The onset of fluid-dynamical behavior in relativistic kinetic theory

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Based on G. Denicol, JN, arXiv:1608.07869 [nucl-th] + arXiv:170x.xxxxx

QUARK MATTER 2017, Chicago, USA

The ubiquitousness of fluid dynamics

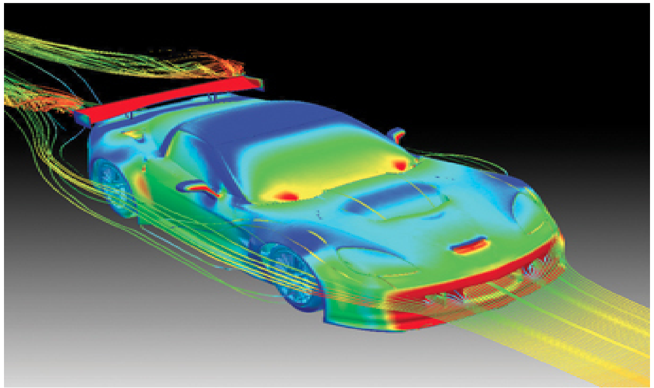
Based on conservations laws + large separation of length scales

Separation of scales $\rightarrow$ macroscopic: L microscopic: ℓ

Knudsen number expansion:

$$K_N \sim \frac{\ell}{L} \ll 1$$

$\rightarrow$ **FLUID**



L ~ 1 m $\ell \sim 10^{-7} m$

Macroscopic: Gradient of velocity field

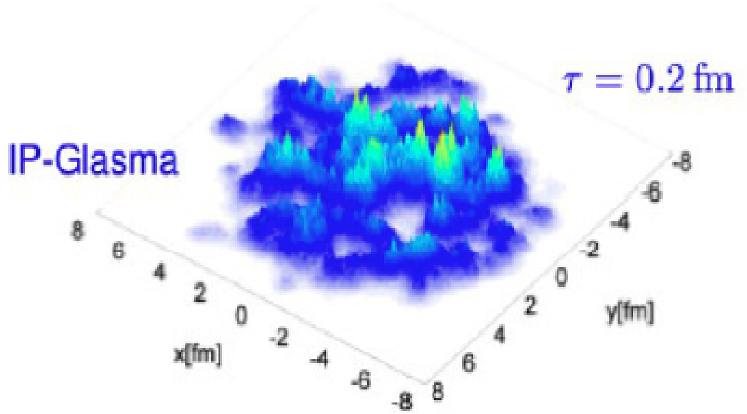
$$\partial v \sim 1/L$$

Example of microscopic scale:

$$\ell \sim 1/(n\sigma)$$ Mean free path

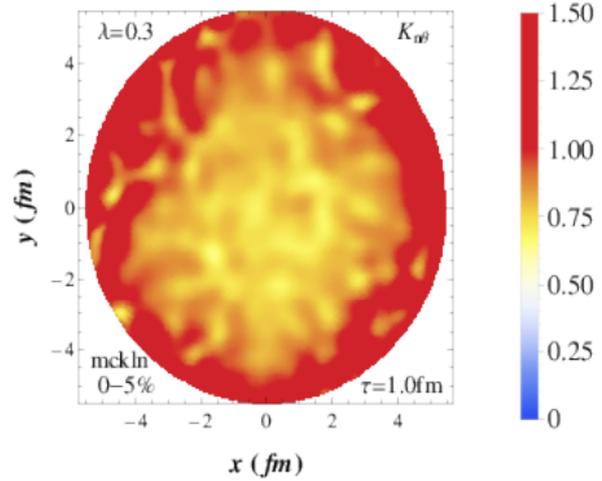
J. Noronha-Hostler, JN, M. Gyulassy, PRC 2016

QGP initial condition

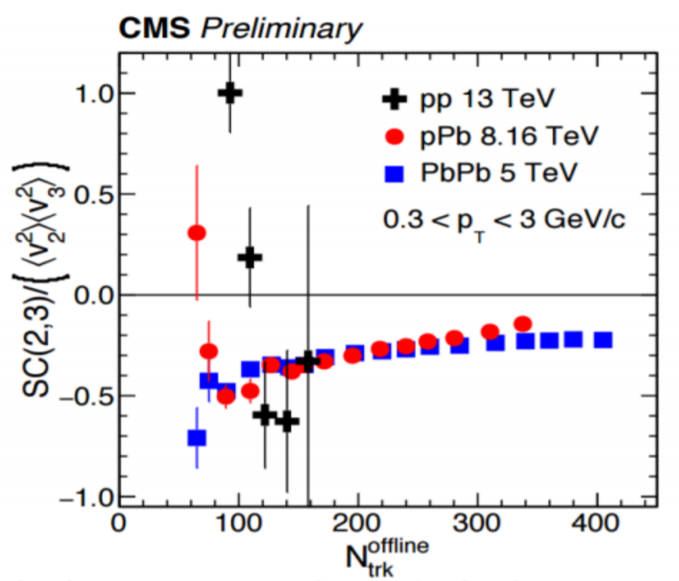


Schenke, Tribedy, Venugopalan, PRL 2012

Knudsen number event-by-event



Hydrodynamic behavior in small systems????



macro scale $\partial\varepsilon/\varepsilon_0 \sim \Lambda_{QCD}$

microscopic scale ????

