Charge redistribution
from novel magneto-vorticity coupling
in anomalous hydrodynamics


Koichi Hattori
Fudan University
Quark Matter 2017 @ Chicago
Strong magnetic field & vorticity/angular momentum induced by heavy-ion collisions

$$\omega = \nabla \times \mathbf{v}_{\text{fluid}}$$
Theoretical estimates

Super-strong B

Deng & Huang (2012), KH & Huang (2016)
Skokov et al. (2009), Voronyuk et al. (2011),

Impact parameter

Vorticity in HIC

Becattini et al., Csernai et al., Huang, Huovinen, Wang

Cf., Long-lived B thanks to back reaction from QGP?
→ Need to know Electrical conductivity in B

KH, S. Li, D. Satow, H.-U. Yee

Extension of AMY’ framework in a strong B.
-- Novel damping mechanism emerging only in B.

Transition from strong B to weak B (Effects of hLL)
How do their effects manifest?
Anomaly-induced transports in a magnetic OR vortex field

\[
\begin{pmatrix}
  j_V^\mu \\
  j_A^\mu
\end{pmatrix} = C_A \begin{pmatrix}
  q_f \mu_A \\
  \mu_V \mu_A
\end{pmatrix} \begin{pmatrix}
  q_f \mu_V \\
  (\mu_V^2 + \mu_A^2)/2 + C_A^{-1}T^2/12
\end{pmatrix} \begin{pmatrix}
  B^\mu \\
  \omega^\mu
\end{pmatrix}
\]

\[
B^\mu = \tilde{F}^{\mu\nu}u_\nu \\
\omega^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}u_\alpha \partial_\beta u_\gamma
\]

Coefficients are time-reversal even.
Non-dissipative transport phenomena is due to topology in quantum anomaly and is nonrenormalizable.

Anomaly relation: \[
\partial_\mu j_A^\mu = q_f^2 C_A E \cdot B
\]

\[
C_A = \frac{1}{2\pi^2}
\]
Spin polarizations from spin-rotation coupling

$$f^{\pm}(\epsilon, \omega) = f_{0}^{\pm}(\epsilon - \mathbf{S} \cdot \omega) \quad f_{0}^{\pm}(\epsilon) = \frac{1}{e^{\pm \beta(\epsilon - \mu)} + 1}$$

Λ polarization

See talks by Upsal, Karpenko, Pang, Q.Wang

Becattini et al., Glastad & Csernai, Gyulassy & Torrieri, Xie,,,
What would be missing?

An interplay $B \otimes \omega$

For dimensional reason, one would get

$$j \sim \# B \cdot \omega$$

Cf., In CVE, it was

$$j^{\mu}_{R/L} = C_A \mu^{2}_{R/L} \omega^{\mu}$$

Could the magneto-vorticity coupling be important??

Q1. Is the coefficient related to any quantum anomaly?

Q2. How is $T$ and/or $\mu$ dependence?

Anomalous hydrodynamics due to Son and Surowka

$$j^{\mu} = \sigma_{\text{Ohm}} E^{\mu} + \xi_B B^{\mu} + \xi_{\omega} \omega^{\mu}$$

The first-order derivative expansion [$A^{\mu} \sim O(\partial^0)$, $\nu^{\mu} \sim O(\partial^0)$]

$$E^{\mu} \sim B^{\mu} \sim \omega^{\mu} \sim O(\partial^1)$$

$\Rightarrow$ Yes, it is important when $B$ is so strong that $B >> O(\partial^1)$.
Dynamics in the lowest Landau levels
-- A quick reminder
Energy levels in $B$ – Chirality eigenstates in the lowest Landau levels

Periodic cyclotron motion $\rightarrow$ Landau level discretization

“Harmonic oscillator” in the transverse plane

$$\epsilon_n^2 = p_z^2 + (2n + 1)eB$$

The lowest Landau levels – Unique ground states

1. Spin directions are “frozen” due to strong $B$.
2. Chirality is locked with the momentum direction.
3. Massless LLL states realize chiral fermions in the (1+1) dim.

$$\epsilon_{n=0} = \pm p_z$$

Zeeman splitting

$$\Delta\epsilon_{Zeeman} = \pm eB \quad (\text{if } g = 2.)$$

$$\epsilon(p_z = m = 0)$$

$\sim \sqrt{eB}$

Spin 0

Spin $\frac{1}{2}$
Chiral anomaly in (1+1) dimension

When a parallel electric field (E//B) is applied,

Creation/annihilation near the surface of the bottomless Dirac sea.

