# A precise first-principles computation of the HVP contribution to $(g-2)_\mu$

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RBC and UKQCD Collaborations

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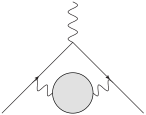
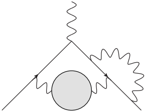
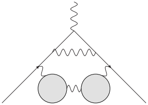
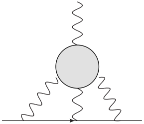
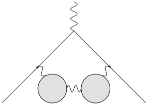
# Hadronic contributions to $a_\mu$

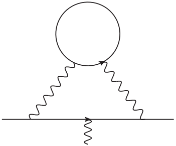
| Contribution | Value $\times 10^{10}$ | Uncertainty $\times 10^{10}$ |
|---|---:|---:|
| QED (5 loops) | 11 658 471.895 | 0.008 |
| EW | 15.4 | 0.1 |
| **HVP LO** | 692.3 | **4.2** |
| HVP NLO | -9.84 | 0.06 |
| HVP NNLO | 1.24 | 0.01 |
| **Hadronic light-by-light** | 10.5 | **2.6** |
| Total SM prediction | 11 659 181.5 | 4.9 |
| BNL E821 result | 11 659 209.1 | 6.3 |
| Fermilab E989 target | | $\approx$ **1.6** |

A reduction of uncertainty for HVP and HLbL is needed. For HLbL only model estimations exist. $\Rightarrow$ First-principles non-perturbative determination desired.

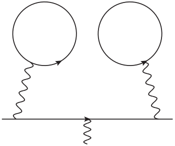
# Classification of hadronic contributions

| HVP LO | HVP NLO | HVP NNLO | HLbL |
|---|---|---|---|
|  |  |  |  |
| |  | ... | Talk by L. Jin |
| | ... | | |

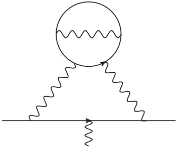
# The lattice approach to HVP LO



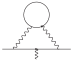
Quark-connected piece with $> 90\%$ of the contribution with by far dominant part from up and down quark loops (Below focus on light contribution only)



Quark-disconnected piece with $\approx 1.5\%$ of the contribution (1/5 suppression already through charge factors); Phys.Rev.Lett. 116 (2016) 232002



QED and isospin-breaking corrections, estimated at the few-per-cent level; work in progress

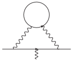


## HVP quark-connected contribution

Biggest challenge to direct calculation at physical point is to control statistics and potentially large finite-volume errors

Statistics: for strange and charm solved issue, for up and down quarks existing methodology less effective

Finite-volume errors are exponentially suppressed in the simulation volume but seem to be sizeable in QCD boxes with $m_\pi L = 4$; (Potentially O(10%) Aubin et al. 2015; for our calculation likely O(3%), correction with FV ChPT may reduce this to $O(0.5\%)$)



## HVP quark-connected contribution

Starting from

$$\sum_x e^{iqx}\langle J_\mu(x)J_\nu(0)\rangle = (\delta_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2) \qquad (1)$$

with vector current $J_\mu(x) = i\sum_f Q_f\overline{\Psi}_f(x)\gamma_\mu\Psi_f(x)$ and using the subtraction prescription of Bernecker-Meyer 2011

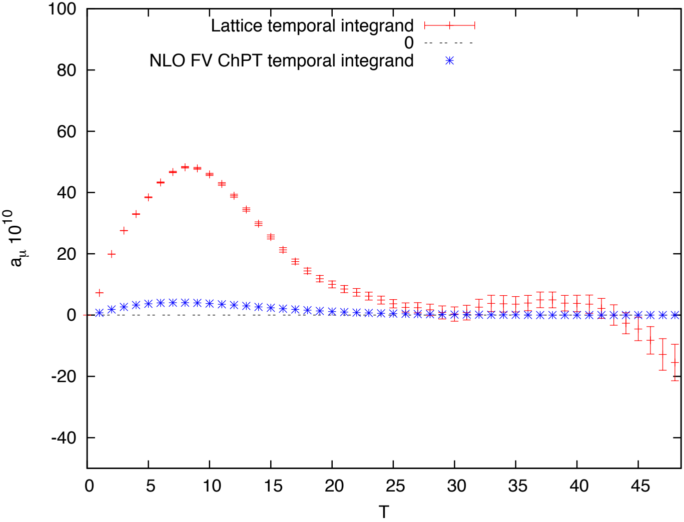
$$\Pi(q^2) - \Pi(q^2=0) = \sum_t \left(\frac{\cos(qt)-1}{q^2} + \frac{1}{2}t^2\right) C(t) \qquad (2)$$