There is no reason to believe that Kn has to be small in this case.

Hydrodynamics as a gradient expansion

Assume deviations from local equilibrium

Boltzmann equation	**Chapman-Enskog series (1939)**
$$p^\mu \partial_\mu f = \mathcal{C}[f,f]$$	$$f = f_{eq}\left(1 + \sum_{n=1}^{\infty} (K_N)^n f_n(x^\mu, p^\mu)\right)$$

Convergence???

- At 0th order in Kn → Ideal hydro, at 1st order → Navier-Stokes

- Very hard to carry out procedure to higher orders

- Solution of Boltzmann completely defined in terms of local hydrodynamic fields:

$$T,\ \mu,\ u^\mu$$

Hydrodynamics as a gradient expansion

- Gradient expansion "solution", though systematic, is highly contrived

- Kn itself depends on the flow properties

- Problems with instabilities and acausality in the relativistic domain

- Procedure cannot describe all possible solutions of Boltzmann …

Ex: <u>Homogeneous relaxation</u> (set all spatial gradients to zero)

$\partial_t f = \mathcal{C}[f, f]$ → Dynamics contains only non-hydro modes

<u>Non-hydro modes</u> → defined by nonzero eigenvalues of collision operator

$$\lim_{k \to 0} \omega_{nh}(k) \neq 0$$

Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Gradient expansion to all orders: Simplest "toy model" of QGP fluid

Bjorken expanding (conformal, transversely homogeneous) fluid:

$$x^\mu = (\tau, x, y, \eta)$$

Spacetime rapidity

$$\eta = \tanh^{-1}(z/t)$$

Milne propertime

$$\tau = \sqrt{t^2 - z^2}$$

Flow velocity

$$u^\mu = (1, 0, 0, 0)$$

Boltzmann equation

$$\partial_\tau f = \mathcal{C}[f, f]$$

By symmetry:

- $f \to f(\tau, k_0, k_\eta)$

- Any gradient $\sim \dfrac{1}{\tau}$

Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Relaxation time τ_R approximation (RTA)

(see Florkowski et al., Nucl.Phys. A916 (2013) 249-259)

$$\frac{\partial_\tau T}{T} + \frac{1}{3\tau} - \frac{\pi}{12\tau} = 0,$$

$$\partial_\tau f_{\mathbf{k}} = -\frac{f_{\mathbf{k}} - f_{\mathrm{eq}}}{\tau_R},$$

Knudsen number

$$K_N = \frac{\tau_R}{\tau}$$

Shear stress tensor

$$\pi^\eta_\eta = \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau} k_0 \left[\frac{1}{3} - \left(\frac{k_\eta}{k_0\tau}\right)^2\right] f_{\mathbf{k}}$$

Landau matching condition

$$\varepsilon = \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau}\, k_0 f_{\mathbf{k}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau}\, k_0 f_{\mathrm{eq}}$$

$$f_{\mathrm{eq}} = \exp\left(-k_0/T\right)$$

Massless particles, constant relaxation time

Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Method of moments: $$\rho_{n,\ell} = \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau} \left(k^0\right)^n \left(\frac{k_\eta}{k^0\tau}\right)^{2\ell} f_{\mathbf{k}}$$

Moments eqs. ⟷ Boltzmann eq.

$$\partial_\tau M_{n,\ell} + \frac{1}{\tau_R} M_{n,\ell} + \frac{6\ell - n}{3\tau} M_{n,\ell} - \frac{n+3}{12\tau} \underline{M_{1,1}\left(1 + M_{n,\ell}\right)}$$
$$+ \frac{1}{\tau} \frac{(n-2\ell)(1+2\ell)}{2\ell+3} M_{n,\ell+1} = -\frac{1}{\tau} \frac{4\ell(n+3)}{3(2\ell+3)}$$

Dimensionless moments

$$M_{1,1} = -\pi/P$$

$$M_{n,\ell} \equiv \frac{\rho_{n,\ell} - \rho_{n,\ell}^{\mathrm{eq}}}{\rho_{n,\ell}^{\mathrm{eq}}}$$

Nonlinearity

Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Solution of Boltzmann → reconstructed via $M_{n,\ell}(\tau)$

Gradient expansion series:

$$M_{n,\ell} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(n,\ell)}}{\hat{\tau}^p}$$

Knudsen number

$$K_N = \frac{\tau_R}{\tau} = \frac{1}{\hat{\tau}}$$

Exact recursive relation

!!!!!

