Choice of moment and derivation of anisotropic dissipative fluid dynamics

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with

E. Molnar, D. H. Rischke

E. Molnar, HN and D. H. Rischke, Phys. Rev. D **94**, no. 12, 125003 (2016)
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Dissipative fluid dynamics: Conservation laws

Basic starting point in fluid dynamics:

Conservation laws

$$\partial_\mu N^\mu = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

- Not closed system
- Need additional approximations
- Match to underlying microscopic theory $\longrightarrow$ transport coefficients and EoS

Anisotropic fluid dynamics:
Barz, Kampfer, Lukacs, Martinas and Wolf, Phys. Lett. B **194**, 15 (1987)
Florkowski, Phys. Lett. B **668**, 32 (2008)
Martinez and Strickland, Phys. Rev. C **81**, 024906 (2010)

How to derive general anisotropic fluid dynamics from Boltzmann equation?

Dissipative fluid dynamics: Tensor decompositions

Introduce flow velocity u^μ:

$$N^\mu = nu^\mu + n^\mu$$

$$T^{\mu\nu} = eu^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}$$

$n = u_\mu N^\mu$	LRF particle density
$n^\mu = \Delta^\mu_\alpha N^\alpha$	particle diffusion current
$e = u_\mu T^{\mu\nu} u_\nu$	LRF energy density
$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$	energy diffusion current
$p(e,n) + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$	isotropic pressure (p_{eq} + *bulk*)
$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$	shear stress tensor

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$T^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$$

Anisotropic fluid dynamics: Tensor decompositions

Introduce a new 4-vector l^μ (direction of anisotropy, $l \cdot u = 0$, $l \cdot l = -1$)

$$N^\mu = n\, u^\mu + n_l\, l^\mu + V_\perp^\mu$$

$$T^{\mu\nu} = e\, u^\mu u^\nu + 2\, M\, u^{(\mu}\, l^{\nu)} + P_l\, l^\mu l^\nu - P_\perp\, \Xi^{\mu\nu} + 2\, W_{\perp u}^{(\mu} u^{\nu)} + 2\, W_{\perp l}^{(\mu}\, l^{\nu)} + \pi_\perp^{\mu\nu}$$

$$\begin{aligned}
n_l &= -N^\mu l_\mu \,, \\
M &= -T^{\mu\nu} u_\mu l_\nu \,, \\
P_l &= T^{\mu\nu} l_\mu l_\nu \,, \\
P_\perp &= -\frac{1}{2}\, T^{\mu\nu} \Xi_{\mu\nu} \,, \\
V_\perp^\mu &= \Xi^\mu_\nu N^\nu \,, \\
W_{\perp u}^\mu &= \Xi^\mu_\alpha T^{\alpha\beta} u_\beta \,, \\
W_{\perp l}^\mu &= -\Xi^\mu_\alpha T^{\alpha\beta} l_\beta \,, \\
\pi_\perp^{\mu\nu} &= \Xi^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} \,.
\end{aligned}$$

- New projection operators orthogonal to l^μ and u^μ:

$$\Xi^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu$$

- $\Xi^{\mu\nu}_{\alpha\beta}$ = symmetric, traceless, orthogonal to l^μ and u^μ

Degrees of freedom same in both cases ($T^{\mu\nu}$ and N^{μ}). Decompositions can be related :

$$P = \frac{1}{3}\left(P_l + 2P_\perp\right) \; .$$

$$\begin{aligned} V^\mu &= n_l\, l^\mu + V_\perp^\mu \; , \\ W^\mu &= M\, l^\mu + W_{\perp u}^\mu \; . \end{aligned}$$

$$\pi^{\mu\nu} = \pi_\perp^{\mu\nu} + 2\, W_{\perp l}^{(\mu}\, l^{\nu)} + \frac{1}{3}\left(P_l - P_\perp\right)\left(2\, l^\mu l^\nu + \Xi^{\mu\nu}\right) \; .$$

As usual (so far) arbitrary u^μ takes a physical meaning when tied to the flow of some conserved quantity:

Eckart frame:

$$V^\mu = 0 \, , \quad V_\perp^\mu = 0 \, , \quad n_l = 0 \, ,$$

Landau frame:

$$W^\mu = 0 \, , \quad W_{\perp u}^\mu = 0 \, , \quad M = 0 \, ,$$

Dissipative (anisotropic) fluid dynamics from the Boltzmann equation

Dissipative fluid dynamics: expand around equilibrium:

$$f_{\mathbf{k}} = f_{0\mathbf{k}}(T,\mu) + \delta f_{\mathbf{k}}$$

- T and μ defined through Landau matching conditions, $e = e_0(T,\mu)$ and $n = n_0(T,\mu)$
- The conditions determine the fictitious equilibrium state (around which we will expand)

Anisotropic dissipative fluid dynamics: expand around anisotropic distribution function

$$f_{\mathbf{k}} = \hat{f}_{0\mathbf{k}}(T,\mu,\hat{\beta}_l) + \delta\hat{f}_{\mathbf{k}}$$

- For convenience require:

$$\lim_{\hat{\beta}_l\to 0} \hat{f}_{0\mathbf{k}}\left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u}, \hat{\beta}_l E_{\mathbf{k}l}\right) = f_{0\mathbf{k}}\left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u}\right)$$