with $C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle$ we may write

$$a_\mu^{\mathrm{HVP}} = \sum_{t=0}^{\infty} w_t C(t)\,, \qquad (3)$$

where $w_t$ captures the QED part of the diagram.

Integrand $w_T C(T)$ for the light-quark connected contribution:

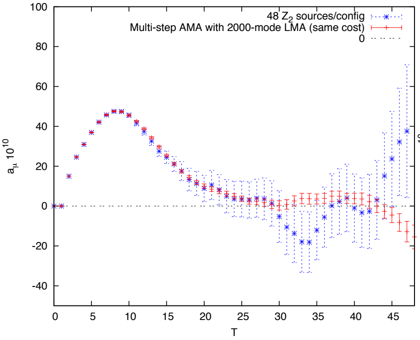


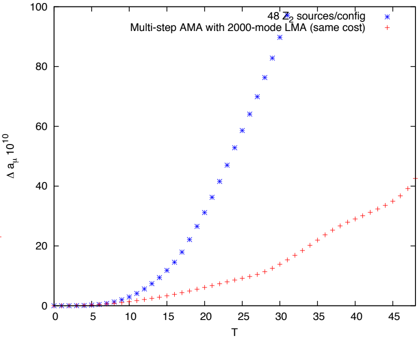

$m_\pi = 140$ MeV, $a = 0.11$ fm (RBC/UKQCD $48^3$ ensemble)

Statistical noise from long-distance region

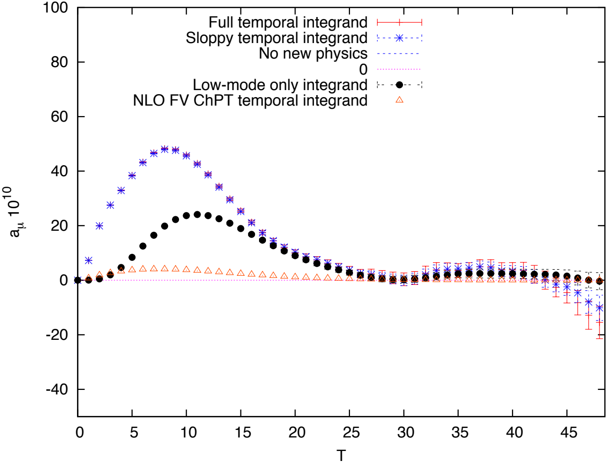
Approaches to the long-distance noise problem:

- Only use short-distance lattice data ($\lesssim$ 0.5fm–1.5fm), beyond that multi-exponentials from fit; our experiments suggest that from data $\lesssim$ 1.5 fm true error is difficult to estimate
- RBC in progress: improved stochastic estimator



Low-mode saturation for physical pion mass (here 2000 modes):



# Complete first-principles analysis

- Currently our statistical uncertainty for a pure first-principles analysis in the continuum limit is at the $\Delta a_\mu \approx 15 \times 10^{-10}$ level

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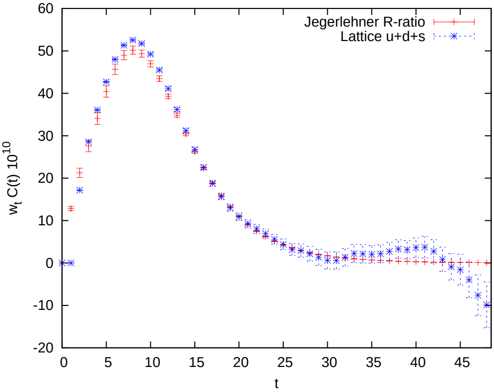
- Sub-percent statistical error likely achievable later this year
- While we are waiting for more statistics ...