$$\alpha_0^{(n,\ell)} = 0$$

$$\alpha_1^{(n,\ell)} = -\frac{4\ell\,(n+3)}{3\,(2\ell+3)}$$

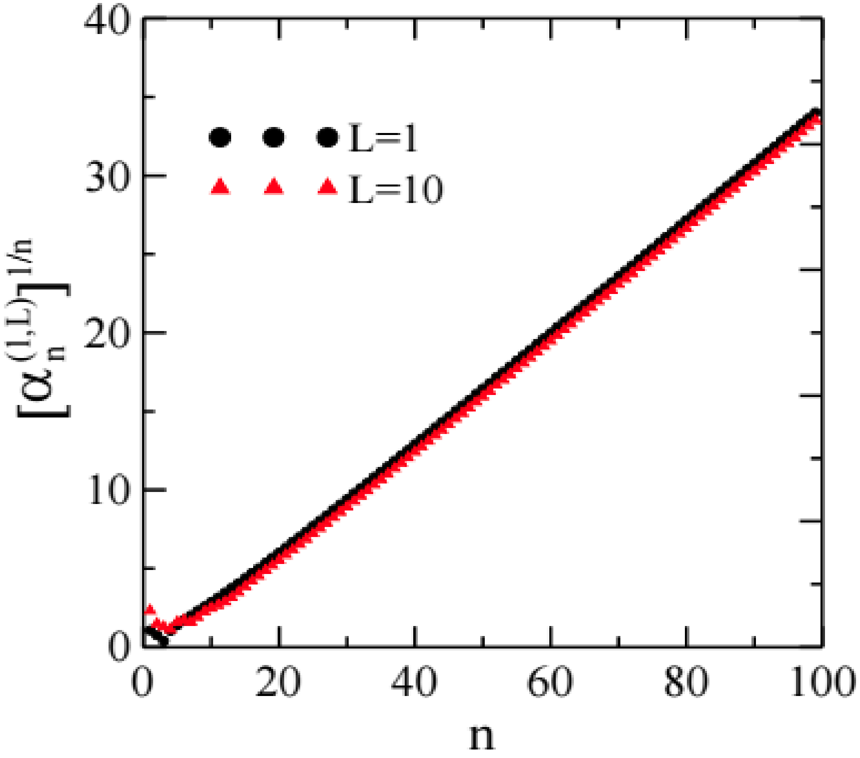
$$\alpha_{m+1}^{(n,\ell)} = -\frac{6\ell - n - 3m}{3}\alpha_m^{(n,\ell)} + \frac{n+3}{12}\alpha_m^{(1,1)} - \frac{(n-2\ell)\,(1+2\ell)}{2\ell+3}\alpha_m^{(n,\ell+1)} + \frac{n+3}{12}\sum_{p=0}^{m}\alpha_p^{(1,1)}\alpha_{m-p}^{(n,\ell)}.$$

Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Gradient (Chapman-Enskog) series is clearly <u>divergent</u> !!!

$$\lim_{m \gg 1} \alpha_m^{(n,\ell)} \sim m!$$



See also Heller, Kurkela, Spalinski, arXiv:1609.04803

For holography see:

arXiv:1302.0697

arXiv:1603.05344

First time this has been shown in relativistic kinetic theory

Hydrodynamics is not just a series in gradients ...

Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Regularity of the system near initial condition: $\tau \to \tau_0$

+

And the 1st order nature of the ODE's for the moments

Show that at early times $M_{n,\ell}(\hat{\tau}) \sim e^{-(\hat{\tau}-\hat{\tau}_0)} \sim e^{-1/K_N}$

Dynamics contains highly non-perturbative terms !!!!

This should also hold for a QCD-like collision kernel

Essential singularity

(given by non-hydro mode)

Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

- New terms not present in the usual gradient series
- They show that initial condition data is not easily “erased”
- They carry information about non-hydro mode dynamics

We propose a novel Generalized Chapman-Enskog (GCE) series

$$M_{n,\ell}(\hat{\tau}) = \sum_{p=0}^{\infty} \frac{\beta_p^{(n,\ell)}(\hat{\tau})}{\hat{\tau}^p}$$

This introduces an expansion parameter in the moments method !!!

Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

CE series: $\alpha_m^{(n,\ell)} \rightarrow$ obey algebraic relations

GCE series: $\beta_m^{(n,\ell)}(\hat{\tau}) \rightarrow$ obey differential equations !!

$$\partial_{\hat{\tau}}\beta_{m+1}^{(n,\ell)} + \beta_{m+1}^{(n,\ell)} = -\frac{4\ell\,(n+3)}{3\,(2\ell+3)}\delta_{m,0}$$

$$\partial_{\hat{\tau}}\beta_0^{(n,\ell)} + \beta_0^{(n,\ell)} = 0$$

$$-\frac{(n-2\ell)\,(1+2\ell)}{2\ell+3}\beta_m^{(n,\ell+1)} - \frac{6\ell-n-3m}{3}\beta_m^{(n,\ell)}$$

$$+\frac{n+3}{12}\beta_m^{(1,1)} + \frac{n+3}{12}\sum_{p=0}^{m}\beta_{m-p}^{(1,1)}\beta_p^{(n,\ell)}.$$

