Anisotropic hydrodynamics for the Gubser flow

Mauricio Martinez Guerrero

Collaborators: *M. McNelis and U. Heinz*
Forthcoming

Quark Matter 2017
Chicago, Illinois, USA
Feb. 5-11, 2017

NC STATE UNIVERSITY

Introduction

- *Relativistic hydrodynamics has become a powerful phenomenological tool to describe the space-time evolution of the fireball created in ultrarelativistic heavy-ion collisions.*

- *Hydrodynamics is an effective theory based on a large separation between the microscopic and macroscopic scales. This hierarchy of scales is not always obvious in heavy ion collisions.*

- *Exactly solvable models can be used as a tools to test the validity and accuracy of the different hydrodynamical approximated models. Furthermore, exact solutions can be used to check your numerical algorithms.*

- *For weakly coupled systems there are exact solutions to the Boltzmann equation for highly non trivial flow velocity patterns*

  Within the Relaxation time approximation (RTA)

  ***Bjorken**: Baym* (1984), *Florkowski, Ryblewski, Strickland* (2013)

  ***Gubser**: Denicol, Heinz, Martinez, Noronha, Strickland* (2014)

  Nonlinear collisional kernel (constant cross section)

  ***FRW**: Bazow, Denicol, Heinz, Martinez, Noronha* (2016)

A bit of kinetic theory

- *Our starting point is the Boltzmann equation in the RTA approximation*

$$p_\mu \partial^\mu f(x^\mu, p_i) = \mathcal{C}[f]$$

$$\mathcal{C}[f] = -\frac{1}{\tau_{rel}} \left(f(x^\mu, p_i) - f_{eq.}(x^\mu, p_i) \right)$$

- *For conformal systems*

$$\tau_{rel} = \frac{c}{T(x^\mu)} \qquad \textit{with} \qquad c = 5\frac{\eta}{\mathcal{S}}$$

- *The relevant macroscopic quantities are obtained by considering the hydrodynamical moments*

$$J^\mu = \langle p^\mu \rangle$$

$$T^{\mu\nu} = \langle p^\mu p^\nu \rangle$$

where $\langle \mathcal{O}(x^\mu, p_i) \rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{-g}\, p^0} \mathcal{O}(x^\mu, p_i) f(x^\mu, p_i)$

Deriving hydrodynamics from kinetic theory I

- *Try to solve the Boltzmann equation by expanding around some particular evolving background*

$$f(x^\mu, p_i) \approx f_0(x^\mu, p_i) + \delta f \qquad\qquad \delta f \lll f_0$$

- *f_0 is the leading order distribution function and δf takes into account dissipative corrections.*
- *We consider that the leading order distribution is given by the Romatschke-Strickland (RS) ansatz (no chemical potential)*

$$f_0(x^\mu, p_i) = f_{RS}\left(\beta_u \sqrt{p_\mu p_\nu \, \Omega^{\mu\nu}}\right) \qquad\qquad \Omega^{\mu\nu} = u^\mu \, u^\nu + \xi \, l^\mu \, l^\nu$$

- *$\Omega_{\mu\nu}$ takes into account the deviations from local isotropy in momentum-space along the l_μ direction. The magnitude of the anisotropy is quantified by the anisotropy parameter ξ.*
- *u_μ defines the local fluid velocity*
- *β_u defines the effective temperature.*

In the LRF $\quad u^\mu = (1,0,0,0) \quad l^\mu = (0,0,0,1) \quad \Rightarrow f_{RS}\left(\beta_u \sqrt{p_\perp^2 + (1+\xi) p_z^2}\right)$

Deriving hydrodynamics from kinetic theory II

- *How to relate* ξ, u_μ , β_u *to macroscopic variables?*
  $\Rrightarrow$ *Landau matching conditions*

$$T^{\mu\nu} u_\nu = \epsilon(\beta_u, \xi)\, u^\mu \qquad \textit{Landau frame}$$

$$\langle\, (-u \cdot p)^2 \,\rangle_{\delta f} \equiv 0 \Rightarrow \epsilon(\beta_u, \xi) = \epsilon_{eq.}(T)$$

No Landau matching prescription for the anisotropy parameter ξ !