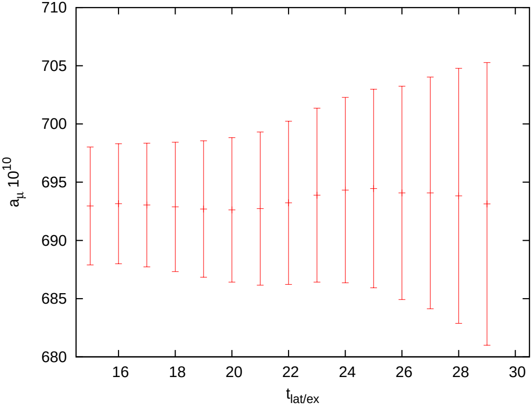
# Combined lattice and dispersive analysis

We can use the dispersion relation to overlay experimental $e^+e^-$ scattering data (Bernecker, Meyer 2011). Below the experimental result is taken from Jegerlehner 2016:




The lattice data here includes finite-volume corrections based on NLO FV ChPT. Continuum limit, charm contribution, and QED/IB correction missing. Work done with T. Izubuchi.

The lattice data is precise at shorter distances and the experimental data is precise at longer distances. We can do a combined analysis with lattice and experimental data:
$a_\mu = \sum_{t=0}^{t_{\mathrm{lat/ex}}} w_t C^{\mathrm{lattice}}(t) + \sum_{t=t_{\mathrm{lat/ex}}+1}^{\infty} w_t C^{\mathrm{exp}}(t)$
As expected a nice plateau region as a function of $t_{\mathrm{lat/ex}}$ is visible.

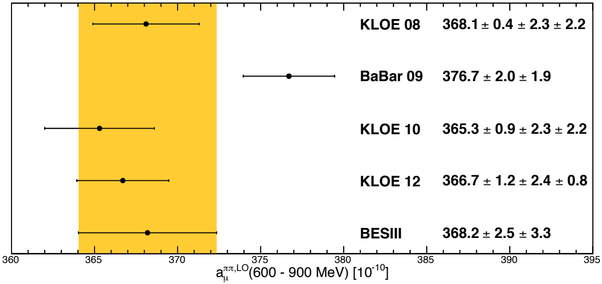



If we took $t_{\mathrm{lat/ex}} = 15a \approx 1.7$ fm, we currently have a statistical error of 0.7%

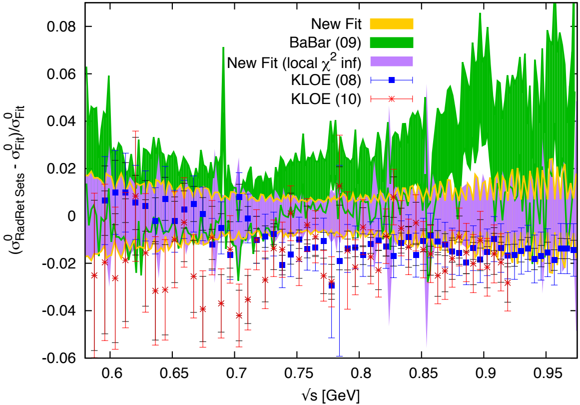
$$a_\mu^{\mathrm{HVP},u,d,s} = 693(5) \times 10^{-10} \, .$$

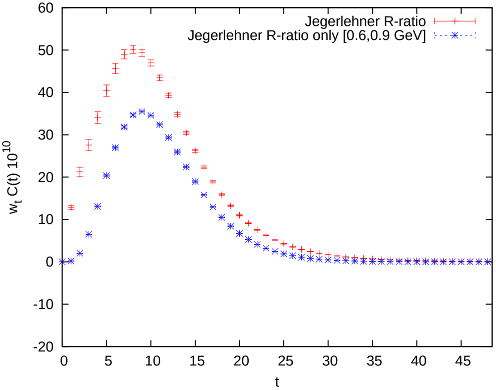
Continuum limit, charm contribution, and QED/IB correction missing. This is a promising way to reduce the overall uncertainty on a short timescale.