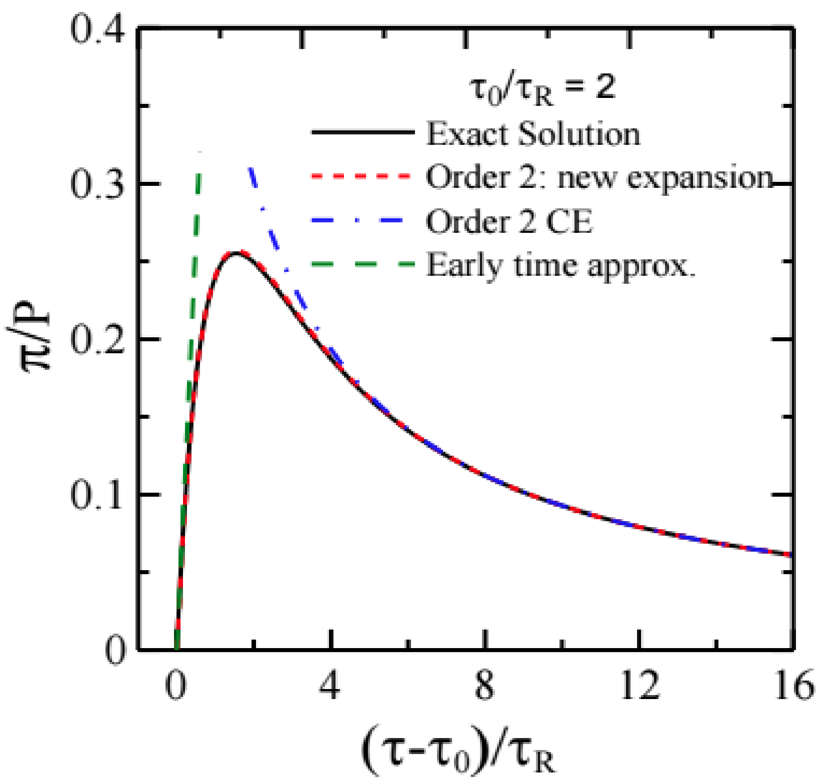
New series describes the whole time evolution since initial condition !!!

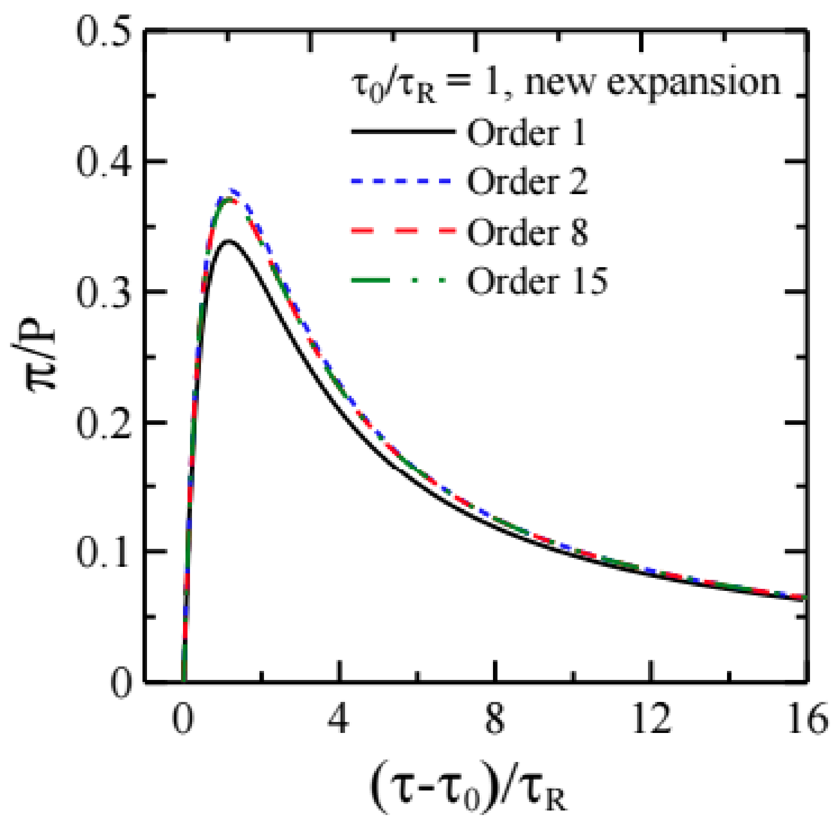
Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Excellent agreement with full exact solution of Boltzmann already when truncated at 2nd order !!!

2nd order truncation not the usual Israel-Stewart theory





Possible implications for heavy ion collisions

Hydrodynamics cannot be just an expansion in gradients since

Dissipative energy-momentum tensor

$$\pi^{\mu\nu} = \underbrace{-\eta\sigma^{\mu\nu}}_{\mathcal{O}(K_n)} + \underbrace{\eta\tau_1\left(D\sigma^{<\mu\nu>} + \frac{\theta}{3}\sigma^{\mu\nu}\right) + \ldots}_{\mathcal{O}(K_n^2)} + \mathcal{O}(K_n^3)$$

any truncation of the theory expansion leads to acausal and unstable dynamics. Radius of convergence of expansion is zero.