- *Energy-momentum conservation can be written as*

$$T^{\mu\nu} = \langle\, p^\mu\, p^\nu \,\rangle = T_0^{\mu\nu} + \delta T^{\mu\nu}$$
$$= T_0^{\mu\nu} + \pi^{\mu\nu}$$

- *Evolution equations of the macroscopic variables are obtained from the conservation laws*

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow$$

Evolution equations for u_μ *and* $\varepsilon(\beta_u, \xi)$ *which couple with* ξ *and* $\pi_{\mu\nu}$

- *Conservation laws do not provide evolution equations for the dissipative corrections* ξ, $\pi_{\mu\nu}$!

Gubser flow

- *In Minkowski space the Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow* (*Gubser* 2010, *Gubser & Yarom* 2010)
- *Gubser flow is better understood in the de Sitter space times a line* $dS_3 \otimes R$

Minkowski space → $dS_3 \otimes R$

$$x^\mu = (\tau, r, \phi, \eta) \quad \Longrightarrow \quad \hat{x}^\mu = (\rho, \theta, \phi, \eta)$$

$$ds^2 = -d\tau^2 + dr^2 + r^2\,d\phi^2 + d\eta^2 \quad \Longrightarrow \quad d\hat{s}^2 = -d\rho^2 + \cosh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right) + d\eta^2$$

- *In* $dS_3 \otimes R$ *the flow velocity profile can be recast as a static flow in the* $(\rho, \theta, \phi, \eta)$ *coordinate system*

$$u^\mu = (u^\tau(\tau, r), u^r(\tau, r), 0, 0) \quad \Longrightarrow \quad \hat{u}^\mu = (1, 0, 0, 0)$$

- *The macroscopic variables (energy density, pressure) depend only on the time-like* ρ *variable*

$$\epsilon(\tau, r) \quad \Longrightarrow \quad \hat{\epsilon}(\rho)$$

Exact solution to the RTA Boltzmann equation

- *In $dS_3 \otimes R$ the dependence of the distribution function is restricted by the symmetries of the Gubser flow*

$$f(\hat{x}^{\mu}, \hat{p}_i) = f\left(\rho, \hat{p}_{\Omega}^2, \hat{p}_{\eta}\right)$$

$$\hat{p}_{\Omega}^2 = \hat{p}_{\theta}^2 + \frac{\hat{p}_{\phi}^2}{\sin^2\theta}$$ → *Total momentum in the* (θ,ϕ) *plane*

$$\hat{p}_{\eta}$$ → *Momentum along the* η *direction*

- *The RTA Boltzmann equation gets reduced to*

$$\frac{\partial}{\partial \rho} f\left(\rho, \hat{p}_{\Omega}^2, \hat{p}_{\eta}\right) = -\frac{\hat{T}(\rho)}{c}\left(f\left(\rho, \hat{p}_{\Omega}^2, \hat{p}_{\eta}\right) - f_{eq}\left(\hat{p}^{\rho}/\hat{T}(\rho)\right)\right)$$

$$c = 5\frac{\eta}{\mathcal{S}}$$

- *The exact solution to this equation is*

$$f\left(\rho, \hat{p}_{\Omega}^2, \hat{p}_{\eta}\right) = D(\rho, \rho_0)\, f_0\left(\rho, \hat{p}_{\Omega}^2, \hat{p}_{\eta}\right) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, \hat{T}(\rho')\, D(\rho, \rho')\, f_{eq}\left(\hat{p}^{\rho}/\hat{T}(\rho)\right)$$

Matching prescriptions

"*Standard" Viscous Hydrodynamics*:
Expansion around the equilibrium (ξ=0)

- ***Denicol-Niemi-Molnar-Rischke* (*DNMR*) *approach*:** *Nonlinear expansion of δf in terms of method of moments together with a systematic power counting in Knudsen and inverse Reynolds number.*

***Viscous Anisotropic Hydrodynamics*:**

Expansion around an anisotropic state ($\xi \neq 0$).

- ***$\mathcal{P}_L$ matching*** (*MNR, See Niemi's talk*): *matches ξ evolution to that of the longitudinal pressure. For the Gubser flow it means that the energy-momentum tensor receives no residual dissipative corrections. The evolution equations can be written in terms of macroscopic variables just as in standard viscous hydrodynamics.*
- ***NSR approach*:** *obtain the evolution equation for ξ by considering a particular combination of the second moment of the Boltzmann equation.*
- $\Rightarrow$ *Pressure anisotropy is not fully captured by the evolution in ξ.*
- ***NLO-NSR approach*:** *same ξ evolution but include residual shear stress in the energy-momentum tensor. This captures the missing part of the pressure anisotropy.*
- *In NLO-NSR and NSR the evolution equations mix macroscopic with microscopic variables.*

Fluid models for the Gubser flow

Conservation Law

$$\frac{d\hat{\epsilon}}{d\rho} + \frac{8}{3}\hat{\epsilon}\,\tanh\rho = \hat{\pi}^{\eta\eta}\,\tanh\rho$$

$$\partial_\rho \hat{\pi}^{\eta\eta} + \frac{\hat{\pi}^{\eta\eta}}{\hat{\tau}_{rel}}\left(1 - \frac{10}{21}\tanh\rho\right) + \frac{8}{3}\hat{\pi}^{\eta\eta}\,\tanh\rho = \frac{16}{45}\hat{\epsilon}\,\tanh\rho$$

Viscous hydrodynamics (DNMR with* 14 *Grad)

$\xi \equiv 0 \quad \beta_u = T^{-1}$

Fluid models for the Gubser flow

Conservation Law

$$\frac{d\hat{\epsilon}}{d\rho} + \frac{8}{3}\hat{\epsilon}\,\tanh\rho = \hat{\pi}^{\eta\eta}\,\tanh\rho$$

$$\partial_\rho \hat{\pi}^{\eta\eta} + \frac{\hat{\pi}^{\eta\eta}}{\hat{\tau}_{rel}}\left(1 - \frac{10}{21}\tanh\rho\right) + \frac{8}{3}\hat{\pi}^{\eta\eta}\,\tanh\rho = \frac{16}{45}\hat{\epsilon}\,\tanh\rho$$

$$\partial_\rho \hat{\pi}^{\eta\eta} + \frac{\hat{\pi}^{\eta\eta}}{\hat{\tau}_{rel}} + \tanh\rho\left(\frac{4}{3}\hat{\pi}^{\eta\eta} + \hat{I}^{RS}_{240}(\hat{\epsilon},\hat{\pi}^{\eta\eta})\right) = \frac{5}{9}\hat{\epsilon}\,\tanh\rho$$

Viscous hydrodynamics (DNMR with 14 Grad)

$\xi \equiv 0 \quad \beta_u = T^{-1}$

Anisotropic hydrodynamics ($\mathcal{P}_L$ matching)

$\xi \neq 0 \quad \beta_u = R(\xi)\,T^{-1}$

$$\hat{\pi}^{\eta\eta} = \hat{\mathcal{P}}_L - \frac{1}{3}\hat{\epsilon}$$

Couple with a higher order RS moment (non-hydrodynamic)

Fluid models for the Gubser flow

Conservation Law → $$\frac{d\hat{\epsilon}(\beta_u,\xi)}{d\rho}+\frac{8}{3}\hat{\epsilon}(\beta_u,\xi)\,\tanh\rho-\hat{\pi}^{\eta\eta}\,\tanh\rho=0$$

Anisotropic hydrodynamics

NSR: Equations for β_u and $\xi\neq 0$ (No δf corrections)

$$\beta_u=R(\xi)\,T^{-1}\,,$$

$$(1+\xi)\left[\partial_\rho\hat{\mathcal{I}}^{RS}_{320}+2\,\tanh\rho\,\hat{\mathcal{I}}^{RS}_{320}\right]-\frac{1}{2}\left[\partial_\rho\hat{\mathcal{I}}^{RS}_{301}+4\tanh\rho\,\hat{\mathcal{I}}^{RS}_{301}\right]=-\frac{1}{\hat{\tau}_r}\left((1+\xi)\left[\hat{\mathcal{I}}^{RS}_{320}-\hat{\mathcal{I}}^{eq.}_{320}\right]-\frac{1}{2}\left[\hat{\mathcal{I}}^{RS}_{301}-\hat{\mathcal{I}}^{eq.}_{301}\right]\right)$$

$$\hat{\pi}^{\eta\eta}\approx\hat{\pi}^{\eta\eta}_{RS}=\langle\,\hat{p}^{\langle\eta}\hat{p}^{\eta\rangle}\,\rangle_{RS}$$

ξ was not matched to the effective shear $\Rightarrow$ Residual dissipative corrections are needed

Fluid models for the Gubser flow

Conservation Law → $$\frac{d\hat{\epsilon}(\beta_u,\xi)}{d\rho} + \frac{8}{3}\hat{\epsilon}(\beta_u,\xi)\,\tanh\rho - \hat{\pi}^{\eta\eta}\,\tanh\rho = 0$$

Viscous Anisotropic hydrodynamics

NLO-NSR: Equations for β_u and $\xi \neq 0$ (Includes δf corrections)

Residual dissipative corrections introduces a new degree of freedom

$$\hat{\pi}^{\eta\eta} = \hat{\pi}^{\eta\eta}_{RS} + \hat{\bar{\pi}}^{\eta\eta}$$

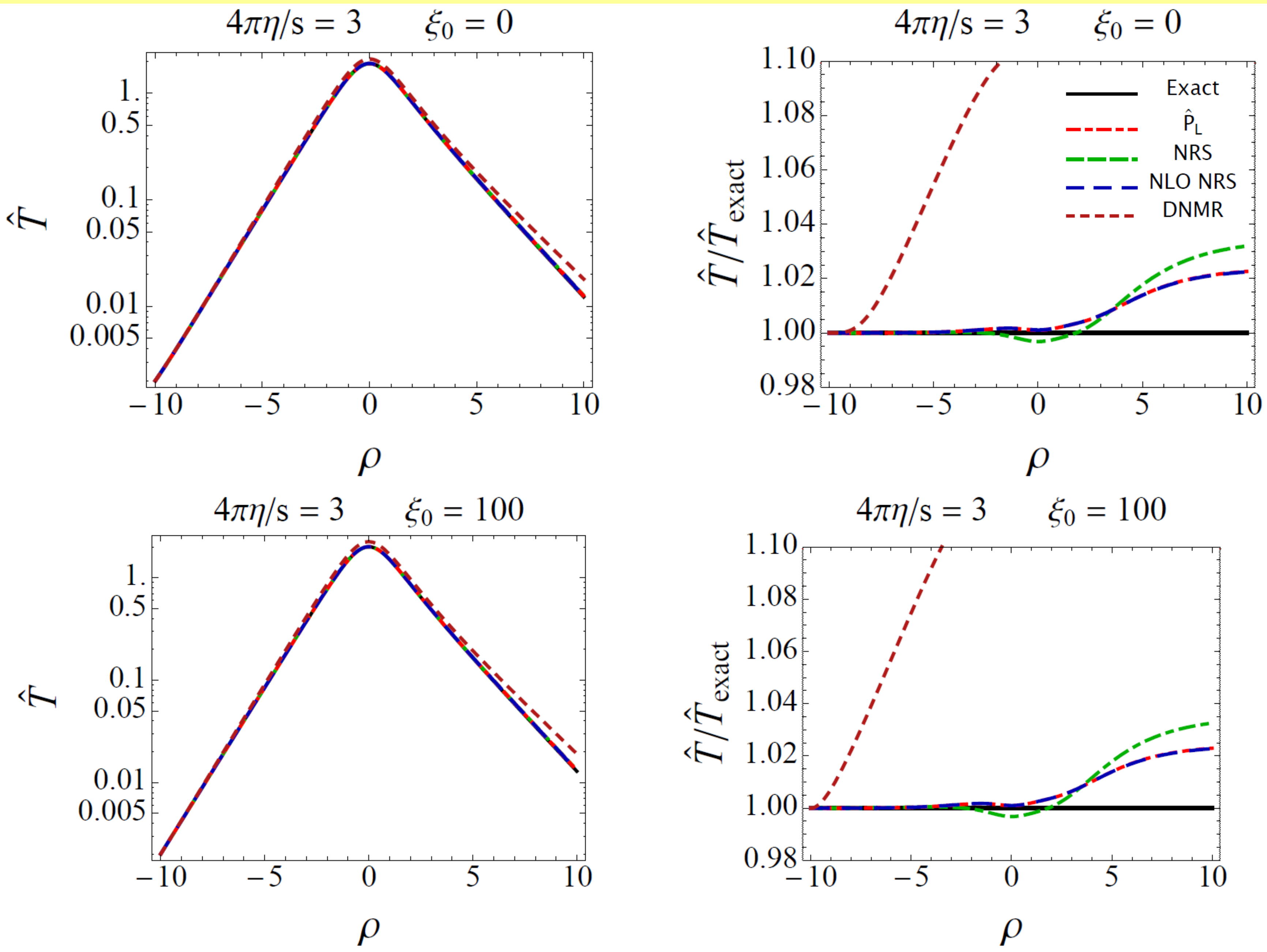
$$\beta_u = R(\xi)\,T^{-1}$$

$$(1+\xi)\left[\partial_\rho \hat{\mathcal{I}}^{RS}_{320} + 2\tanh\rho\,\hat{\mathcal{I}}^{RS}_{320}\right] - \frac{1}{2}\left[\partial_\rho \hat{\mathcal{I}}^{RS}_{301} + 4\tanh\rho\,\hat{\mathcal{I}}^{RS}_{301}\right] = -\frac{1}{\hat{\tau}_r}\left((1+\xi)\left[\hat{\mathcal{I}}^{RS}_{320} - \hat{\mathcal{I}}^{eq.}_{320}\right] - \frac{1}{2}\left[\hat{\mathcal{I}}^{RS}_{301} - \hat{\mathcal{I}}^{eq.}_{301}\right]\right)$$