## BESIII 2015 update:

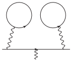


Hagiwara et al. 2011:






Problematic experimental region can readily be replaced by precise lattice data. Lattice also can be arbiter regarding different experimental data sets.



## HVP quark-disconnected contribution

First results at physical pion mass with a statistical signal
Phys.Rev.Lett. 116 (2016) 232002

Statistics is clearly the bottleneck; calculation was a potential road-block of a first-principles calculation for a long time; due to very large pion-mass dependence calculation at physical pion mass is crucial.

New stochastic estimator allowed us to get result

$$a_\mu^{\mathrm{HVP\ (LO)\ DISC}} = -9.6(3.3)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-10} \tag{4}$$

from 20 configurations at physical pion mass and 45 propagators/configuration.

Our setup:

$$C(t) = \frac{1}{3V} \sum_{j=0,1,2} \sum_{t'} \langle \mathcal{V}_j(t+t') \mathcal{V}_j(t') \rangle_{\mathrm{SU(3)}} \tag{5}$$

where $V$ stands for the four-dimensional lattice volume, $\mathcal{V}_\mu = (1/3)(\mathcal{V}_\mu^{u/d} - \mathcal{V}_\mu^s)$, and

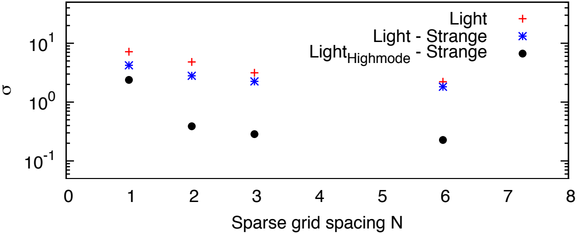
$$\mathcal{V}_\mu^f(t) = \sum_{\vec{x}} \mathrm{Im}\,\mathrm{Tr}[D^{-1}_{\vec{x},t;\vec{x},t}(m_f)\gamma_\mu]\,. \tag{6}$$

We separate 2000 low modes (up to around $m_s$) from light quark propagator as $D^{-1} = \sum_n v^n (w^n)^\dagger + D^{-1}_{\mathrm{high}}$ and estimate the high mode stochastically and the low modes as a full volume average Foley 2005.

We use a sparse grid for the high modes similar to Li 2010 which has support only for points $x_\mu$ with $(x_\mu - x_\mu^{(0)})$ mod $N = 0$; here we additionally use a random grid offset $x_\mu^{(0)}$ per sample allowing us to stochastically project to momenta.

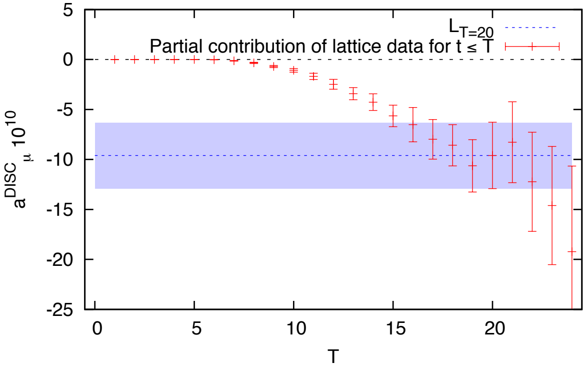
Combination of both ideas is crucial for noise reduction at physical pion mass!

Fluctuation of $\mathcal{V}_\mu$ ($\sigma$):




Since $C(t)$ is the autocorrelator of $\mathcal{V}_\mu$, we can create a stochastic estimator whose noise is potentially reduced linearly in the number of random samples, hence the normalization in the lower panel

Study $L_T = \sum_{t=T+1}^{\infty} w_t C(t)$ and use value of $T$ in plateau region (here $T = 20$) as central value. Use a combined estimate of a resonance model and the two-pion tail to estimateas systematic uncertainty.