Nielsen-Ninomiya, Ambjorn-Greensite-Peterson
“Density” of cyclotron motions

In a homogeneous B, there is a translational invariance.

There are no preferred positions for the center of the cyclotron motion.

Degeneracy in the transverse plane:

\[ \rho = \frac{N_{\text{state}}}{S} = \frac{eB}{2\pi} \]
Consequences of a magneto-vorticity coupling
Shift of thermal distribution functions by the spin-vorticity coupling

Spin-vorticity coupling

\[ f^\pm(\epsilon, \omega) = f_0^\pm(\epsilon - \mathbf{S} \cdot \mathbf{\omega}) \quad f_0^\pm(\epsilon) = \frac{1}{e^{\pm\beta(\epsilon - \mu)} + 1} \]

Landau & Lifshitz, Becattini et al.

In the LLL, the spin direction is aligned along a magnetic field.

\[ \Delta \epsilon^\pm \equiv -\mathbf{S} \cdot \mathbf{\omega} = \mp \text{sgn}(q_f) \frac{1}{2} \hat{B} \cdot \mathbf{\omega} \quad - \text{for particle} \]
\[ + \text{ for antiparticle} \]

Number density

\[ n_R = \frac{|q_f B|}{2\pi} \left[ \int_0^\infty \frac{dp_z}{2\pi} f^+(p) + \int_{-\infty}^0 \frac{dp_z}{2\pi} f^-(p) \right] \]

At the LO in the energy shift \( \Delta \epsilon \)

\[ \Delta n_R = \frac{|q_f B|}{2\pi} \left[ \Delta \epsilon^+ \int_0^\infty \frac{dp_z}{2\pi} \frac{\partial f_0^+(p_z)}{\partial p_z} + \Delta \epsilon^- \int_{-\infty}^0 \frac{dp_z}{2\pi} \frac{\partial f_0^-(p_z)}{\partial p_z} \right] \]
$$\Delta n_R = q_f \frac{C_A}{4} B \cdot \omega [f_0^+(0) + f_0^-(0)]$$

$$= q_f \frac{C_A}{4} B \cdot \omega$$

$$f_0^+(0) + f_0^-(0) = 1 \text{ identically for any } T \text{ and } \mu.$$ 

The shift is independent of the chirality, and depends only on the spin direction.

$$\Delta n_L = \Delta n_R$$

In the V-A basis,

$$\Delta n_V = \Delta n_R + \Delta n_L = q_f \frac{C_A}{2} B \cdot \omega$$

$$\Delta n_A = \Delta n_R - \Delta n_L = 0$$

Spatial components of the current

$$\Delta j_R^3 = v_R \Delta n_R \quad j^1 = j^2 = 0 \text{ for the LLL}$$

Velocity: $v_{R/L} = \pm \text{sgn}(q_f B)$

The shift depends on the chirality through the velocity.

$$\Delta j_R^3 = -\Delta j_L^3$$

In the V-A basis, $\Delta j_V^3 = 0$

$$\Delta j_A^3 = |q_f| \text{sgn}(B) \frac{C_A}{2} B \cdot \omega$$
Field-theoretical computation by Kubo formula

Perturbative $\omega$ in a strong $B$

$$\lambda = -2i \lim_{q_x \to 0} \frac{\partial}{\partial q_x} \langle n_V(x) T^{02}(x') \rangle \theta(t - t')$$

Similar to the Kubo formula used to get the $T^2$ term in CVE (Landsteiner, Megias, Pena-Benitez)

We confirm
1. the previous results obtained from the shift of distributions.
2. a relation of $\langle n_V T^{02} \rangle$ to the chiral anomaly diagram in the $(1+1)$ dim.

$$\Pi_{AV}^{\mu \nu} = j^{\mu}_A j^{\nu}_V$$

$q_\mu \Pi_{AV}^{\mu \nu} \neq 0$ !!!