- Israel-Stewart theory (used in 100% of hydro models):

$$\tau_\pi D\pi^{\langle\mu\nu\rangle} + \pi^{\langle\mu\nu\rangle} = -\eta\sigma^{\mu\nu} + \ldots \quad \rightarrow \text{non-hydro mode} \quad \tau_\pi$$

Just a possible resummation, not unique (or universal).

Conclusions

- The transition from Kn << 1 to Kn ~ 1 is a fundamental problem in fluid dynamics.

- This problem can be fully solved in a simple model for the QGP fluid within kinetic theory.

- In this case, the gradient / Knudsen series **diverges.** Same happens at strong coupling (see backup slides).

- Novel non-perturbative non-hydro modes are needed to make sense of fluid dynamics (at least in the relativistic regime).

Relativistic fluid dynamics is not just a series in gradients

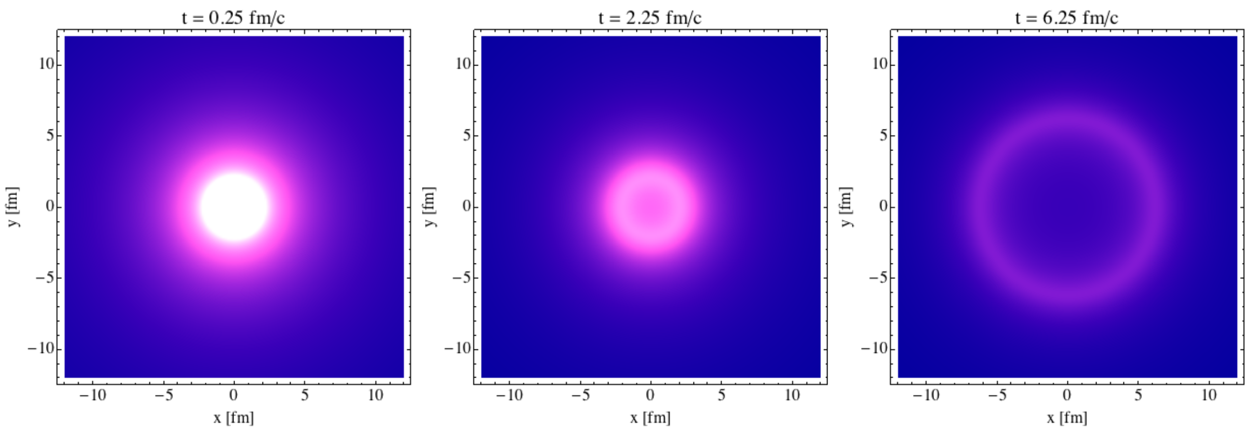
- New effective theory of fluid dynamics, valid at any valid of the coupling, including both hydro + non-hydro modes must still be constructed.

- This should be relevant for understanding the fluid-like behavior of small systems in heavy ions.