Combined with an estimate of discretization errors, we find

$$a_\mu^{\mathrm{HVP\ (LO)\ DISC}} = -9.6(3.3)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-10} \,. \qquad (7)$$

Status and prospects:

- Improved statistical estimators both for connected light and disconnected contributions at physical point.
- For the connected light contribution our method reduces noise in the long-distance part of the correlator by an order of magnitude compared to same-cost $Z_2$ sources.
- For the disconnected contributions our method allowed for a precise calculation at physical pion mass with a total of 45 propagators on 20 configurations.

- Disconnected HVP contribution at physical point calculated at $\Delta a_\mu^{\mathrm{HVP,\ disc}} = 4 \times 10^{-10}$ (BNL E821 has $\Delta a_\mu = 6.3 \times 10^{-10}$, FNAL E989 aims at $\Delta a_\mu = 1.6 \times 10^{-10}$). Further improvements with our methodology straightforward.

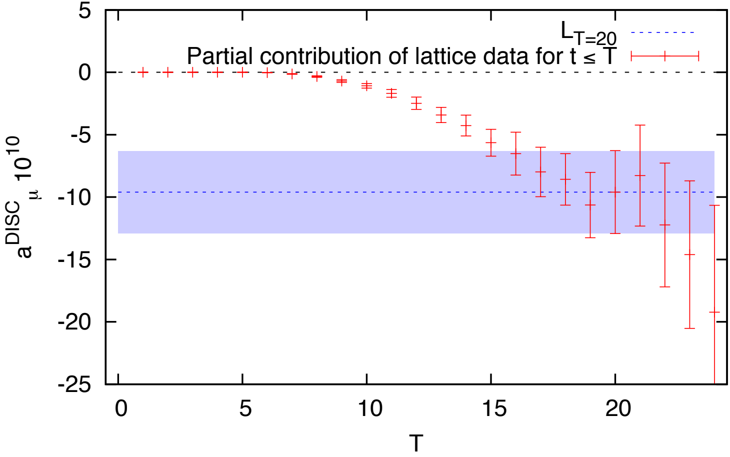
- A combined analysis of lattice and R-ratio data yields a very precise result with current data (0.7% statistical uncertainty); sub-percent continuum limit lattice-only result is also within reach.

- Continuum limit, charm contribution, QED/isospin-breaking corrections are still missing.

- Finite-volume behavior so far is consistent with NLO FV ChPT prediction within errors, designated study of fixed $T$ and different spatial volume in progress.

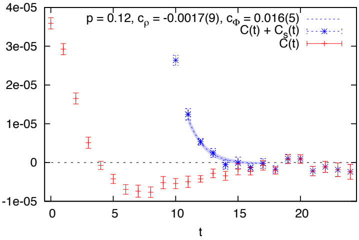
# Thank you

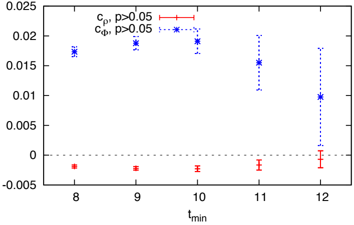
Result for partial sum $L_T = \sum_{t=0}^{T} w_t C(t)$:




For $t \geq 15$ $C(t)$ is consistent with zero but the stochastic noise is $t$-independent and $w_t \propto t^4$ such that it is difficult to identify a plateau region based only on this plot

Resulting correlators and fit of $C(t) + C_s(t)$ to $c_\rho e^{-E_\rho t} + c_\phi e^{-E_\phi t}$ in the region $t \in [t_{\min}, \ldots, 17]$ with fixed energies $E_\rho = 770$ MeV and $E_\phi = 1020$. $C_s(t)$ is the strange connected correlator.