There is no $T$ or $\mu$ correction in the massless limit, since it is related to the chiral anomaly!
Collective excitations and instabilities of chiral fluid with CME
Magneto-hydrodynamics (MHD) in conducting plasmas

Alfven’s theorem = “Frozen-in” condition:
Magnetic flux is frozen in a fluid volume and moves together with the fluid.

Transverse fluctuation

Oscillation

Tension of B-field = Restoring force
Fluid mass density = Inertia

Transverse Alfven wave
Alfvén wave from a linear analysis

\[ B_0 \neq 0, \ T > 0, \ \mu_V = 0 \]

Navier-Stokes + Maxwell eqs.

\[
\partial_\mu T^{\mu\nu} = F^{\nu\alpha} j_\alpha \\
\partial_\mu F^{\mu\nu} = j^\nu, \ \partial_\mu \tilde{F}^{\mu\nu} = 0 \\
j^\mu = \sigma_{\text{Ohm}} E^\mu
\]

Stationary solutions

\[ u^\mu = (1, 0), \ B^\mu = (0, B_0), \ j^\mu = (0, 0) \]

Transverse perturbations

\[ B_0 \perp \delta v, \delta B, \delta j(z) \]
as functions of z.

Wave equation

\[
\partial_t^2 \delta B(t, z) = \frac{B_0^2}{\epsilon + \rho} \partial_z^2 \delta B(t, z)
\]

Alfvén wave velocity

Transverse wave propagating along background \( B_0 \)

Alfvén wave propagating along \( k \)

Same wave equation for \( \delta v \)

\( \rightarrow \) Fluctuations of B and v propagate together.
Anomalous MHD in conducting plasmas

\[ \mu_A \neq 0, \ B_0 \neq 0, \ T > 0, \ \mu_V = 0 \]

A finite CME current without CVE:

\[ j^\mu = \sigma_{\text{Ohm}} E^\mu + \sigma_{\text{CME}} B^\mu \]

Wave equation

\[ \partial_t^2 \delta B(t, z) = v_{\text{Alf}}^2 \partial_z^2 \delta B(t, z) + \sigma_{\text{CME}} \nabla \times \partial_t \delta B \]

and a similar wave equation for \( \delta v \).

Helicity decomposition (Circular R/L polarizations)

\[ \nabla \times e_{R/L} = \pm e_{R/L} \]

Two modes for each helicity propagating in parallel to B AND in antiparallel to B.

In total, there are 4 modes.
R-helicity

L-helicity

$\nu_{\text{Alf}} = 0.2$

$\sigma_{\text{CME}} = 0$

$\sigma_{\text{CME}} = 0.2\sigma_{\text{Ohm}}$

$\sigma_{\text{CME}} = 0.6\sigma_{\text{Ohm}}$

Exponential growth

Always damping

Chiral imbalance btw R and L fermions

Chiral Plasma Instability (CPI)

Magnetic helicity

Fluid helicity (structures of vortex strings)

What would be a fate of the instability in the chiral fluid?

See also Y. Hirono’s talk on Feb. 10.
Summary 1

A magneto-vorticity coupling $B \otimes \omega$ induces charge redistributions without $\mu_A$.

- Related to the chiral anomaly in (1+1) dimensions.
- No $T$ or $\mu$ correction.

When $B \cdot \omega \neq 0$,

$$j^0_{EM,V} = q_f^2 C_A^2 (B \cdot \omega)$$

$$j^3_{EM,A} = \text{sgn}(q_f) q_f^2 C_A^2 (B \cdot \omega) \hat{B}$$

Emerges even without $\mu_A$. 
Summary 2

We observed an onset of instabilities in both B and fluid velocity v in anomalous hydrodynamics, when a chiral imbalance induces a CME current.

Amplitudes of both B and v grow exponentially in time.

Effects of the viscosity act to suppress the instability and change the wave velocity. [KH, Y. Hirono, H.-U. Yee, Y. Yin, In preparation.]
Causality problem:
A rigid rotation of an infinite-size system breaks causality in the peripheral region.

One must have a finite-size system with an exterior boundary or a local vortex field.

Local shift of the charge density
\rightarrow Redistribution of charge in the system

E.g., when \( v \to 0 \) sufficiently fast,
\[
\Delta n_{\text{net}} \propto \int d^3 x \mathbf{B} \cdot \boldsymbol{\omega} = \frac{1}{2} \int d^3 x \nabla \cdot (\mathbf{B} \times \mathbf{v}) = \frac{1}{2} \int_{\partial S} dS \cdot (\mathbf{B} \times \mathbf{v}) = 0
\]
Consistent predictions from fundamental and effective theories

QFT with Kubo formula

Anomalous hydrodynamics due to Son and Surowka

\[ j^\mu = \sigma_{\text{Ohm}} E^\mu + \xi_B B^\mu + \xi_\omega \omega^\mu \]

The first-order derivative expansion \([A^\mu \sim O(\partial^0), \nu^\mu \sim O(\partial^0)]\)

\[ E^\mu \sim B^\mu \sim \omega^\mu \sim O(\partial^1) \]

**B**\(^\mu\) and **ω**\(^\mu\) terms would violate the second law of thermodynamics if \(\sigma_{\text{CME}}\) and \(\sigma_{\text{CVE}}\) were arbitrary.

\[ \partial^\mu s_\mu = X(\xi_B, \xi_\omega, C_A) B^\mu + Y(\xi_B, \xi_\omega, C_A) \omega^\mu + (\text{dissipative parts}) \]

Cured by requiring the coefficients vanish identically.

⇒ Leads to the CME and CVE conductivities.

\[ \xi_B = \sigma_{\text{CME}}, \xi_\omega = \sigma_{\text{CVE}} \text{ up to the frame choice} \]
Field-theoretical computation by Kubo formula confirms the simplest calculation

Perturbative $\omega$ in a strong $B$

$$\lambda = -2i \lim_{q_x \to 0} \frac{\partial}{\partial q_x} \langle n_V(x) T^{02}(x') \rangle \theta(t - t')$$

Similar to the Kubo formula for CVE by Landsteiner, Megias, Pena-Benitez

$$n_V(x) = \bar{\psi}_{LLL}(x) \gamma^0 \psi_{LLL}(x)$$

$$T^{0i}(x) = \frac{i}{2} \bar{\psi}_{LLL}(x)(\gamma^0 D^i + \gamma^i D^0)\psi_{LLL}(x)$$

$$D^\mu = \partial^\mu + iq_f A^\mu_{ext}$$

Retarded correlator in the coordinate space

$$\langle n_V(x) T^{02}(x') \rangle \theta(t - t') = \frac{1}{2i} \text{tr} \left[ \gamma^0 P + S_{LLL}(x, x') \gamma^0 D_{x'}^{i=2} P + S_{LLL}(x', x) \right]$$

The correlator has a manifest gauge invariance wrt the gauge of $B_{ext}$

Cf., $S_{LLL}(x', x) \rightarrow e^{-i\alpha(x')} S_{LLL}(x', x) e^{i\alpha(x)}$  \quad $P_+ = (1 + isgn(q_f B)\gamma^1\gamma^2)/2$
$S_{LLL}(p) = 2e^{-\frac{|p_{\perp}|^2}{q_f B}} \frac{i}{\not{p}_{\parallel} + m_f} P_+$

$\langle n_V T^{02} \rangle = i \frac{q_f B}{8\pi} q_x \Pi^{00}_{1+1} \quad \Pi_{1+1}^\mu = \frac{1}{\pi} \frac{1}{q_{\parallel}^2} (q_{\parallel} g_{\parallel \parallel} - q_{\parallel} q_{\parallel}^\nu)$

We confirm $n_V = q_f \frac{c_A}{2} B \cdot \omega$ and $j_A = |q_f| \frac{c_A}{2} (B \cdot \omega) \hat{B}$.

Consistent with the previous observation from the shift of distributions.

Easy to establish a relation to the anomaly diagram

$\dot{j}_A^\mu = -\epsilon^{\mu \nu}_{\parallel} j_V^\nu \quad \epsilon^{03}_{\parallel} = -\epsilon^{30}_{\parallel} = 1 \quad \Pi_{AV}^{\mu \nu} = \epsilon^{\mu \alpha}_{\parallel} (\Pi_{1+1})^\nu_{\alpha}$

$p_{\parallel} \langle j_A \rangle_\mu = p_{\parallel} \epsilon_{\parallel}^{\mu \alpha} (\Pi_{1+1})_{\alpha \beta} A^\beta = \frac{1}{2\pi} \epsilon^{\mu \nu}_{\parallel} F_{\mu \nu} \quad \langle j_V^\mu \rangle = -\Pi^{\mu \nu}_{1+1} A^\nu$

for $q_f B > 0$. An overall minus sign appears when $q_f B < 0$.

There is no T or $\mu$ correction in the massless limit!
Alfvén wave from a linear analysis

\[ B_0 > 0, \ T > 0, \ \mu_V = 0 \]

**EoM**

\[
\begin{align*}
\partial_\mu T^{\mu\nu} &= F^{\nu\alpha} j_\alpha \\
\partial_\mu F^{\mu\nu} &= j^\nu, \ \partial_\mu \tilde{F}^{\mu\nu} = 0 \\
j^\mu &= \sigma_{\text{Ohm}} E^\mu
\end{align*}
\]

Stationary solutions

\[ u^\mu = (1, 0), \ B^\mu = (0, B_0), \ j^\mu = (0, 0) \]

Transverse perturbations

\[ B_0 \perp \delta v, \delta B, \delta j(z) \]

as functions of \( z \).

Linearized eqs in an ideal hydro.

\[
\begin{align*}
\partial_t \delta B &= B_0 \partial_z \delta v \\
(\epsilon + p) \partial_t \delta v &= B_0 \partial_z \delta B
\end{align*}
\]

Transverse mode propagating along \( B_0 \)

\[ B_0 \parallel k \]

Wave equation for the Alfvén wave

\[
\partial_t^2 \delta B(t, z) = \frac{B_0^2}{\epsilon + p} \partial_z^2 \delta B(t, z)
\]

Alfvén wave velocity
Anomalous MHD in conducting plasmas

\[ \mu_A \neq 0, \ B_0 \neq 0, \ T > 0, \ \mu_V = 0 \]

A finite CME current without CVE:

\[ j^\mu = \sigma_{\text{Ohm}} E^\mu + \sigma_{\text{CME}} B^\mu \]

Linearized eqs in an ideal hydro.

\[
\begin{align*}
\partial_t \delta B &= B_0 \partial_z \delta \nu + \sigma_{\text{CME}} \nabla \times \delta B \\
(\epsilon + p) \partial_t \delta \nu &= B_0 \partial_z \delta B
\end{align*}
\]

\[
\partial_t \begin{pmatrix} \delta B_R \\ \delta \nu_R \end{pmatrix} = \begin{pmatrix} \sigma_{\text{Ohm}}^{-1} (\sigma_{\text{CME}} k - k^2) & ikv^2_{\text{Alf}} \\ ik & 0 \end{pmatrix} \begin{pmatrix} \delta B_R \\ \delta \nu_R \end{pmatrix} \quad v^2_{\text{Alf}} = \frac{B_0^2}{\epsilon + p}
\]

\[
\omega = -\frac{i}{2} \Xi k \pm k \sqrt{v^2_{\text{Alf}} - \frac{1}{4} \Xi^2} \quad \Xi = \frac{\sigma_{\text{CME}} - k}{\sigma_{\text{Ohm}}}
\]

When \( \Xi > 0 \), instabilities arise both in \( \delta B \) and \( \delta \nu \)!!

Effects of the viscosity act to suppress the instability and change the wave velocity.