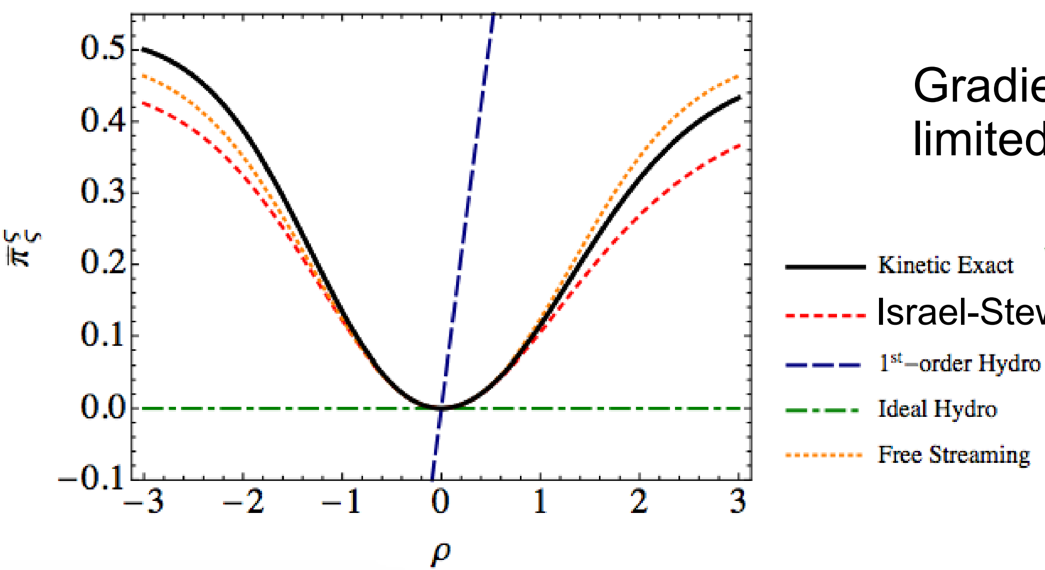
EXTRA SLIDES

Importance of non-hydro mode: Gubser flow

Phys.Rev.Lett. 113 (2014) no.20, 202301



Not a simple Longitudinal Bjorken expansion



Israel-Stewart-like (DNMR) → contains non-hydro mode

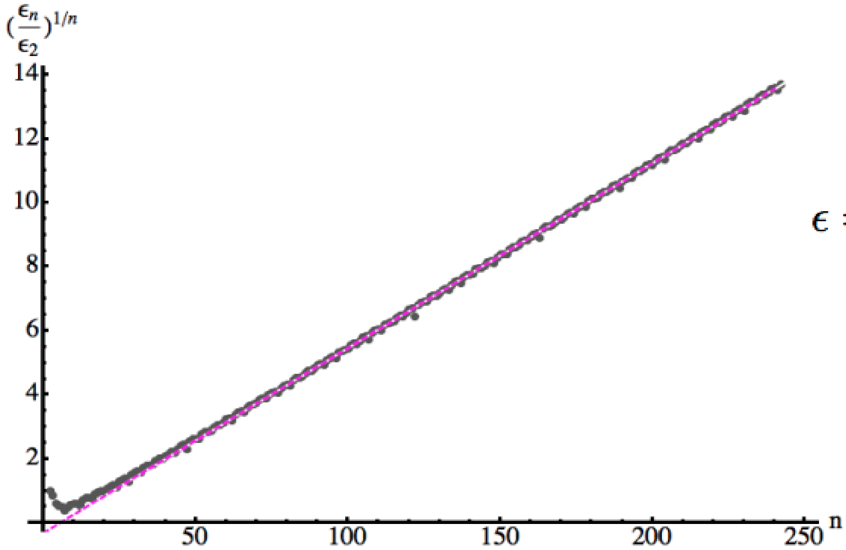
Gradient expansion solution extremely limited

Divergence of the gradient expansion at strong coupling

$$K_N \sim \frac{\ell}{L} \ll 1$$

→ Fluid/gravity correspondence (aka Chapman-Enskog at strong coupling) developed by Minwalla, Hubeny, Rangamani, and etc

Gradient expansion at large orders at strong coupling done by Heller et al., PRL (2013), for N=4 SYM + Bjorken expansion



Energy density

$$\epsilon = \frac{3}{8} N_c^2 \pi^2 \frac{1}{\tau^{4/3}} \left(\epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \dots \right)$$

Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Entropy production in N=2* theory (FLRW flow)

$$\frac{d(a^3 s)}{dt} = \frac{N^2}{16\pi} a^{7-2\Delta} \mu^2 \delta_\Delta^2 (4-\Delta)^2 s_\Delta \times \Omega_\Delta^2,$$

$$\Omega_\Delta \equiv \sum_{n=0}^{\infty} \mathcal{T}_{\Delta,n+1}[a] \, \frac{F_{\Delta,n}(1)}{\mu^n}.$$

$$\mathcal{T}_{\Delta,n}[a] = \left(-\frac{1}{2} - \frac{3\omega}{2}\right)^n \Gamma\left(n + \frac{2(\Delta-4)}{1+3\omega}\right) a^n H^n.$$ → **1st Analytical proof of divergence!!!**

Black hole QNM's = non-hydro modes → singularities in the Borel plane

(after resummation)

Divergence of the hydrodynamic series

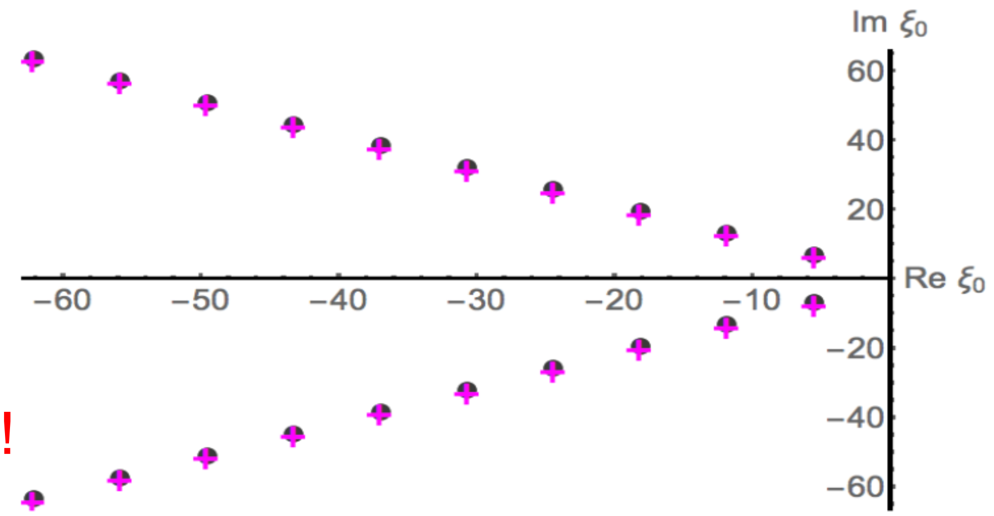
Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Hydrodynamic series

$$\Omega_\Delta = \sum_{n=0}^{\infty} c_n g^n, \qquad c_n \equiv \frac{\Gamma(n+4-\Delta)F_{\Delta,n}(1)}{(4\pi)^n} \text{ and } g \equiv \frac{H}{T} = \frac{4\pi}{\mu} aH.$$

Borel sum

$$\Omega_\Delta^{(B)}(\xi) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\xi^n$$



Borel singularities = are the black hole quasinormal modes !!!

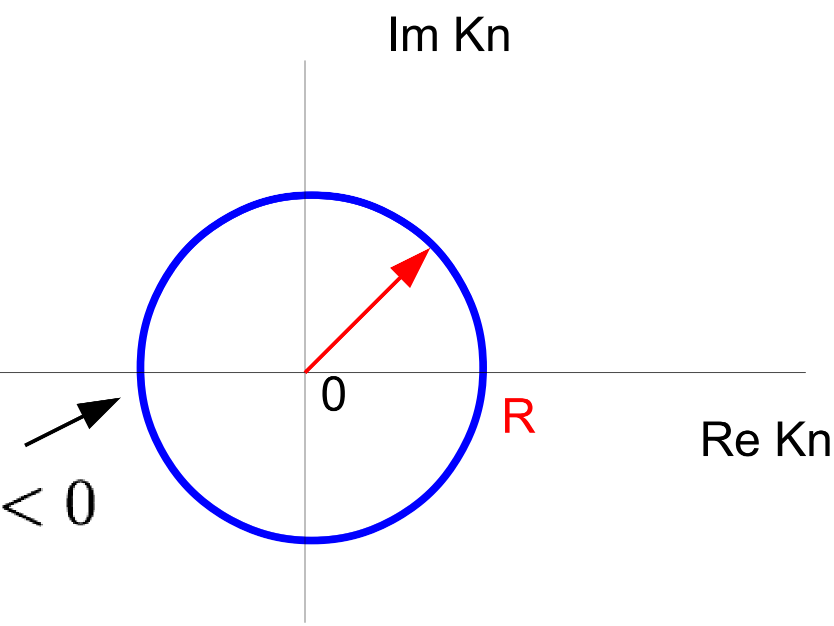
Divergence of the gradient expansion in kinetic theory

Simple argument to show that the series must diverge (a la Dyson)

If series converged around $K_N = \frac{\tau_R}{\tau} \to 0$, there would be a nonzero radius of convergence R

But for RTA $\partial_\tau f = -\frac{(f - f_{eq})}{\tau_R}$

Since $\tau > 0$



$K_N < 0 \to \tau_R < 0$

INSTABILITY !!!!!!

Hydrodynamics as a series expansion


D. Hilbert, 1912

Scaled Boltzmann equation

$$p^\mu \partial_\mu f = \frac{\mathcal{C}[f,f]}{K_N}$$

Knudsen number

$$K_N \sim \frac{\ell}{L}$$

Formal solution via a series

Hilbert Series

$$f(x^\mu, p^\mu, K_N) = \sum_{n=0}^{\infty} (K_N)^n f_n(x^\mu, p^\mu)$$

$f_0 \sim e^{-p \cdot u/T}$ $\rightarrow$ derives (does not assume) ideal fluid dynamics

No statement about existence of series is made

Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

1st analytical proof of the divergence of gradient expansion:

→ Knudsen gradient series has zero radius of convergence

→ Knudsen series leads to acausal and unstable dynamics

→ There must be a new way to define hydrodynamics
beyond the gradient expansion

→ A recent way to understand that involves resurgence.

Friedmann-Robertson-Lemaitre-Walker (FRLW) spacetime

Maximally (spatially) symmetric spacetime

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - K\,r^2} + r^2\,d\Omega^2\,.\right]$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\,T_{\mu\nu}$$

Einstein's equations

K ~ 0 (spatially flat –> our universe)

K = 1, -1

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon - \frac{K}{a^2}$$

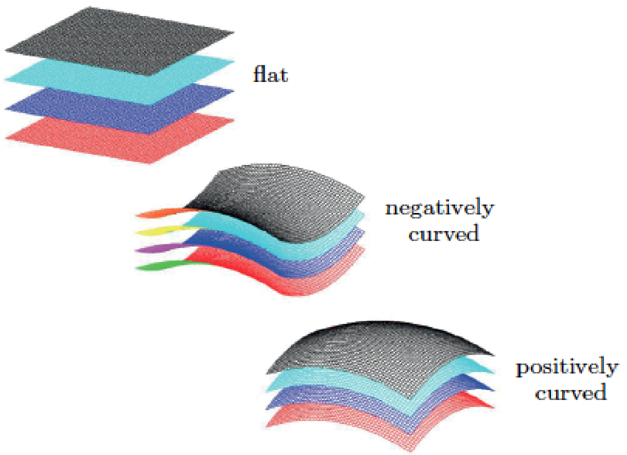
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3P)$$

$$\varepsilon \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^{0} & \text{vacuum} \end{cases}$$

FLRW spacetime

Universe

Spatial isotropy + homogeneity



Isotropic and homogeneous expanding FLRW spacetime

(zero spatial curvature)

Ex: metric

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

Determined from Einstein's equations

Friedmann-Lemaitre-Robertson-Walker spacetime

We consider an isotropic and homogeneous expanding FRW spacetime

(zero spatial curvature)

metric

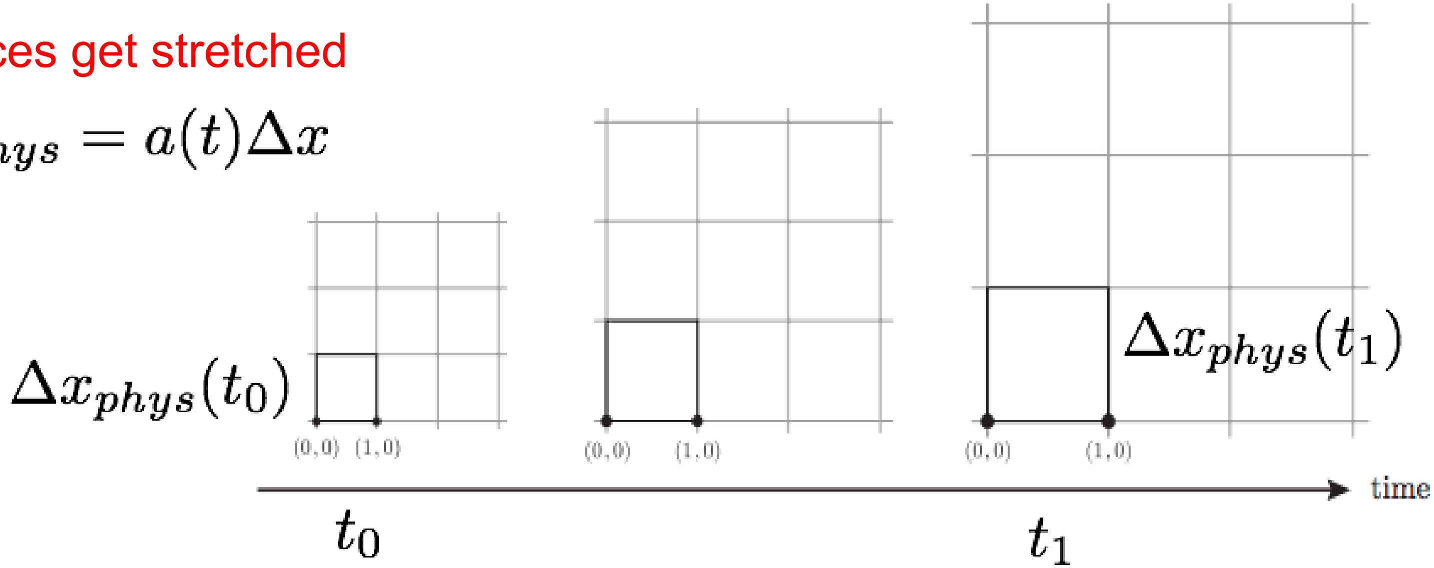
$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

Cosmological scale factor (e.g., radiation) $a(t) \sim t^{1/2}$

Hubble parameter $H = \dot{a}/a > 0$

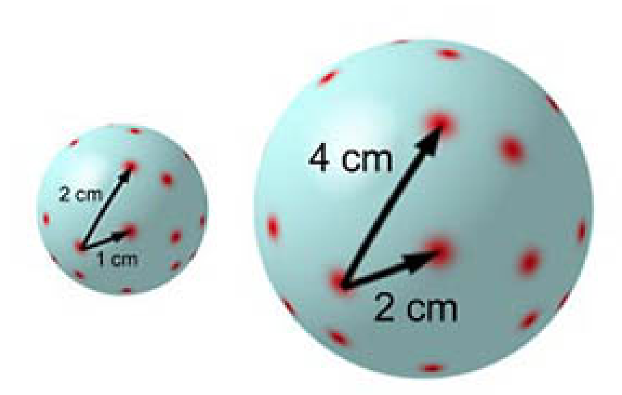
Distances get stretched

$$\Delta x_{phys} = a(t)\Delta x$$



- Locally static system corresponding to an expanding, homogeneous Universe

- Provides a simple case to study the validity of gradient expansion at strong coupling



Universe expands

Flow locally static

Shear tensor = 0

Heat flow = 0

FLRW spacetime is conformal to Minkowski and, thus, one can perform gauge/gravity calculations in which the gauge theory lives in a curved 4-dimensional spacetime!!!!

Given that heavy ion data indicates that T ~ QCD transition the QGP is a nearly perfect fluid …

There must have been nearly perfect fluidity in the early universe

Experimental consequences of that are not yet known (are there any??)

Given that around those temperatures QCD is not conformal, we would like to use a nonconformal gravity dual in a FLRW spacetime

This was done by A. Buchel, M. Heller, JN in arXiv:1603.05344 [hep-th] PRD (2016)

Toy model for QCD: N=2* gauge theory

Pilch, Warner, Buchel, