Particle production in \( pA \) collisions beyond leading order

Edmond Iancu
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w/ A.H. Mueller and D.N. Triantafyllopoulos, arXiv:1608.05293

\[
\begin{align*}
\eta &= 3.2 (\times 0.1) \\
\eta &= 2.2 \\
\eta &= 2.2, 3.2
\end{align*}
\]

BRAHMS \( \eta = 2.2, 3.2 \)

\[
\begin{align*}
\text{LO} \\
\text{NLO} \\
\text{data}
\end{align*}
\]
Particle production in $pp$ and $pA$ collisions at forward rapidities explores the physics of high gluon densities at small-$x$

- non-linear phenomena: gluon saturation, multiple scattering
- resummations based on the eikonal approximation (Wilson lines)
- non-linear evolution equations: BK, B-JIMWLK

Effective theory derived in pQCD: Color Glass Condensate

The CGC formalism is now being promoted to NLO

- NLO versions for the BK and B-JIMWLK equations
  \cite{Balitsky2008,Chirilli2013,Kovner2013}
- NLO impact factor for particle production in $pA$ collisions
  \cite{Chirilli2012,Mueller2012}

But the strict NLO approximations turned out to be problematic
“Negative growth” of the dipole scattering amplitude

Not really a surprise

- similar problems for NLO BFKL
- large transverse logarithms
- collinear resummations
- Mellin representation

(Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)

Lappi, Mäntysaari, arXiv:1502.02400
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Collinear improvement for NLO BK (transverse coordinates)


Evolution becomes stable with promising phenomenology

- excellents fits to DIS (Iancu et al, 2015; Albacete, 2015)
Particle production in \( d+Au \) collisions (RHIC)

- Very good agreement at low \( p_\perp \) 😊 ... but negative at larger \( p_\perp \) 😞

\[
\frac{d^3N}{dy dp_\perp} \quad \text{[GeV}^{-2}] \\
\eta = 2.2, 3.2 \\
\]

Is this a real problem?

- “small-\( x \) resummations do not apply at large \( p_\perp \)”
- but \( p_\perp \sim Q_s \) is not that large!

- Likely related to the rapidity subtraction in NLO impact factor

Stasto, Xiao, and Zaslavsky, arXiv:1307.4057
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Various proposals which alleviate the problem (pushed to higher $p_{\perp}$)

- Kang, Vitev, and Xing, arXiv:1403.5221
- Altinoluk, Armesto, Beuf, Kovner, and Lublinsky, arXiv:1411.2869
- Ducloué, Lappi, and Zhu, arXiv:1604.00225
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![Graph showing BRAHMS data for $\eta = 2.2, 3.2$]

- Is this a real problem?
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  - but $p_\perp \sim Q_s$ is not that large!

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- A reorganization of the perturbative expansion which avoids the rapidity subtraction (E.I., A. Mueller and D. Triantafyllopoulos, 2016)

- Ongoing numerical implementation which looks promising (Ducloué, Lappi, and Zhu, w.i.p.)
A quark initially collinear with the proton acquires a transverse momentum $p_\perp$ via multiple scattering off the saturated gluons.

\[
\begin{align*}
x_p \equiv \frac{p^+}{q^+} &= \frac{p_\perp}{\sqrt{s}} e^\eta \\
X_g \equiv \frac{p^-}{P^-} &= \frac{p_\perp}{\sqrt{s}} e^{-\eta}
\end{align*}
\]

$X_g \ll x_p$ when $\eta > 0$

- $\eta$: quark rapidity in the COM frame
- $x_p$: longitudinal fraction of the quark in the proton
- $X_g$: longitudinal fraction of the gluon in the target

Gluons in the nucleus have a typical transverse momentum $k_\perp \sim Q_s(X_g)$.
Multiple scattering

- Eikonal approximation \(\implies\) the transverse coordinate representation

\[
\begin{align*}
\mathcal{M}_{ij}(k_\perp) &\equiv \int d^2x_\perp e^{-ix_\perp \cdot k_\perp} V_{ij}(x_\perp) \\
V(x_\perp) &= \text{P exp} \left\{ ig \int dx^+ A^-_a(x^+, x_\perp) t^a \right\}
\end{align*}
\]

- \(A^-_a\): color field representing small-\(x\) gluons in the nucleus
Multiple scattering

Amplitude: \[ M_{ij}(k_\perp) \equiv \int d^2x_\perp \, e^{-ix_\perp \cdot k_\perp} \, V_{ij}(x_\perp) \]

Cross-section: \[ \frac{d\sigma}{dy \, d^2k_\perp} \simeq x_p q(x_p, Q^2) \frac{1}{N_c} \left\langle \sum_{ij} |M_{ij}(k_\perp)|^2 \right\rangle_{X_g} \]

- Average over the color fields \( A^- \) in the target (CGC)
- Two Wilson lines at different transverse coordinates, traced over color
- Equivalently: the elastic $S$-matrix for a $q\bar{q}$ color dipole

\[
S(x, y; X_g) \equiv \frac{1}{N_c} \langle \text{tr}[V(x)V^+(y)] \rangle_{X_g}
\]

\[
\frac{d\sigma}{dy \, d^2k} \simeq x_p q(x_p) \int_{x, y} e^{-i(x-y) \cdot k} \, S(x, y; X_g)
\]

- The Fourier transform $S(k, X_g)$: “unintegrated gluon distribution”
Equivalently: the elastic $S$-matrix for a $q\bar{q}$ color dipole

\[ S(x, y; X_g) \equiv \frac{1}{N_c} \left\langle \text{tr} \left[ V(x) V^\dagger(y) \right]\right\rangle_{X_g} \]

\[ \frac{d\sigma}{dy \, d^2k} \simeq x_p q(x_p) \int_{x, y} e^{-i(x-y) \cdot k} S(x, y; X_g) \]

‘Hybrid factorization’: collinear fact. for $p$ & CGC fact. for $A$

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\[
\frac{d\sigma}{dy \, d^2k} \simeq x_q q(x_p) \int_{x, y} e^{-i(x-y) \cdot k} \, S(x, y; X_g)
\]

The dipole picture is preserved by the high-energy evolution up to NLO 

(Kovchegov and Tuchin, 2002; Mueller and Munier, 2012)
BK equation (leading order)

- Probability $\sim \alpha_s \ln \frac{1}{x}$ to radiate a soft gluon with $x \equiv \frac{p^+}{k^+} \ll 1$

- Evolution equation for the dipole $S$–matrix $S_{xy}(Y)$ with $Y \equiv \ln(1/x)$

$$\frac{\partial S_{xy}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(y - z)^2} [S_{xz}S_{zy} - S_{xy}]$$

- Dipole kernel: probability for the dipole to emit a soft gluon at $z$

- Large-$N_c$ approximation to the Balitsky-JIMWLK hierarchy

- Saturation momentum $Q_s(Y)$: $S(r, Y) = 0.5$ when $r = 1/Q_s(Y)$
Adding running coupling: rcBK

- The evolution speed: saturation exponent $\lambda_s \equiv \frac{d \ln Q_s^2}{dY}$

- At LO, $\lambda_s \sim 1$ is way too large: $\lambda_{\text{HERA}} = 0.2 \div 0.3$

- Including running coupling dramatically slows down the evolution

- ... but there are other, equally important, NLO corrections!
Fit parameters: initial condition for the rcBK equation + $K$-factors

\[
\left. \frac{dN}{dy \, d^2k} \right|_{LO} = K^h \int_{x_p}^{1} \frac{dz}{z^2} \frac{x_p}{z} q \left( \frac{x_p}{z} \right) S \left( \frac{k}{z}, X_g \right) D_{h/q}(z)
\]
Particle production beyond leading order

- **LO approximation:** any number $n \geq 0$ of soft emissions $\Rightarrow (\alpha_s Y)^n$

  - $x_\perp$
  - $y_\perp$

  \[ x \ll 1 \]

  \[ n = 0 \Rightarrow O(\alpha_s Y) \]

- **NLO corrections to the evolution:** 2 soft gluons, with similar values of $x$

  - $x_1 \sim x_2 \ll 1$

  \[ n = 1 \Rightarrow O(\alpha_s^2 Y) \]

- **NLO correction to impact factor:** the first gluon can be hard

  - $x \leq 1$

  \[ n = 0 \Rightarrow O(\alpha_s) \]
The first gluon contributes both to the evolution (when $x \ll 1$) and to the NLO impact factor (generic $x$): How to avoid over counting?

$k_{\perp}$-factorization: use a ‘rapidity subtraction’

- the method used by Chirilli, Xiao, and Yuan (arXiv:1203.6139)
- leads to a negative cross-section at semi-hard $k_{\perp}$

Our proposal (E.I., A. Mueller and D. Triantafyllopoulos, arXiv:1608.05293)
- separate the first gluon emission from the evolution and compute it with the exact kinematics

The integral representation of the BK equation is useful in that sense.
\[ S_{xy}(X_g) = S_{xy}(X_0) + \bar{\alpha}_s \int_{X_g}^{1} \frac{dx}{x} \int_{x}^{1} \frac{(x-y)^2}{(x-z)^2(y-z)^2} \left[ S_{xz}S_{zy} - S_{xy} \right] (X(x)) \]

- N.B. Except for the first gluon, the evolution is associated with the nucleus

\[ X(x) : \text{energy fraction in the target} \]

\[ X(x) \approx \frac{k_{\perp}^2}{xs} = \frac{X_g}{x} \]

\[ X \leq 1 \implies x \geq X_g \]

- Even at LO, a running coupling is generally understood

\[ \bar{\alpha}_s \to \bar{\alpha}_s(r_{\text{min}}), \text{ where } r_{\text{min}} \equiv \min \{|x-y|, |x-z|, |y-z|\} \]
**BK evolution in integral form**

- In more compact, but formal, notations (coordinate or momentum space)

\[ S(X_g) = S_0 + \bar{\alpha}_s \int_{X_g}^{1} \frac{dx}{x} K(0) S(X(x)) \; ; \; \quad X(x) \equiv \frac{X_g}{x} \]

- \( K(0) \): LO emission kernel for a soft gluon (including \( \perp \) non-locality)

\[ S \text{ (solution to LO BK equation)} \]

- The target is not explicit anymore

- The presence of a **running coupling** complicates the Fourier transform
Adding the NLO impact factor

- Compute (only) the first gluon emission with the **exact kinematics**

\[
\frac{dN}{d^2 k} = S_0(k) + \bar{\alpha}_s \int_{x_g}^1 \frac{dx}{x} K(x) S(k, X(x)) ; \quad X(x) \approx \frac{X_g}{x}
\]

- **\( K(x) \)**: kernel for emitting a gluon with exact kinematics \((x \leq 1)\)

  (see e.g. Dominguez, Marquet, Xiao, and Yuan, arXiv:1101.0715)

- This cross-section is almost obviously **positive definite**

- LO evolution + NLO impact factor ... still mixed with each other
Recovering $k_{\perp}$-factorization

- Add and subtract the LO BK equation

\[
\frac{dN}{d^2k} = S(k, X_g) + \bar{\alpha}_s \int_{X_g}^{1} \frac{dx}{x} \left[ K(x) - K(0) \right] S(k, X(x))
\]

- To NLO accuracy, one can replace $S(X(x)) \simeq S(X_g)$ ...
  - the integral over $x$ is now dominated by $x \sim 1$
  - ... and then set $X_g \rightarrow 0$ in the lower limit ('plus prescription')

Local in rapidity : $k_{\perp}$-factorization in the form presented by CXY

*(Chirilli, Xiao, and Yuan, arXiv:1203.6139 [hep-ph]*)
Recovering $k_\perp$-factorization

- Add and subtract the LO BK equation

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\frac{dN}{d^2k} = S(k, X_g) + \bar{\alpha}_s \int_0^1 \frac{dx}{x} [K(x) - K(0)] S(k, X_g)
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- To NLO accuracy, one can replace $S(X(x)) \simeq S(X_g)$ ...
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- Local in rapidity : $k_\perp$-factorization in the form presented by CXY
  
  (Chirilli, Xiao, and Yuan, arXiv:1203.6139 [hep-ph])
The fine-tuning problem

\[
\frac{dN}{d^2k} = S(k, X_g) + \bar{\alpha}_s \int_{X_g}^1 \frac{dx}{x} \left[ K(x) - K(0) \right] S(k, X(x))
\]

- A mathematical identity ... which may be dangerous in practice
  - one adds and subtracts a large, LO, contribution!

- Any approximation/numerical error in the BK solution or in the subtraction procedure \(\implies\) mismatch between the ‘added’ and ‘subtracted’ pieces

- Potential sources of mismatch (of formally higher order):
  - the ‘plus’ prescription \([\text{N.B. } S(X_g) > S(X(x)) \text{ for any } x < 1]\)
  - the treatment of the running coupling, e.g. in the Fourier transform
Fixed coupling: both methods work

Running coupling: rapidity subtraction leads to negativity
Completing the NLO evolution

(E.I., A. Mueller and D. Triantafyllopoulos, arXiv:1608.05293)

- Recall: the NLO BK evolution also involves 2-loop graphs

\[ \frac{dN}{d^2k} = S_0 + \bar{\alpha}_s \int_{x_g}^1 \frac{dx}{x} \mathcal{K}(x) S(X(x)) + \bar{\alpha}_s^2 \int_{x_g}^1 \frac{dx}{x} \mathcal{K}_2(0) S(X(x)) \]

- \( \mathcal{K}_2(0) \): NLO correction to the BK kernel with collinear improvement

(Balitsky and Chirilli, 2008; Iancu et al, 2015)
THANK YOU
‘Real amplitude’: the gluon is produced in the final state

LC energy conservation:

\[
\frac{k_{\perp}^2}{2(1-x)q_0^+} + \frac{p_{\perp}^2}{2xq_0^+} = XP^{-}
\]

\[\implies X = X(x, p_{\perp})\]

simplifies when \(k_{\perp} \simeq p_{\perp} \gg Q_s\)

\[
X(x) \simeq \frac{k_{\perp}^2}{xs} = \frac{X_g}{x}
\]

\[X \leq 1 \implies x \geq X_g\]

Equivalently: gluon lifetime should be larger than the target width

The same condition holds for the ‘virtual’ corrections

- non-trivial cancellations required by probability conservation
The negativity problem

(Stasto, Xiao, and Zaslavsky, arXiv:1307.4057)

- Sudden drop in the numerical estimate at momenta $p_{\perp}$ of order $Q_s$

```
| $\eta = 3.2$ | $\times 0.1$ |
| $\eta = 2.2$ |

BRAHMS $\eta = 2.2, 3.2$
```

- “NLO evolution is notoriously unstable”
- Sure, but in this calculation $S \approx S_{rcBK}$
  - $rcBK$ evolution is well behaved
  - the actual “LO approx” in practice
    $\left. \frac{dN}{dy \, d^2k} \right|_{LO} = S_{rcBK}(k, X_g)$

- The NLO correction to the impact factor is negative (not a real surprise) ... and dominates over the LO result at sufficiently large $k_{\perp}$
Some proposals to solve the problem

- General idea: the ‘subtracted’ term performs an over-subtraction
- Strategy: reduce the longitudinal ($x$) phase-space for the ‘hard’ gluon
  - factorization scale $x_0$ separating ‘evolution’ from ‘impact factor’
    - (Kang, Vitev, and Xing, arXiv:1403.5221)
    \[
    \int_0^1 \frac{dx}{x} \left[ K(x) - K(0) \right] \implies \int_0^{x_0} \frac{dx}{x} \left[ K(x) - K(0) \right]
    \]
  - $x_0$ can depend upon $k_{\perp}$, say to account for ‘time-ordering’
    - (Ducloué, Lappi, and Zhu, arXiv:1604.00225)
- In principle, it shouldn’t matter that much
  - the $x_0$–dependence must cancel in a complete calculation
- In practice, it only pushes the problem up to somewhat higher $k_{\perp}$
  - also, strongly dependent upon the precise implementation of $x_0$
Energy conservation ("Ioffe's time")

(Altinoluk, Armesto, Beuf, Kovner, and Lublinsky, arXiv:1411.2869)

- $x$ cannot be arbitrarily small since constrained by energy conservation

\[ k_{0\perp} = 0 \]

\[ \frac{k_0^+}{k_0^-} \]

\[ p^- p^+ = x q_0^+ \]

\[ \Delta x^+ \sim \frac{2 x q_0^+}{p_-^2} \]

- Gluon lifetime should be larger than the target width

\[ \frac{2 x q_0^+}{p_-^2} > \frac{1}{P^-} \implies x > \frac{p_-^2}{s} \]
Implementing the constraint

(Watanabe, Xiao, Yuan, and Zaslavsky, arXiv:1505:05183)

- It matters for the subtraction scheme only if \( k_\perp \gg p_\perp \)

Once again, it pushes the problem to higher \( k_\perp \)

- ... and strongly dependent upon the model/evolution chosen for \( S \)
Why is this a problem?

- An extreme example: GBW saturation model $S_{\text{GBW}}(k, X) \propto e^{-\frac{k^2}{Q_s^2}}$
  - the ‘added’ piece is exponentially suppressed at $k_\perp \gg Q_s$
  - the ‘subtracted’ piece develops a power-law tail $\propto \frac{1}{k^4_\perp}$
  - the overall result becomes negative at sufficiently large $k_\perp$

(Ducloué, Lappi, and Zhu, arXiv:1604.00225)