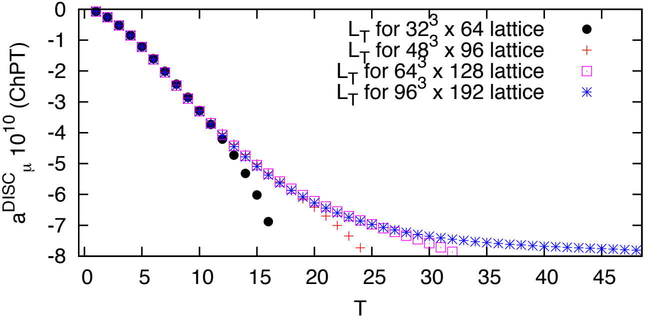



We fit to $C(t) + C_s(t)$ instead of $C(t)$ since the former has a spectral representation.

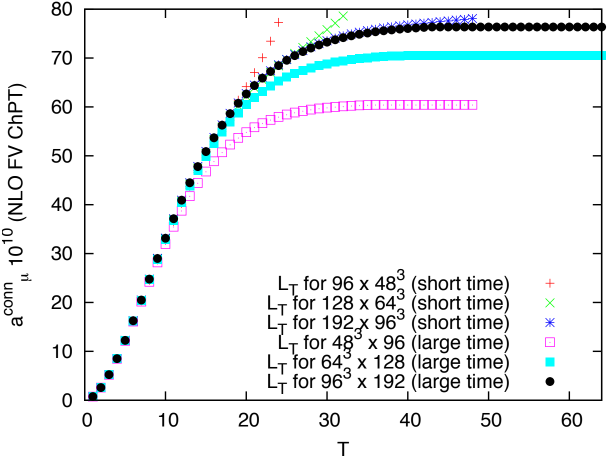
We could use this model alone for the long-distance tail to help identify a plateau but it would miss the two-pion tail

We therefore additionally calculate the two-pion tail for the disconnected diagram in ChPT:



A closer look at the NLO FV ChPT prediction (1-loop sQED):

We show the partial sum $\sum_{t=0}^{T} w_t C(t)$ for different geometries and volumes:



# The dispersive approach to HVP LO

The dispersion relation

$$\begin{aligned} \Pi_{\mu\nu}(q) &= i\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2) \\ \Pi(q^2) &= -\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\frac{\mathrm{Im}\Pi(s)}{q^2-s}\,. \end{aligned}$$

allows for the determination of $a_\mu^{\mathrm{HVP}}$ from experimental data via

$$a_\mu^{\mathrm{HVP\ LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2\left[\int_{4m_\pi^2}^{E_0^2} ds\,\frac{R_\gamma^{\mathrm{exp}}(s)\hat{K}(s)}{s^2} + \int_{E_0^2}^{\infty} ds\,\frac{R_\gamma^{\mathrm{pQCD}}(s)\hat{K}(s)}{s^2}\right],$$

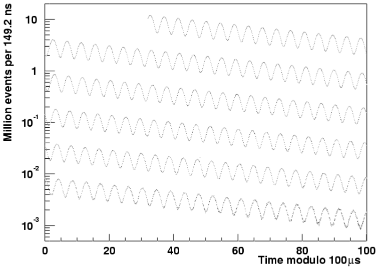
$$R_\gamma(s) = \sigma^{(0)}(e^+e^- \to \gamma^* \to \mathrm{hadrons})/\frac{4\pi\alpha^2}{3s}$$

Experimentally with or without additional hard photon (ISR: $e^+e^- \to \gamma^*(\to \mathrm{hadrons})\gamma$)

Experimental setup: muon storage ring with tuned momentum of muons to cancel leading coupling to electric field

$$\vec{\omega}_a = -\frac{q}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right]$$

Because of parity violation in weak decay of muon, a correlation between muon spin and decay electron direction exists, which can be used to measure the anomalous precession frequency $\omega_a$:

