mu\nu}V$. This then is how one constructs matrix elements in general that are invariant under complex Lorentz transformations.

Complex Lorentz invariance is just as natural as real Lorentz invariance.

5 Relation of CPT to Complex Lorentz Transformations

On coordinates PT implements $x^\mu \to -x^\mu$, and thus so does CPT since the coordinates are charge conjugation even (i.e. unaffected by a charge conjugation transformation). With a boost in the x_1-direction implementing $x_1' = x_1\cosh\xi + t\sinh\xi$, $t' = t\cosh\xi + x_1\sinh\xi$, with $\xi = i\pi$ we obtain

$$\begin{aligned}
\Lambda^0{}_1(i\pi): &\qquad x_1 \to -x_1, \qquad t \to -t,\\
\Lambda^0{}_2(i\pi): &\qquad x_2 \to -x_2, \qquad t \to -t,\\
\Lambda^0{}_3(i\pi): &\qquad x_3 \to -x_3, \qquad t \to -t,\\
\pi\tau = \Lambda^0{}_3(i\pi)\Lambda^0{}_2(i\pi)\Lambda^0{}_1(i\pi): &\qquad x^\mu \to -x^\mu.
\end{aligned} \tag{17}$$

The complex $\pi\tau$ thus implements the linear part of a PT and thus CPT transformation on the coordinates.

With $\Lambda^0{}_i(i\pi)$ implementing $e^{-i\pi\gamma^0\gamma_i/2} = -i\gamma^0\gamma_i$ in the Dirac gamma matrix space, on introducing

$$\hat{\pi}\hat{\tau} = \hat{\Lambda}^0{}_3(i\pi)\hat{\Lambda}^0{}_2(i\pi)\hat{\Lambda}^0{}_1(i\pi) = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5, \tag{18}$$

we obtain

$$\hat{\pi}\hat{\tau}\psi_1(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_1(-x), \qquad \hat{\pi}\hat{\tau}\psi_2(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = \gamma^5\psi_2(-x). \tag{19}$$

Thus up to an overall complex phase, quite remarkably we recognize this transformation as acting as none other than the **linear** part of a CPT transformation since $\hat{C}\hat{P}\hat{T}[\psi_1(x) + i\psi_2(x)]\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} = i\gamma^5[\psi_1(-x) - i\psi_2(-x)]$. **Thus CPT is naturally associated with the complex Lorentz group**.

With the Lagrangian density $L(x)$ being spin zero, $\hat{\pi}\hat{\tau}$ effects $\hat{\pi}\hat{\tau}L(x)\hat{\tau}^{-1}\hat{\pi}^{-1} = L(-x)$ up to a phase. We will show below that the phase is one. Thus, with K denoting complex conjugation, when acting on a spin zero Lagrangian we can identify $\hat{C}\hat{P}\hat{T} = K\hat{\pi}\hat{\tau}$. On applying $\hat{\pi}\hat{\tau}$ we obtain

$$\begin{aligned}
\hat{C}\hat{P}\hat{T}\int d^4x L(x)\hat{T}^{-1}\hat{P}^{-1}\hat{C}^{-1} &= K\hat{\pi}\hat{\tau}\int d^4x L(x)\hat{\tau}^{-1}\hat{\pi}^{-1}K\\
&= K\int d^4x L(-x)K = K\int d^4x L(x)K = \int d^4x L^*(x).
\end{aligned} \tag{20}$$

Establishing the CPT theorem is thus reduced to showing that $L(x) = L^*(x)$.

6 CPT Theorem Without Hermiticity

	C	P	T	CP	CT	PT	CPT
$\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}\gamma^0\psi$	-	+	+	-	-	+	-
$\bar{\psi}\gamma^i\psi$	-	-	-	+	+	+	-
$\bar{\psi}\gamma^0\gamma^5\psi$	+	-	+	-	+	-	-
$\bar{\psi}\gamma^i\gamma^5\psi$	+	+	-	+	-	-	-
$\bar{\psi}i[\gamma^0,\gamma^i]\psi$	-	-	+	+	-	-	+
$\bar{\psi}i[\gamma^i,\gamma^j]\psi$	-	+	-	-	+	-	+
$\bar{\psi}[\gamma^0,\gamma^i]\gamma^5\psi$	-	+	-	-	+	-	+
$\bar{\psi}[\gamma^i,\gamma^j]\gamma^5\psi$	-	-	+	+	-	-	+