- As before T and μ can be defined through Landau matching conditions, $e = e_0(T,\mu)$ and $n = n_0(T,\mu)$
- Need one more mathing condition to determine $\hat{\beta}_l$

$$f_{\mathbf{k}} = \hat{f}_{0\mathbf{k}}(T, \mu, \hat{\beta}_l) + \delta\hat{f}_{\mathbf{k}}$$

$\delta\hat{f}_{\mathbf{k}}$ can be expanded using irreducible basis: $k^{\{\mu_1} \cdots k^{\mu_\ell\}} = \Xi^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} k^{\nu_1} \cdots k^{\nu_\ell}$, and polynomials of scalars $E_{\mathbf{k}u} = k \cdot u$ and $E_{\mathbf{k}l} = -k \cdot l$. The expansion coefficients are the moments of $\delta\hat{f}_{\mathbf{k}}$

$$\hat{\rho}^{\mu_1\cdots\mu_\ell}_{ij} \equiv \int dK\, E^i_{\mathbf{k}u}\, E^j_{\mathbf{k}l}\, k^{\{\mu_1} \cdots k^{\mu_\ell\}} \delta\hat{f}_{\mathbf{k}}$$

Some of the moments can be identified with the ones appearing in $T^{\mu\nu}$ and N^μ:

$$\hat{\rho}_{10} \equiv n - \hat{n}\,, \quad \hat{\rho}_{20} \equiv e - \hat{e}\,, \quad \hat{\rho}_{01} \equiv n_l - \hat{n}_l\,, \quad \hat{\rho}_{11} \equiv M - \hat{M}\,, \quad \hat{\rho}_{02} \equiv P_l - \hat{P}_l$$
$$\hat{\rho}^{\mu}_{00} \equiv V^\mu_\perp\,, \quad \hat{\rho}^{\mu}_{10} \equiv W^\mu_{\perp u}\,, \quad \hat{\rho}^{\mu}_{01} \equiv W^\mu_{\perp l}\,, \quad \hat{\rho}^{\mu\nu}_{00} \equiv \pi^{\mu\nu}_\perp\,.$$

Boltzmann equation $k^\mu \partial_\mu f_{\mathbf{k}} = C[f] \longrightarrow$ Equations of motion for $\hat{\rho}^{\mu_1\cdots\mu_\ell}_{ij}$

$$\frac{d}{d\tau}\hat{\rho}^{\{\mu_1\cdots\mu_\ell\}}_{ij} \equiv \Xi^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} \frac{d}{d\tau}\hat{\rho}^{\nu_1\cdots\nu_\ell}_{ij}$$

scalar

$$
\begin{aligned}
D\hat{\rho}_{ij} &= \mathcal{C}_{i-1,j} - D\hat{\mathcal{I}}_{ij} + D_l\hat{\mathcal{I}}_{i-1,j+1} + \left(i\,\hat{\mathcal{I}}_{i-1,j+1} + j\,\hat{\mathcal{I}}_{i+1,j-1}\right) l_\alpha Du^\alpha - \left[(i-1)\,\hat{\mathcal{I}}_{i-2,j+2} + (j+1)\,\hat{\mathcal{I}}_{ij}\right] l_\alpha D_l u^\alpha \\
&+ \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\mathcal{I}}_{i-2,j} - (i+1)\,\hat{\mathcal{I}}_{ij} + (i-1)\,\hat{\mathcal{I}}_{i-2,j+2}\right]\tilde{\theta} - \frac{1}{2}\left[m_0^2\,j\,\hat{\mathcal{I}}_{i-1,j-1} - j\,\hat{\mathcal{I}}_{i+1,j-1} + (j+2)\,\hat{\mathcal{I}}_{i-1,j+1}\right]\tilde{\theta}_l \\
&+ D_l\hat{\rho}_{i-1,j+1} - \tilde{\nabla}_\mu\hat{\rho}^\mu_{i-1,j} + \left[(i-1)\,\hat{\rho}^\mu_{i-2,j+1} + j\,\hat{\rho}^\mu_{i,j-1}\right] l_\alpha\tilde{\nabla}_\mu u^\alpha + \left(i\,\hat{\rho}_{i-1,j+1} + j\,\hat{\rho}_{i+1,j-1}\right) l_\alpha Du^\alpha \\
&- \left[(i-1)\,\hat{\rho}_{i-2,j+2} + (j+1)\,\hat{\rho}_{ij}\right] l_\alpha D_l u^\alpha + i\,\hat{\rho}^\mu_{i-1,j} Du_\mu - (i-1)\,\hat{\rho}^\mu_{i-2,j+1} D_l u_\mu \\
&- j\,\hat{\rho}^\mu_{i,j-1} Dl_\mu + (j+1)\,\hat{\rho}^\mu_{i-1,j} D_l l_\mu + \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\rho}_{i-2,j} - (i+1)\,\hat{\rho}_{ij} + (i-1)\,\hat{\rho}_{i-2,j+2}\right]\tilde{\theta} \\
&- \frac{1}{2}\left[m_0^2\,j\,\hat{\rho}_{i-1,j-1} - j\,\hat{\rho}_{i+1,j-1} + (j+2)\,\hat{\rho}_{i-1,j+1}\right]\tilde{\theta}_l + (i-1)\,\hat{\rho}^{\mu\nu}_{i-2,j}\tilde{\sigma}_{\mu\nu} - j\,\hat{\rho}^{\mu\nu}_{i-1,j-1}\tilde{\sigma}_{l,\mu\nu}\,, \qquad (110)
\end{aligned}
$$