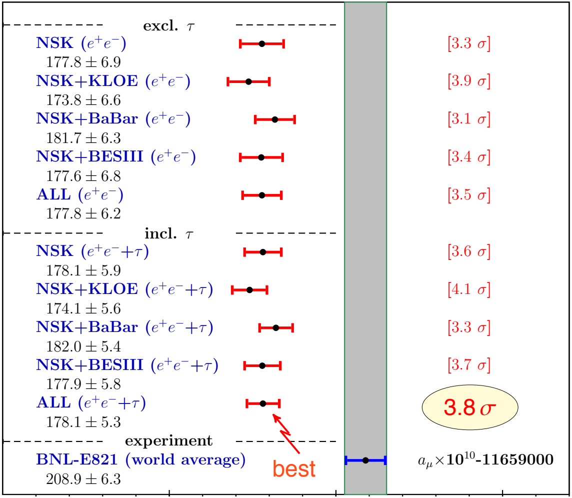


# Jegerlehner FCCP2015 summary:

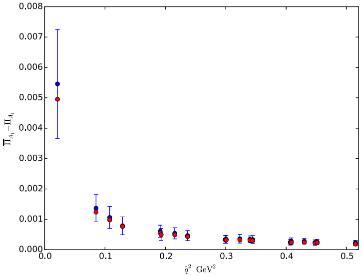
| final state | range (GeV) | $a_\mu^{\mathrm{had}(1)} \times 10^{10}$ (stat) (syst) [tot] | rel | abs |
|---|---|---|---|---|
| $\rho$ | ( 0.28, 1.05) | 507.55 ( 0.39) ( 2.68)[ 2.71] | 0.5% | 39.9% |
| $\omega$ | ( 0.42, 0.81) | 35.23 ( 0.42) ( 0.95)[ 1.04] | 3.0% | 5.9% |
| $\phi$ | ( 1.00, 1.04) | 34.31 ( 0.48) ( 0.79)[ 0.92] | 2.7% | 4.7% |
| $J/\psi$ | | 8.94 ( 0.42) ( 0.41)[ 0.59] | 6.6% | 1.9% |
| $\Upsilon$ | | 0.11 ( 0.00) ( 0.01)[ 0.01] | 6.8% | 0.0% |
| had | ( 1.05, 2.00) | 60.45 ( 0.21) ( 2.80)[ 2.80] | 4.6% | 42.9% |
| had | ( 2.00, 3.10) | 21.63 ( 0.12) ( 0.92)[ 0.93] | 4.3% | 4.7% |
| had | ( 3.10, 3.60) | 3.77 ( 0.03) ( 0.10)[ 0.10] | 2.8% | 0.1% |
| had | ( 3.60, 9.46) | 13.77 ( 0.04) ( 0.01)[ 0.04] | 0.3% | 0.0% |
| had | ( 9.46,13.00) | 1.28 ( 0.01) ( 0.07)[ 0.07] | 5.4% | 0.0% |
| pQCD | (13.0,$\infty$) | 1.53 ( 0.00) ( 0.00)[ 0.00] | 0.0% | 0.0% |
| data | ( 0.28,13.00) | 687.06 ( 0.89) ( 4.19)[ 4.28] | 0.6% | 0.0% |
| total | | 688.59 ( 0.89) ( 4.19)[ 4.28] | 0.6% | 100.0% |

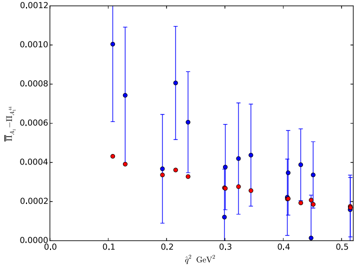
Results for $a_\mu^{\mathrm{had}(1)} \times 10^{10}$. Update August 2015, incl SCAN[NSK]+ISR[KLOE10,KLOE12,BaBar,BESIII]

# Jegerlehner FCCP2015 summary ($\tau \leftrightarrow e^+e^-$):



# From Aubin et al. 2015 (arXiv:1512.07555v2)



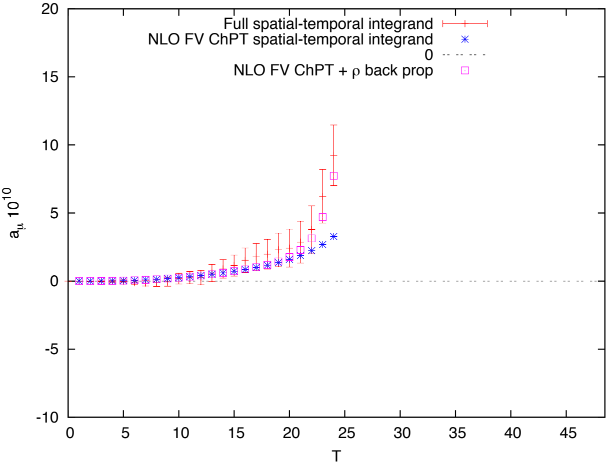




MILC lattice data with $m_\pi L = 4.2$, $m_\pi \approx 220$ MeV; Plot difference of $\Pi(q^2)$ from different irreps of 90-degree rotation symmetry of spatial components versus NLO FV ChPT prediction (red dots)