Table 1: C, P, and T assignments for fermion bilinears

	C	P	T	CP	CT	PT	CPT
$\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}\psi\bar{\psi}\psi$	+	+	+	+	+	+	+
$\bar{\psi}\psi\bar{\psi}i\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}i\gamma^5\psi\bar{\psi}i\gamma^5\psi$	+	+	+	+	+	+	+
$\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$	+	+	+	+	+	+	+
$\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\gamma^5\psi$	-	-	+	+	-	-	+
$\bar{\psi}\gamma^\mu\gamma^5\psi\bar{\psi}\gamma_\mu\gamma^5\psi$	+	+	+	+	+	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\psi\bar{\psi}i[\gamma_\mu,\gamma_\nu]\psi$	+	+	+	+	+	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\psi\bar{\psi}[\gamma_\mu,\gamma_\nu]\gamma^5\psi$	+	-	-	-	-	+	+
$\bar{\psi}i[\gamma^\mu,\gamma^\nu]\gamma^5\psi\bar{\psi}i[\gamma_\mu,\gamma_\nu]\gamma^5\psi$	+	+	+	+	+	+	+

Table 2: C, P, and T assignments for fermion bilinears and quadrilinears that have spin zero

CPT phase alternates with spin. All spin zero quantities have even CPT. Also all are real (Mannheim 2015).

Since probability conservation requires an antilinear symmetry, we have $K\int d^4xL(x)K = \int d^4xL^*(x) = \int d^4xL(x)$. Thus infer that all the numerical coefficients in $L(x)$ are real, that $L(x) = L^*(x)$, and that $\int d^4xL(x)$ is CPT invariant, **with the CPT theorem thus being extended to non-Hermitian Hamiltonians.**

7 Some Implications

(1) In the complex conjugate energy case time-independent transitions occur between decaying and growing states. A decay such as $K^+ \to \pi^+\pi^0$ can thus occur if the Hamiltonian has an antilinear symmetry, even though it would be forbidden if the Hamiltonian is Hermitian. Then the CPT theorem in the antilinear case ensures that its rate is equal to that of $K^- \to \pi^-\pi^0$. We thus extend the CPT theorem to unstable states.

(2) In those cases in which charge conjugation is separately conserved CPT reduces to PT, even if the Hamiltonian is not Hermitian. (Even for non-Hermitian Hamiltonians CPT plus C implies PT.) In such cases we recover the non-Hermitian PT program of Bender and collaborators, and thus put the PT symmetry program on a quite firm theoretical foundation.

(3) Our derivation of the CPT theorem leads to $L = L^*$ and thus to $H = H^*$. In contrast, in the standard derivation of the CPT theorem $H = H^\dagger$ is input. Here $H = H^*$ is output, with it being probability conservation plus complex Lorentz invariance that is input. Now in one of the standard derivations of the CPT theorem (see e.g. Weinberg Quantum Field Theory I) one notes that all spin zero multilinears are Hermitian. Then a Hermiticity assumption requires all numerical coefficients be real and the CPT theorem follows. Remarkably then, both types of derivation lead to the very same functional form for the action, with real numerical coefficients in each case. So how can we tell them apart.

(4) So consider as an example $I_{\rm S} = \int d^4x[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2]/2$ with Hamiltonian $H = \int d^3x[\dot{\phi}^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + m^2\phi^2]/2$. Solutions to the wave equation obey $\omega_{\vec{k}}^2 = \vec{k}^2 + m^2$, $\phi(\vec{x},t) = \sum[a(\vec{k})\exp(-i\omega_k t + i\vec{k}\cdot\vec{x}) + a^\dagger(\vec{k})\exp(+i\omega_k t - i\vec{k}\cdot\vec{x})]$, and the Hamiltonian is given by $H = \sum[\vec{k}^2 + m^2]^{1/2}[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})]/2$. If $m^2 > 0$ all energies are real, and both H and $\phi(\vec{x},t)$ are Hermitian. However, if $m^2 = -n^2 < 0$, now $\omega_{\vec{k}}^2 = \vec{k}^2 - n^2$, the $k < n$ energies come in complex conjugate pairs and neither H nor $\phi(\vec{x},t)$ is Hermitian. Despite this, the standard derivation of the CPT theorem would have identified $I_{\rm S} = \int d^4x[\partial_\mu\phi\partial^\mu\phi + n^2\phi^2]/2$ as being a Hermitian theory. **But it is not, and one cannot tell by inspection.** One needs to solve the theory and get the solutions first. Nonetheless, in both the $m^2 > 0$ and $m^2 < 0$ cases $\phi(\vec{x},t)$ is a CPT even field and H is CPT invariant (since m^2 is real), and that one can tell by inspection. Thus CPT symmetry is input, and H and $\phi(\vec{x},t)$ will only be Hermitian for certain values of parameters (reminiscent of our two-dimensional example where $E_\pm = 1 \pm (s^2-1)^{1/2}$).

Hermiticity never needs to be postulated, with it being output in those cases in which it is found to occur. Probability conservation and complex Lorentz invariance entail CPT invariance not Hermiticity.

Antilinear CPT symmetry thus has primacy over Hermiticity, and it is antilinearity that should be taken as the guiding principle for quantum theory.