vector

$$
\begin{aligned}
D\hat{\rho}^{\{\mu\}}_{ij} &= \mathcal{C}^{\{\mu\}}_{i-1,j} - \frac{1}{2}\tilde{\nabla}^\mu\left(m_0^2\,\hat{\mathcal{I}}_{i-1,j} - \hat{\mathcal{I}}_{i+1,j} + \hat{\mathcal{I}}_{i-1,j+2}\right) \\
&+ \frac{1}{2}\left[m_0^2\,i\,\hat{\mathcal{I}}_{i-1,j} - (i+2)\,\hat{\mathcal{I}}_{i+1,j} + i\,\hat{\mathcal{I}}_{i-1,j+2}\right]\Xi^\mu_\alpha Du^\alpha - \frac{1}{2}\left[m_0^2\,j\,\hat{\mathcal{I}}_{i,j-1} - j\,\hat{\mathcal{I}}_{i+2,j-1} + (j+2)\,\hat{\mathcal{I}}_{i,j+1}\right]\Xi^\mu_\alpha Dl^\alpha \\
&- \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\mathcal{I}}_{i-2,j+1} - (i+1)\,\hat{\mathcal{I}}_{i,j+1} + (i-1)\,\hat{\mathcal{I}}_{i-2,j+3}\right]\Xi^\mu_\alpha D_l u^\alpha \\
&+ \frac{1}{2}\left[m_0^2\,(j+1)\,\hat{\mathcal{I}}_{i-1,j} - (j+1)\,\hat{\mathcal{I}}_{i+1,j} + (j+3)\,\hat{\mathcal{I}}_{i-1,j+2}\right]\Xi^\mu_\alpha D_l l^\alpha \\
&+ \frac{1}{2}\left[(i-1)\left(m_0^2\,\hat{\mathcal{I}}_{i-2,j+1} - \hat{\mathcal{I}}_{i,j+1} + \hat{\mathcal{I}}_{i-2,j+3}\right) + j\left(m_0^2\,\hat{\mathcal{I}}_{i,j-1} - \hat{\mathcal{I}}_{i+2,j-1} + \hat{\mathcal{I}}_{i,j+1}\right)\right] l_\alpha\tilde{\nabla}^\mu u^\alpha \\
&+ \frac{1}{2}\left[m_0^2\,i\,\hat{\rho}_{i-1,j} - (i+2)\,\hat{\rho}_{i+1,j} + i\,\hat{\rho}_{i-1,j+2}\right]\Xi^\mu_\alpha Du^\alpha - \frac{1}{2}\left[m_0^2\,j\,\hat{\rho}_{i,j-1} - j\,\hat{\rho}_{i+2,j-1} + (j+2)\,\hat{\rho}_{i,j+1}\right]\Xi^\mu_\alpha Dl^\alpha \\
&- \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\rho}_{i-2,j+1} - (i+1)\,\hat{\rho}_{i,j+1} + (i-1)\,\hat{\rho}_{i-2,j+3}\right]\Xi^\mu_\alpha D_l u^\alpha \\
&+ \frac{1}{2}\left[m_0^2\,(j+1)\,\hat{\rho}_{i-1,j} - (j+1)\,\hat{\rho}_{i+1,j} + (j+3)\,\hat{\rho}_{i-1,j+2}\right]\Xi^\mu_\alpha D_l l^\alpha \\
&- \frac{1}{2}\tilde{\nabla}^\mu\left(m_0^2\,\hat{\rho}_{i-1,j} - \hat{\rho}_{i+1,j} + \hat{\rho}_{i-1,j+2}\right) + \Xi^\mu_\alpha D_l\hat{\rho}^\alpha_{i-1,j+1} \\
&+ \frac{1}{2}\left[(i-1)\left(m_0^2\,\hat{\rho}_{i-2,j+1} - \hat{\rho}_{i,j+1} + \hat{\rho}_{i-2,j+3}\right) + j\left(m_0^2\,\hat{\rho}_{i,j-1} - \hat{\rho}_{i+2,j-1} + \hat{\rho}_{i,j+1}\right)\right] l_\alpha\tilde{\nabla}^\mu u^\alpha \\
&+ \left[i\,\hat{\rho}^\mu_{i-1,j+1} + j\,\hat{\rho}^\mu_{i+1,j-1}\right] l_\alpha Du^\alpha + \hat{\rho}_{ij,\nu}\,\tilde{\omega}^{\mu\nu} + \hat{\rho}_{i-1,j+1,\nu}\,\tilde{\omega}^{\mu\nu}_l \\
&+ \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\rho}^\mu_{i-2,j} - (i+2)\,\hat{\rho}^\mu_{ij} + (i-1)\,\hat{\rho}^\mu_{i-2,j+2}\right]\tilde{\theta} - \frac{1}{2}\left[m_0^2\,j\,\hat{\rho}^\mu_{i-1,j-1} - j\,\hat{\rho}^\mu_{i+1,j-1} + (j+3)\,\hat{\rho}^\mu_{i-1,j+1}\right]\tilde{\theta}_l \\
&+ \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\rho}_{i-2,j,\nu} - (i+1)\,\hat{\rho}_{ij,\nu} + (i-1)\,\hat{\rho}_{i-2,j+2,\nu}\right]\tilde{\sigma}^{\mu\nu} \\
&- \frac{1}{2}\left[m_0^2\,j\,\hat{\rho}_{i-1,j-1,\nu} - j\,\hat{\rho}_{i+1,j-1,\nu} + (j+2)\,\hat{\rho}_{i-1,j+1,\nu}\right]\tilde{\sigma}^{\mu\nu}_l - \left[(i-1)\,\hat{\rho}^\mu_{i-2,j+2} + (j+1)\,\hat{\rho}^\mu_{ij}\right] l_\alpha D_l u^\alpha \\
&- \Xi^\mu_\alpha\tilde{\nabla}_\nu\hat{\rho}^{\alpha\nu}_{i-1,j} - (i-1)\,\hat{\rho}^{\mu\nu}_{i-2,j+1} D_l u_\nu + (j+1)\,\hat{\rho}^{\mu\nu}_{i-1,j} D_l l_\nu + i\,\hat{\rho}^{\mu\nu}_{i-1,j} Du_\nu - j\,\hat{\rho}^{\mu\nu}_{i,j-1} Dl_\nu \\
&+ \left[(i-1)\,\hat{\rho}^{\mu\nu}_{i-2,j+1} + j\,\hat{\rho}^{\mu\nu}_{i,j-1}\right] l_\alpha\tilde{\nabla}_\nu u^\alpha + (i-1)\,\hat{\rho}^{\mu\nu\lambda}_{i-2,j}\tilde{\sigma}_{\nu\lambda} - j\,\hat{\rho}^{\mu\nu\lambda}_{i-1,j-1}\tilde{\sigma}_{l,\nu\lambda}\,, \qquad (111)
\end{aligned}
$$

