Lattice Quantum Chromodynamics and Quark-Gluon Plasma

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QCD at high temperature and density

Elements of finite-temperature field theory

Lattice QCD

Some recent results

Related talks

Conclusion
QCD phase diagram

- Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase

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Study response of the system to change of external parameters, i.e. temperature and baryon density, asymptotic freedom suggests a weakly interacting phase\(^1\)

- Experimental program: RHIC, LHC, FAIR, NICA
- Hadronic phase: confinement, chiral symmetry breaking
- QGP phase: deconfinement, restoration of chiral symmetry

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Quantum Chromodynamics

- Minkowski → Euclidean space-time: $t \rightarrow -i\tau$

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Quantum Chromodynamics

- Minkowski → Euclidean space-time: \( t \rightarrow -i\tau \)
- The QCD Lagrangian:

\[
L_{QCD}^E = L_{gluon}^E + L_{fermion}^E = -\frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a - \sum_{f=u,d,s...} \bar{\psi}_f^\alpha(x) \left( D_{\alpha\beta}^E + m_f \delta_{\alpha\beta} \right) \psi_f^\beta(x)
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$$= -\frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}$$

$$- \sum_{f=u,d,s\ldots} \bar{\psi}^\alpha_f(x) \left( D^E_{\alpha\beta} + m_f \delta_{\alpha\beta} \right) \psi^\beta_f(x)$$

- The field strength tensor $F^a_{\mu\nu}$:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

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Quantum Chromodynamics

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$$- \sum_{f=u,d,s...} \bar{\psi}_f^\alpha(x) \left( \Phi^E_{\alpha\beta} + m_f \delta_{\alpha\beta} \right) \psi_f^\beta(x)$$

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$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

- The covariant derivative $\Phi^E$:

$$\Phi^E = \gamma^E_\mu D^E_\mu = \left( \partial_\mu + ig \frac{\lambda^a}{2} A^a_\mu \right) \gamma^E_\mu$$

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Path-integral representation

- The grand canonical partition function of QCD:

\[ Z(T, V, \bar{\mu}) = \int \prod_{\mu} D A_{\mu} \prod_{f=u,d,s...} D \psi_f D \bar{\psi}_f e^{-S_E(T, V, \bar{\mu})} \]

- The Euclidean action

\[ S_E(T, V, \bar{\mu}) \equiv -\frac{1}{T} \int_0^{1/T} d\chi_0 \int_V d^3x \mathcal{L}^E(\bar{\mu}) \]
Path-integral representation

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S_E(T, V, \mu) \equiv -\frac{1}{T} \int_0^{\tau} d\tau \int d^3 x L^E(\mu)
\]

- Quark chemical potentials:

\[
L^E(\mu) = L^E_{QCD} + \sum_{f=u,d,s...} \mu_f \bar{\psi}_f \gamma_0 \psi_f
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Path-integral representation

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\[ \mathcal{L}^E(\vec{\mu}) = \mathcal{L}_{QCD}^E + \sum_{f=u,d,s} \mu_f \bar{\psi}_f \gamma_0 \psi_f \]

- The expectation value of a physical observable \( \mathcal{O} \):

\[ \langle \mathcal{O} \rangle = \frac{1}{Z(T, V, \vec{\mu})} \int \prod_\mu DA_\mu \prod_f D\psi_f D\bar{\psi}_f \mathcal{O} \ e^{-S_E(T, V, \vec{\mu})} \]
Strong coupling constant

- If there is a small parameter (coupling constant) – we can write $\langle O \rangle$ as a series expansion (e.g. works in QED, $\alpha \sim 1/137$) and evaluate it order by order.
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- In QCD the coupling constant is large in the region of interest (i.e. on the energy scales of few hundred MeV)
Lattice QCD
Lattice gauge theory

- Lattice gauge theory\(^3\) is not an approximation of QFT, but a non-perturbative regularization scheme of QFT

\(^3\)Wilson (1974)
Lattice gauge theory

- Lattice gauge theory\(^3\) is not an approximation of QFT, but a non-perturbative regularization scheme of QFT.
- Discrete space-time domain \(N_s^3 \times N_T\) (figure from Lepage, hep-lat/0506036):

\[ \text{site} \rightarrow \bullet \bullet \bullet \bullet \]
\[ \text{link} \rightarrow \bullet \bullet \bullet \bullet \]

\(N_T\)

\(a\)

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Lattice gauge theory

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- Discrete space-time domain \(N_s^3 \times N_\tau\) (figure from Lepage, hep-lat/0506036):

- Gauge links:

\[
U_{x,\hat{\mu}} \equiv \mathcal{P} \exp \left\{ ig \int_x^{x+a\hat{\mu}} dy \nu A_\nu(y) \right\} \approx \exp \left\{ iagA_\mu(x) \right\}
\]

- Lattice spacing \(a\) provides an **ultra-violet cutoff**: momenta are restricted up to \(\sim \pi/a\)