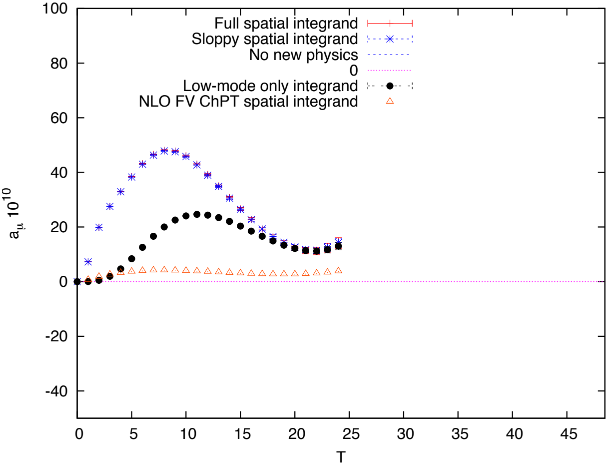
While the absolute value of $a_\mu$ is poorly described by the two-pion contribution, the volume dependence may be described sufficiently well to use ChPT to control FV errors at the 1% level; this needs further scrutiny

Aubin et al. find an $O(10\%)$ finite-volume error for $m_\pi L = 4.2$ based on the $A_1 - A_1^{44}$ difference (right-hand plot)

Compare difference of integrand of $48 \times 48 \times 96 \times 48$ (spatial) and $48 \times 48 \times 48 \times 96$ (temporal) geometries with NLO FV ChPT $(A_1 - A_1^{44})$:



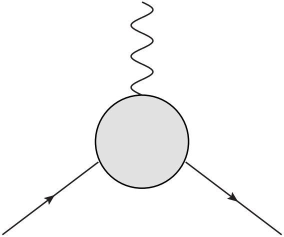

$m_\pi = 140$ MeV, $p^2 = m_\pi^2/(4\pi f_\pi)^2 \approx 0.7\%$



Full spatial integrand
Sloppy spatial integrand
No new physics
0
Low-mode only integrand
NLO FV ChPT spatial integrand
100
80
60
40
20
0
-20
-40
0
5
10
15
20
25
30
35
40
45
$a_\mu\ 10^{10}$
T

# The anomalous magnetic moment

The anomalous magnetic moment $a$ can be expressed in terms of scattering of particle off a classical photon background



For external photon index $\mu$ with momentum $q$ the scattering amplitude can be generally written as

$$(-ie)\left[\gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q^\nu}{2m}F_2(q^2)\right] \tag{8}$$

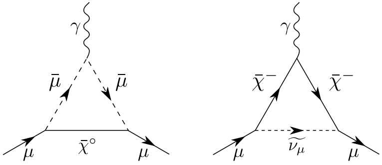
with $F_2(0) = a$.

# The muon anomalous magnetic moment

The muon anomalous magnetic moment promises to be useful to discover new physics beyond the standard model (SM) of particle physics.

In general, new physics contributions to $a_\ell$ are given by $a_\ell - a_\ell^{\mathrm{SM}} \propto (m_\ell^2/\Lambda_{\mathrm{NP}}^2)$ for lepton $\ell = e, \mu, \tau$ and new physics scale $\Lambda_{\mathrm{NP}}$.

With $\ell = \tau$ being experimentally inaccessible, $\ell = \mu$ promises good sensitivity to new physics.




Example contributions: one-loop MSSM neutralino/smuon and chargino/sneutrino contributions to $a_\mu$