Additional equation for the residual shear stress

$$\begin{aligned}
\partial_\rho \hat{\bar{\pi}}^{\eta\eta} = &-\tanh\rho\left[\frac{4}{3}\hat{\bar{\pi}}^{\eta\eta} + \hat{\alpha}\,\hat{\mathcal{I}}^{RS}_{240} - \hat{\beta}\,\hat{\mathcal{I}}^{RS}_{340} + \frac{4}{3}\hat{\omega}\,\hat{\mathcal{I}}^{RS}_{440} + \frac{\hat{\omega}_{\langle\eta\eta\rangle}}{2}\left(3\,\hat{\mathcal{I}}^{RS}_{460} - \hat{\mathcal{I}}^{RS}_{440}\right)\right] \\
&- \frac{\hat{\bar{\pi}}^{\eta\eta} + \hat{\pi}^{\eta\eta}_{RS}}{\hat{\tau}_{rel}} - \beta_u^2 \partial_\rho \hat{\beta}_u \left(\hat{\mathcal{I}}^{RS}_{330} - \frac{1}{3}\hat{\mathcal{I}}^{RS}_{310}\right) \\
&- \beta_u \tanh\rho\left(\hat{\mathcal{I}}^{RS}_{310} - \hat{\mathcal{I}}^{RS}_{330} + \frac{1}{3}\left(\hat{\mathcal{I}}^{RS}_{310} - \hat{\mathcal{I}}^{RS}_{3,-1,0}\right)\right) + \frac{\partial_\rho \xi}{2}\beta_u\left(\hat{\mathcal{I}}^{RS}_{330} - \frac{1}{3}\hat{\mathcal{I}}^{RS}_{310}\right),
\end{aligned}$$

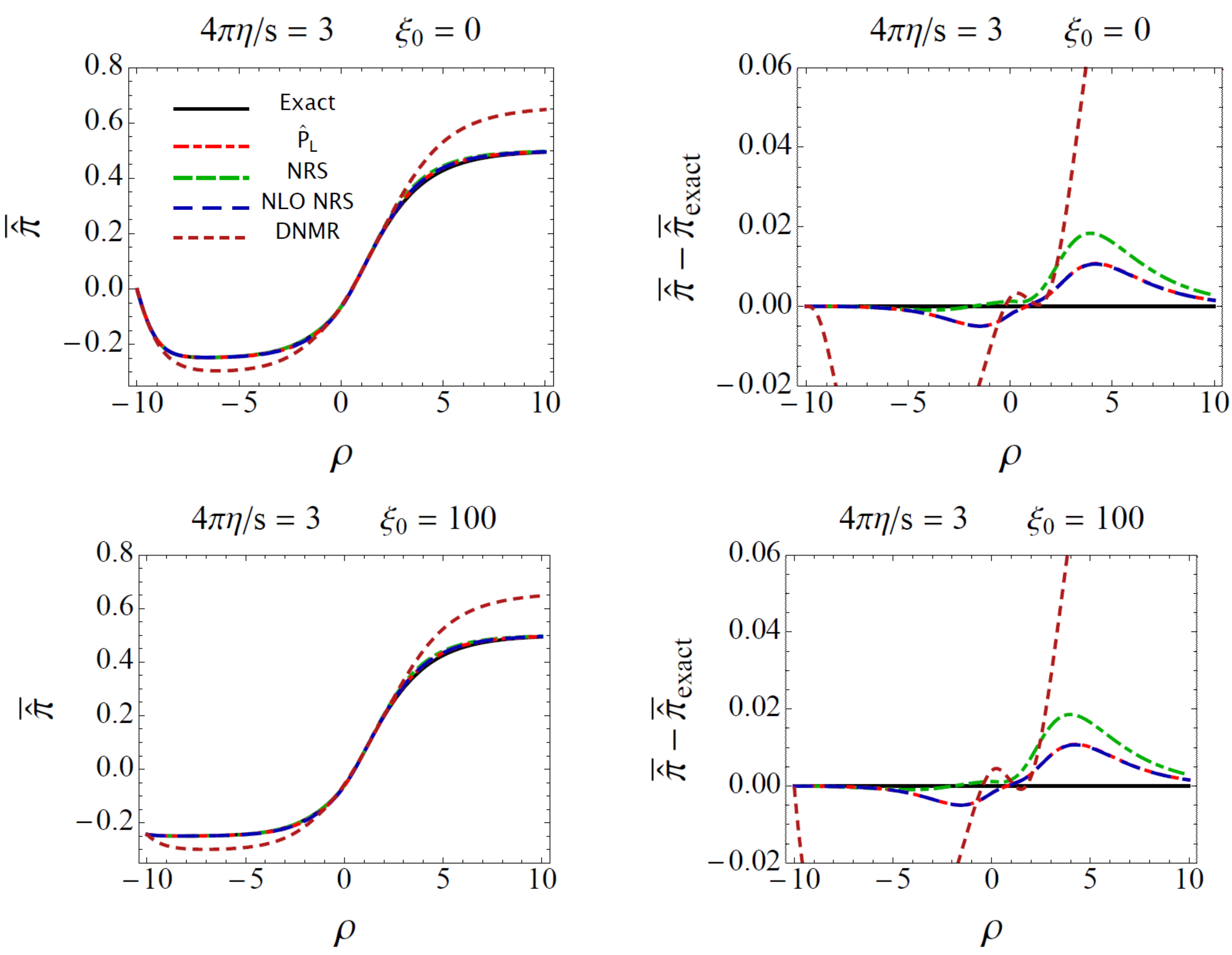
Results



$4\pi\eta/s = 3 \quad \xi_0 = 0$
$\hat{T}$
ρ
$4\pi\eta/s = 3 \quad \xi_0 = 0$
$\hat{T}/\hat{T}_{exact}$
ρ
Exact
$\hat{P}_L$
NRS
NLO NRS
DNMR
$4\pi\eta/s = 3 \quad \xi_0 = 100$
$\hat{T}$
ρ
$4\pi\eta/s = 3 \quad \xi_0 = 100$
$\hat{T}/\hat{T}_{exact}$
ρ

Results

$$\bar{\hat{\pi}} = \frac{3}{4}\frac{\hat{\pi}^{\eta\eta}}{\hat{\epsilon}}$$



Conclusions

- *Exact solution to the RTA Boltzmann equation for the Gubser flow enables us to test the accuracy and validity of hydrodynamic models for different initial conditions and values of* $\eta/\mathcal{S}$.
- *The numerical results show that different prescriptions of anisotropic hydrodynamics work better than DNMR viscous hydro for different values of* $\eta/\mathcal{S}$ *and far-from-equilibrium initial conditions*.
- *The* $\mathcal{P}_L$ *matching (MNR) provides the simplest description and best match to the exact solution to the RTA Boltzmann equation for the Gubser flow. However, after correcting the NSR scheme for the missing residual shear stress, it provides an equally precise hydrodynamic prescription*.

Wilke van der Schee, MIT/Utrecht

AN APOLOGY TO THOSE WHO CANNOT ATTEND

Backup slides

Definition of the RS moments

$$\hat{I}^{RS}_{nlq} = \int \frac{d^3\hat{p}}{(2\pi)^3\sqrt{-g}\,\hat{p}^{\rho}} \left(\hat{p}^{\rho}\right)^n \left(\hat{p}_{\eta}\right)^l \left(\frac{\hat{p}_{\Omega}^2}{\cosh^2\rho}\right)^q f_{RS}\left(\beta_u\sqrt{\frac{\hat{p}_{\Omega}^2}{\cosh^2\rho} + (1+\xi)\hat{p}_{\eta}^2}\right)$$

Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

Special Conformal transformations + rotation along the beam line ($SO(3)_q$)

Boost invariance ($SO(1,1)$)

Reflections along the beam line (Z_2)

Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

In polar Milne Coordinates (τ, r, ϕ, η)

$$u^\mu = \left(\cosh\kappa(\tau,r), \sinh\kappa(\tau,r), 0, 0\right)$$

$$\kappa(\tau,r) = \tanh^{-1}\left(\frac{2q^2\tau r}{1+(qr)^2+(q\tau)^2}\right)$$

q is a scale parameter