\(^3\)Wilson (1974)
Possible terms in the action$^4$:

\begin{align*}
&U_x(x,y) \\
&U_y(x,y) \\
&U_x(x,y+3) \\
&U_y(x+4,y)
\end{align*}

\textsuperscript{4}figure from Gupta, hep-lat/9807028
The lattice regularized partition function:

\[
\mathcal{Z}(T, V, \vec{\mu}) = \int \prod_{x, \hat{\mu}} \text{d}U_{x, \hat{\mu}} \prod_{x, f} \text{d}\psi_{x, f} \text{d}\bar{\psi}_{x, f} \ e^{-S_f - S_g}
\]

\[
= \int \prod_{x, \hat{\mu}} \text{d}U_{x, \hat{\mu}} \prod_{f} \text{det}M_f(\mu_f) \ e^{-S_g}
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Fermions can be integrated out because they enter as bilinears:

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Lattice QCD

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\[ S_f \sim \sum_{x, y} \bar{\psi}_x M_{x, y}[U] \psi_y \]

If \( \prod_f \det M_f(\mu_f) \) is neglected this is called quenched approximation.
Fermions are notoriously difficult to deal with:

- There is fermion doubling problem (more species on the lattice than in the original theory).
- The determinant $\det M_f$ makes the action non-local and global updating algorithms (such as molecular dynamics) need to be used in conjunction with iterative solvers for inverting $M_f$.
- Fermion discretizations may not respect the symmetries of the original theory.

Various fermion discretization schemes currently in use:

- Staggered: preserve a part of the chiral symmetry, computationally cheap, require taking 4-th root of the Dirac operator.
- Wilson: explicitly break chiral symmetry (additive mass renormalization).
- Domain-wall: amount of symmetry breaking is controlled by the fifth dimension, exact in $L_s \to \infty$ limit.
- Overlap: exact chiral symmetry.
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Lattice QCD: example

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- We have $32^3 \times 8 \times 4 \times 18 \approx 2 \times 10^7$ degrees of freedom (over whose values we need to integrate)

A smarter way to integrate – importance sampling: randomly choose those field configurations ($\{U_i\}$) that contribute the most into the path integral and then approximate:

$$\langle O \rangle \approx \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} O(\{U_i\})$$
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Markov Chain Monte Carlo

- The problem of evaluating integrals (or sums) of high dimension is quite common across many fields.
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- Consider a quantity in the canonical ensemble:

\[
\langle O \rangle = \frac{1}{Z} \sum_{\{q\}} O(\{q\}) e^{-\beta E(\{q\})}
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Rewrite using the state probabilities:

$$\langle O \rangle = \sum_{\{q\}} O(\{q\}) P(\{q\})$$

$$P(\{q\}) = \frac{1}{Z} e^{-\beta E(\{q\})}$$

We could sample the states with highest probabilities, but that requires evaluating the partition function $Z$. 
Metropolis algorithm\(^5\)

▶ Given a particular state \(\{q\}\) of the system, propose a new state \(\{q'\}\)

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Metropolis algorithm\(^5\)

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- Accept it with the probability:

\[
P = \min \left\{ 1, \frac{P(\{q'\})}{P(\{q\})} \right\} = \min \left\{ 1, \frac{e^{-\beta E(\{q'\})}}{e^{-\beta E(\{q\})}} \right\}
\]

- Notice: \(Z\) drops out!

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Metropolis algorithm\textsuperscript{5}

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\]

▶ Notice: \(Z\) drops out!

▶ One can prove that the sequence of states produced with this Markov Chain Monte Carlo process converges to the canonical ensemble

▶ The essential technical step is to develop efficient algorithms for the transition \(\{q\} \rightarrow \{q'\}\)

▶ The main algorithm in use for dynamical fermions is Rational Hybrid Monte Carlo (RHMC)

\textsuperscript{5}Metropolis et al. (1953)
Markov Chain Monte Carlo is applicable when the action is real (so the weight factor $e^{-S}$ has probabilistic meaning).
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This is not the case in several physically relevant cases, such as QCD at non-zero baryon density and the Hubbard model away from half-filling
“Sign problem”

- Markov Chain Monte Carlo is applicable when the action is real (so the weight factor $e^{-S}$ has probabilistic meaning).
- This is not the case in several physically relevant cases, such as QCD at non-zero baryon density and the Hubbard model away from half-filling.
- There are several approaches to deal with the sign problem in QCD (limited to not too large values of $\mu$):
  - Taylor expansion in $\mu/T$
  - Simulations at imaginary chemical potential $i\mu$
  - Reweighting
Some recent results
Chiral symmetry restoration

- Spontaneous breaking of the global $SU(N_f)_L \times SU(N_f)_R$ symmetry in massless QCD leads to non-zero quark condensate in the hadronic phase

$$\langle \bar{\psi} \psi \rangle_f = \frac{T}{V} \frac{\partial \ln Z}{\partial m_f}$$

$$\chi_{m,l}(T) = \frac{\partial \langle \bar{\psi} \psi \rangle_l}{\partial m_l}$$

- The chiral crossover temperature $T_c = 154 \pm 9$ MeV
Deconfinement: Polyakov loop