rank 2 tensor

$$
\begin{aligned}
D\hat{\rho}^{\{\mu\nu\}}_{ij} &= \mathcal{C}^{\{\mu\nu\}}_{i-1,j} \\
&+ \frac{1}{4}\left\{m_0^4\,(i-1)\,\hat{\mathcal{I}}_{i-2,j} - 2m_0^2\left[(i+1)\,\hat{\mathcal{I}}_{ij} - (i-1)\,\hat{\mathcal{I}}_{i-2,j+2}\right] - 2\,(i+1)\,\hat{\mathcal{I}}_{i,j+2} + (i+3)\,\hat{\mathcal{I}}_{i+2,j} + (i-1)\,\hat{\mathcal{I}}_{i-2,j+4}\right\}\tilde{\sigma}^{\mu\nu} \\
&- \frac{1}{4}\left\{m_0^4\,j\,\hat{\mathcal{I}}_{i-1,j-1} - 2m_0^2\left[j\,\hat{\mathcal{I}}_{i+1,j-1} - (j+2)\,\hat{\mathcal{I}}_{i-1,j+1}\right] - 2\,(j+2)\,\hat{\mathcal{I}}_{i+1,j+1} + j\,\hat{\mathcal{I}}_{i+3,j-1} + (j+4)\,\hat{\mathcal{I}}_{i-1,j+3}\right\}\tilde{\sigma}^{\mu\nu}_l \\
&+ \frac{1}{4}\left\{m_0^4\,(i-1)\,\hat{\rho}_{i-2,j} - 2m_0^2\left[(i+1)\,\hat{\rho}_{ij} - (i-1)\,\hat{\rho}_{i-2,j+2}\right] - 2\,(i+1)\,\hat{\rho}_{i,j+2} + (i+3)\,\hat{\rho}_{i+2,j} + (i-1)\,\hat{\rho}_{i-2,j+4}\right\}\tilde{\sigma}^{\mu\nu} \\
&- \frac{1}{4}\left\{m_0^4\,j\,\hat{\rho}_{i-1,j-1} - 2m_0^2\left[j\,\hat{\rho}_{i+1,j-1} - (j+2)\,\hat{\rho}_{i-1,j+1}\right] - 2\,(j+2)\,\hat{\rho}_{i+1,j+1} + j\,\hat{\rho}_{i+3,j-1} + (j+4)\,\hat{\rho}_{i-1,j+3}\right\}\tilde{\sigma}^{\mu\nu}_l \\
&+ \frac{1}{2}\left[m_0^2\,i\,\hat{\rho}^{\{\mu}_{i-1,j} - (i+4)\,\hat{\rho}^{\{\mu}_{i+1,j} + i\,\hat{\rho}^{\{\mu}_{i-1,j+2}\right] Du^{\nu\}} - \frac{1}{2}\left[m_0^2\,j\,\hat{\rho}^{\{\mu}_{i,j-1} - j\,\hat{\rho}^{\{\mu}_{i+2,j-1} + (j+4)\,\hat{\rho}^{\{\mu}_{i,j+1}\right] Dl^{\nu\}} \\
&+ \frac{1}{2}\Xi^{\mu\nu}_{\alpha\beta}\left[(i-1)\left(m_0^2\,\hat{\rho}^\alpha_{i-2,j+1} - \hat{\rho}^\alpha_{i,j+1} + \hat{\rho}^\alpha_{i-2,j+3}\right) + j\left(m_0^2\,\hat{\rho}^\alpha_{i,j-1} - \hat{\rho}^\alpha_{i+2,j-1} + \hat{\rho}^\alpha_{i,j+1}\right)\right] l_\gamma\tilde{\nabla}^\beta u^\gamma \\
&- \frac{1}{2}\Xi^{\mu\nu}_{\alpha\beta}\left[m_0^2\,(i-1)\,\hat{\rho}^\alpha_{i-2,j+1} - (i+3)\,\hat{\rho}^\alpha_{i,j+1} + (i-1)\,\hat{\rho}^\alpha_{i-2,j+3}\right] D_l u^\beta \\
&+ \frac{1}{2}\Xi^{\mu\nu}_{\alpha\beta}\left[m_0^2\,(j+1)\,\hat{\rho}^\alpha_{i-1,j} - (j+1)\,\hat{\rho}^\alpha_{i+1,j} + (j+5)\,\hat{\rho}^\alpha_{i-1,j+2}\right] D_l l^\beta \\
&- 2\,\tilde{\omega}_\lambda{}^{\{\mu}\,\hat{\rho}^{\nu\}\lambda}_{ij} - 2\,\tilde{\omega}_{l,\lambda}{}^{\{\mu}\,\hat{\rho}^{\nu\}\lambda}_{i-1,j+1} - \frac{1}{2}\tilde{\nabla}^{\{\mu}\left(m_0^2\,\hat{\rho}^{\nu\}}_{i-1,j} - \hat{\rho}^{\nu\}}_{i+1,j} + \hat{\rho}^{\nu\}}_{i-1,j+2}\right) \\
&+ \Xi^{\mu\nu}_{\alpha\beta} D_l\hat{\rho}^{\alpha\beta}_{i-1,j+1} + \left[i\,\hat{\rho}^{\mu\nu}_{i-1,j+1} + j\,\hat{\rho}^{\mu\nu}_{i+1,j-1}\right] l_\alpha Du^\alpha - \left[(i-1)\,\hat{\rho}^{\mu\nu}_{i-2,j+2} + (j+1)\,\hat{\rho}^{\mu\nu}_{ij}\right] l_\alpha D_l u^\alpha \\
&+ \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\rho}^{\mu\nu}_{i-2,j} - (i+3)\,\hat{\rho}^{\mu\nu}_{ij} + (i-1)\,\hat{\rho}^{\mu\nu}_{i-2,j+2}\right]\tilde{\theta} + \frac{2}{3}\left[m_0^2\,(i-1)\,\hat{\rho}^{\kappa\{\mu}_{i-2,j} - (i+2)\,\hat{\rho}^{\kappa\{\mu}_{ij} + (i-1)\,\hat{\rho}^{\kappa\{\mu}_{i-2,j+2}\right]\tilde{\sigma}^{\nu\}}_\kappa \\
&- \frac{1}{2}\left[m_0^2\,j\,\hat{\rho}^{\mu\nu}_{i-1,j-1} - j\,\hat{\rho}^{\mu\nu}_{i+1,j-1} + (j+4)\,\hat{\rho}^{\mu\nu}_{i-1,j+1}\right]\tilde{\theta}_l - \frac{2}{3}\left[m_0^2\,j\,\hat{\rho}^{\kappa\{\mu}_{i-1,j-1} - j\,\hat{\rho}^{\kappa\{\mu}_{i+1,j-1} + (j+3)\,\hat{\rho}^{\kappa\{\mu}_{i-1,j+1}\right]\tilde{\sigma}^{\nu\}}_{l,\kappa} \\
&- \Xi^{\mu\nu}_{\alpha\beta}\tilde{\nabla}_\lambda\hat{\rho}^{\alpha\beta\lambda}_{i-1,j} + i\,\hat{\rho}^{\mu\nu\gamma}_{i-1,j} Du_\gamma - j\,\hat{\rho}^{\mu\nu\gamma}_{i,j-1} Dl_\gamma - (i-1)\,\hat{\rho}^{\mu\nu\lambda}_{i-2,j+1} D_l u_\lambda + (j+1)\,\hat{\rho}^{\mu\nu\lambda}_{i-1,j} D_l l_\lambda \\
&+ \left[(i-1)\,\hat{\rho}^{\mu\nu\lambda}_{i-2,j+1} + j\,\hat{\rho}^{\mu\nu\lambda}_{i,j-1}\right] l_\alpha\tilde{\nabla}_\lambda u^\alpha + (i-1)\,\hat{\rho}^{\mu\nu\lambda\kappa}_{i-2,j}\tilde{\sigma}_{\lambda\kappa} - j\hat{\rho}^{\mu\nu\lambda\kappa}_{i-1,j-1}\tilde{\sigma}_{l,\lambda\kappa}\,. \qquad (112)
\end{aligned}
$$

Boltzmann equation in terms of moments

To reduce the (infinite) degrees of freedom to the fluid dynamical ones, this set of equations need to be truncated.

Simple example: 0+1 dimensional Bjorken expansion

- Romatschke-Strickland anisotropic distribution function
- Boltzmann equation with relaxation time approximation (RTA)
- RTA can be solved exactly:
  Florkowski, Ryblewski and Strickland, Nucl. Phys. A **916**, 249 (2013)
  Florkowski, Ryblewski and Strickland, Phys. Rev. C **88**, 024903 (2013)
  (Thanks to M. Strickland for showing us how)
- $\delta\hat{f}$ decouples from the evolution (up to 14-moment approximation)

The Romatschke-Strickland distribution function

Romatschke and Strickland, Phys. Rev. D **68**, 036004 (2003)

The RS-distribution function

$$\hat{f}_{RS}(\alpha_{RS}, \beta_{RS}, \xi) \equiv \left[\exp\left(-\alpha_{RS} + \beta_{RS}\sqrt{E_{\mathbf{k}u}^2 + \xi E_{\mathbf{k}l}^2}\right) + a\right]^{-1},$$

ξ is the so-called anisotropy parameter

The conserved quantities

$$\begin{aligned}
\hat{N}^{\mu}_{RS} &= \hat{n} u^{\mu}, \\
\hat{T}^{\mu\nu}_{RS} &= \hat{e} u^{\mu} u^{\nu} + \hat{P}_l l^{\mu} l^{\nu} - \hat{P}_{\perp} \Xi^{\mu\nu}, \\
\hat{n} &\equiv \hat{I}^{RS}_{100} = n_0(\alpha_{RS}, \beta_{RS}) R_{100}(\xi), \qquad \hat{e} \equiv \hat{I}^{RS}_{200} = e_0(\alpha_{RS}, \beta_{RS}) R_{200}(\xi), \\
\hat{P}_l &\equiv \hat{I}^{RS}_{220} = e_0(\alpha_{RS}, \beta_{RS}) R_{220}(\xi), \qquad \hat{P}_{\perp} \equiv \hat{I}^{RS}_{201} = P_0(\alpha_{RS}, \beta_{RS}) R_{201}(\xi).
\end{aligned}$$

where in general moments of $\hat{f}_{RS}$ can be defined as

The thermodynamic integrals

$$\hat{I}^{RS}_{nrq} \equiv \frac{(-1)^q}{(2q)!!} \int dK E_{\mathbf{k}u}^{n-r-2q} E_{\mathbf{k}l}^{r} \left(\Xi^{\mu\nu} k_{\mu} k_{\nu}\right)^q \; \hat{f}_{RS}$$

Any of these moments can be taken as dynamical variable

The Romatschke-Strickland distribution function

Landau matching

$$(\hat{N}^{\mu}_{RS} - N^{\mu}_{0})u_{\mu} = 0 \quad \Rightarrow \quad \hat{n}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = n_0\left(\alpha_0, \beta_0\right),$$

$$(\hat{T}^{\mu\nu}_{RS} - T^{\mu\nu}_{0})u_{\mu}u_{\nu} = 0 \quad \Rightarrow \quad \hat{e}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = e_0\left(\alpha_0, \beta_0\right).$$

Conservation of particles

$$\beta_0 = \beta_{RS}\frac{R_{100}\left(\xi\right)}{R_{200}\left(\xi\right)}, \qquad \lambda_0 = \lambda_{RS}\frac{\left[R_{100}\left(\xi\right)\right]^4}{\left[R_{200}\left(\xi\right)\right]^3},$$

$$\hat{I}^{RS}_{nrq}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = I_{nq}\left(\alpha_0, \beta_0\right) R_{nrq}\left(\xi\right) \frac{\left[R_{200}\left(\xi\right)\right]^{1-n}}{\left[R_{100}\left(\xi\right)\right]^{2-n}}.$$

Non-Conservation of particles

$$\beta_0 = \frac{\beta_{RS}}{\left[R_{200}\left(\xi\right)\right]^{1/4}}$$

$$\hat{I}^{RS}_{nrq}\left(\beta_{RS}, \xi\right) = I_{nq}\left(\beta_0\right)\frac{R_{nrq}\left(\xi\right)}{\left[R_{200}\left(\xi\right)\right]^{(n+2)/4}}.$$

0+1D Bjorken expansion

$$u^\mu \equiv \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right) = (\cosh\eta, 0, 0, \sinh\eta)\,,$$
$$l^\mu \equiv \left(\frac{z}{\tau}, 0, 0, \frac{t}{\tau}\right) = (\sinh\eta, 0, 0, \cosh\eta)\,.$$

General equation of motion

$$\frac{\partial \hat{I}_{i+j,j,0}}{\partial\tau} + \frac{1}{\tau}\left[(j+1)\,\hat{I}_{i+j,j,0} + (i-1)\,\hat{I}_{i+j,j+2,0}\right] = \hat{\mathcal{C}}_{i-1,j},$$
$$\hat{\mathcal{C}}_{i-1,j} = -\frac{1}{\tau_{eq}}\left(\hat{I}_{i+j,j,0} - I_{i+j,j,0}\right)\,, \qquad \tau_{eq}(\tau) = 5\beta_0(\tau)\frac{\eta}{s}.$$

Conservation equations

$$\frac{\partial n_0\left(\alpha_0,\beta_0\right)}{\partial\tau} + \frac{1}{\tau} n_0\left(\alpha_0,\beta_0\right) = 0,$$
$$\frac{\partial e_0\left(\alpha_0,\beta_0\right)}{\partial\tau} + \frac{1}{\tau}\left[e_0\left(\alpha_0,\beta_0\right) + \hat{P}_l\left(\alpha_{RS},\beta_{RS},\xi\right)\right] = 0.$$

Note: Evolution of e_0 depends only on P_l.

0+1D Bjorken expansion: choice of eom/matching

$$\frac{\partial \hat{P}_l}{\partial \tau} + \frac{1}{\tau}\left(3\hat{P}_l - \hat{I}^{RS}_{240}\right) = -\frac{1}{\tau_{eq}}\left(\hat{P}_l - P_0\right),$$

$$\frac{\partial \hat{n}}{\partial \tau} + \frac{1}{\tau}\hat{n} = -\frac{1}{\tau_{eq}}\left(\hat{n} - n_0\right),$$

$$\frac{\partial \hat{I}^{RS}_{000}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}^{RS}_{000} - \hat{I}^{RS}_{020}\right) = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{000} - I_{000}\right),$$

$$\frac{\partial \hat{I}^{RS}_{300}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}^{RS}_{300} + 2\hat{I}^{RS}_{320}\right) = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{300} - I_{300}\right),$$

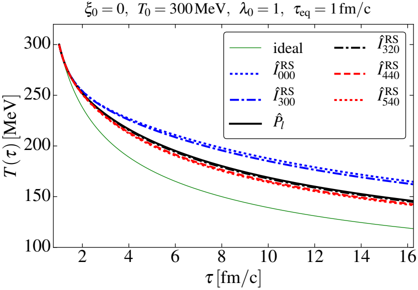
$$\frac{\partial \hat{I}^{RS}_{320}}{\partial \tau} + \frac{3}{\tau}\hat{I}^{RS}_{320} = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{320} - I_{320}\right)$$

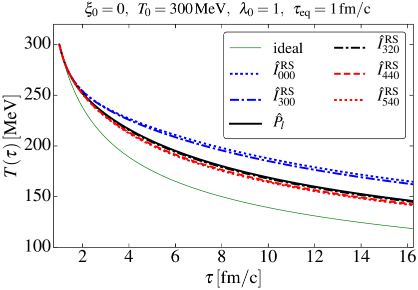
$$\frac{\partial \hat{I}^{RS}_{440}}{\partial \tau} + \frac{1}{\tau}\left(5\hat{I}^{RS}_{440} - \hat{I}^{RS}_{460}\right) = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{440} - I_{440}\right),$$

$$\frac{\partial \hat{I}^{RS}_{540}}{\partial \tau} + \frac{5}{\tau}\hat{I}^{RS}_{540} = -\frac{1}{\tau_{eq}}\left(\hat{I}^{RS}_{540} - I_{540}\right).$$

Note: Choosing any of these also implies a matching conditions for ξ
Note2: Up to 14-moment approximation: no corrections from $\delta\hat{f}_{\mathbf{k}}$

0+1D Bjorken expansion - Temperature

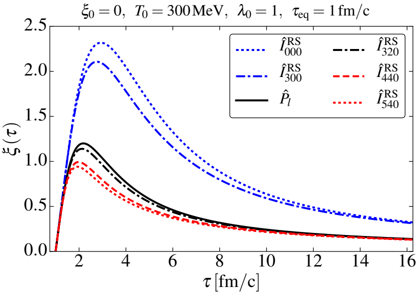


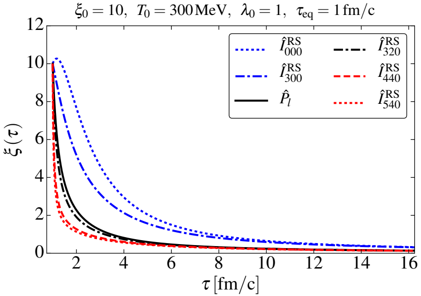


- The evolution of temperature for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of temperature for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion - Anisotropy

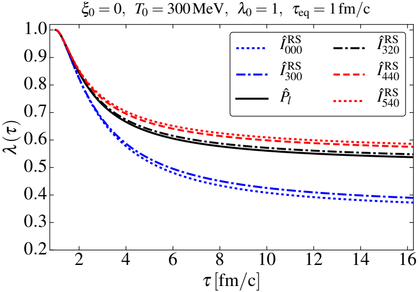


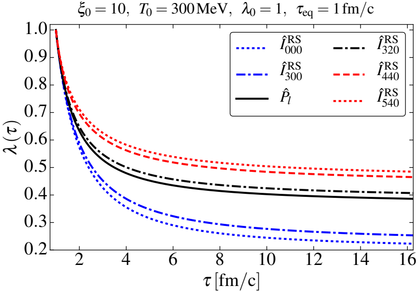


- The evolution of the anisotropy parameter for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of anisotropy parameter for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion - Fugacity

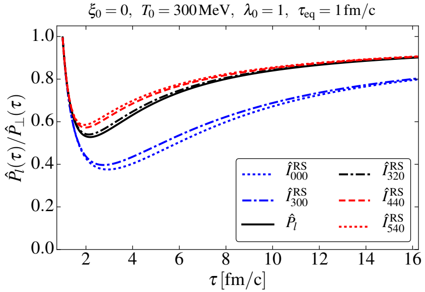


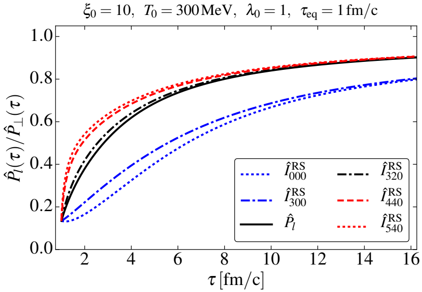


- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion - Pressure

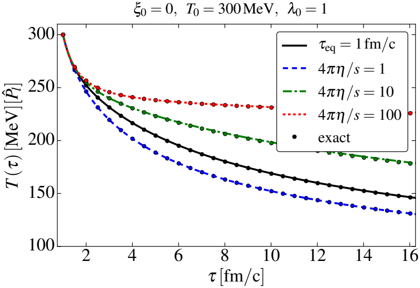


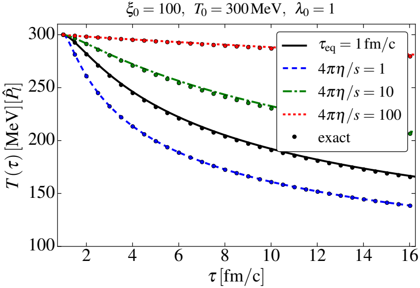


- The evolution of the L/T pressure for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the L/T pressure for $\xi(\tau_0) = \xi_0 = 10$.

0+1D Bjorken expansion - Comparisons to the exact solution

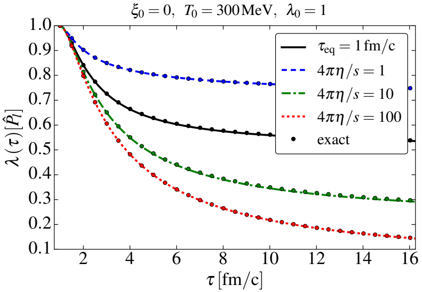


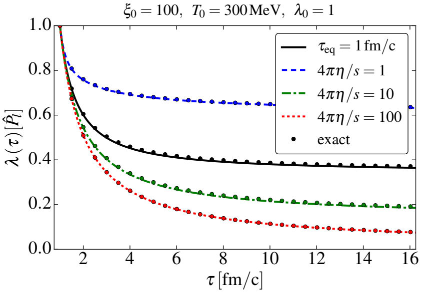


- The evolution of the temperature for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the temperature for $\xi(\tau_0) = \xi_0 = 100$.

0+1D Bjorken expansion - Comparisons to the exact solution

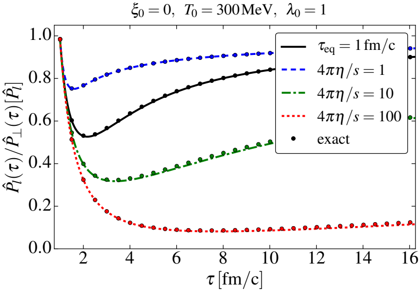


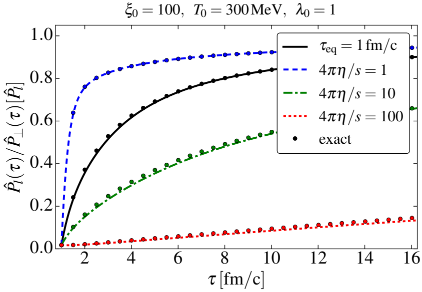


- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the fugacity for $\xi(\tau_0) = \xi_0 = 100$.

0+1D Bjorken expansion - Comparisons to the exact solution

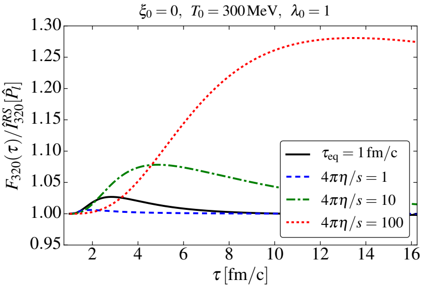


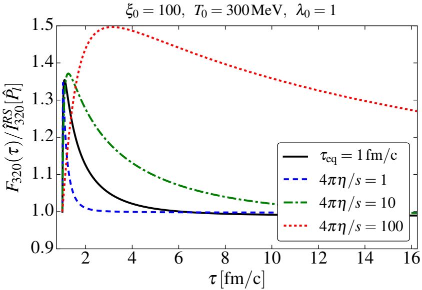


- The evolution of the L/T pressure for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the L/T pressure for $\xi(\tau_0) = \xi_0 = 100$.

0+1D Bjorken expansion - Comparisons to the exact solution





- The evolution of the I_{320} for $\xi(\tau_0) = \xi_0 = 0$.

- The evolution of the I_{320} for $\xi(\tau_0) = \xi_0 = 100$.

$$F_{nrq} = \frac{(-1)^q}{(2q)!!} \int dK E_{\mathbf{k}u}^{n-r-2q} E_{\mathbf{k}l}^{r} \left(\Xi^{\mu\nu} k_\mu k_\nu\right)^q f_{\mathbf{k}}$$

Summary

- Derived general moment expansion around anisotropic state
- Ambiguity in closing the equations investigated in 0+1D Bjorken expansion
- Excellent agreement with the (RTA) Boltzmann equation, when eom written directly for P_l