- The expectation value of the (renormalized) Polyakov loop:

\[ L_{\text{ren}}(T) = e^{-c(g^2)N_\tau} \cdot \frac{1}{V_N c} \sum_{\vec{x}} \left\langle \text{Tr} \prod_{x_0=1}^{N_\tau} U(x_0,\vec{x}),\hat{0} \right\rangle \]

- Related to the free energy of static quark anti-quark pair

\[ L_{\text{ren}}(T) = \exp(-F_\infty(T)/2T) \]
Fluctuations of conserved charges

▶ Pressure

\[
\frac{P}{T^4} = \frac{1}{VT^3} \ln Z(T, V, \vec{\mu})
\]

▶ can be expanded in the chemical potentials

\[
\frac{P}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{u^d s}^{ij\mu}(T) \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_d}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k
\]

\[
\chi_{u^d s}^{ij\mu}(T) = \left. \frac{\partial i+j+k P/ T^4}{\partial \hat{\mu}_u^i \partial \hat{\mu}_d^j \partial \hat{\mu}_s^k} \right|_{\vec{\mu}=0}
\]

▶ The generalized susceptibilities \( \chi_{ij\mu}^{u^d s}(T) \) can be evaluated at zero chemical potential

▶ Higher order susceptibilities are more sensitive to criticality
Fluctuations of conserved charges

- Fluctuations of strangeness $\chi_2^S$ and electric charge $\chi_2^Q$
- Notice that cutoff effects in $\chi_2^Q$ are more significant
The trace anomaly $\Theta_{\mu\mu}$ is evaluated directly on the lattice, then the pressure can be calculated as an integral of $\Theta_{\mu\mu}$

$$\Theta_{\mu\mu} \equiv \varepsilon - 3p = -\frac{V}{T^4} \frac{d \ln Z}{d \ln a} \quad \Rightarrow \quad \frac{p}{T^4} - \frac{p_0}{T_0^4} = \int_{T_0}^{T} dT' \frac{\varepsilon - 3p}{T'^5}$$

Rapid growth of the pressure signals the rise in the number of degrees of freedom (i.e. deconfinement)
QCD equation of state at $\mu_B > 0$

- Pressure evaluated as an expansion up to $O(\mu_B^6)$ at several values of the chemical potential.
- This requires evaluation of various generalized susceptibilities of high order, which is a very computationally demanding problem.
Screening properties of the QGP medium

The singlet free energy of static quark anti-quark pair as function of their separation $r$

Decrease of $F_1(r)$ with increasing temperature is related to the Debye screening effect in QGP
Bayesian inference methods are used for solving the integral equation that relates the Euclidean correlation function $G(\tau, \vec{p})$ and the spectral function $\rho(\omega, \vec{p})$:

$$G(\tau, \vec{p}) = \int_0^\infty d\omega \rho(\omega, \vec{p})K(\omega, \tau)$$

$$K(\omega, \tau) = \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)}$$
Related talks
Related talks

- Tue, 14:40, par 3.1, J. Günther: The QCD equation of state at finite density from analytical continuation
- Tue, 15:00, par 3.1, A. Rothkopf: High statistics study of in-medium S- and P-wave quarkonium states in lattice Non-relativistic QCD
- Tue, 15:20, par 3.1, S. Sharma: The QCD Equation of State and critical end-point estimates at 6th order in chemical potentials
- Tue, 15:40, par 3.1, A. Pasztor: Lattice QCD thermodynamics up to the perturbative regime
- Tue, 15:40, par 3.3, P. Petreczky: Lattice calculations of the heavy quark potential at non-zero temperature
- Tue, 17:30, par 4.3, H. Ohno: Quarkonium spectral functions and heavy quark diffusion of charm and bottom quarks from lattice QCD at finite temperature
Related talks

- Wed, 09:30, par 5.3, G. Basar: Going with the flow: a new solution to the sign problem
- Wed, 12:00, par 6.3, M. P. Lombardo: Topology and axion’s properties from lattice QCD with a dynamical charm
- Wed, 14:40, par 7.2, F. Karsch: Conserved charge fluctuations at vanishing and non-vanishing baryon chemical potential from lattice QCD
- Wed, 16:30, par 8.1, S. Borsanyi: Fluctuations of conserved charges at zero and finite density
- Thu, 12:30, plenary, O. Kaczmarek: Lattice QCD results on soft and hard probes of strongly interacting matter
Conclusion
What to take away from this talk

- Lattice provides a non-perturbative regularization of a quantum field theory
- The problem of evaluating quantum mechanical averages of various observables is reduced to computing integrals of very high dimension
- This can be done with importance sampling, realized with a Markov Chain Monte Carlo process
- Due to theoretical and algorithmic developments as well as growth in computational power lattice QCD simulations nowadays can be done at the physical light quark masses and the continuum limit can often be taken
- Finite-temperature lattice QCD has recently provided results on the chiral crossover temperature, fluctuations and correlations of various conserved charges, equation of state at zero and finite baryon chemical potential and many more!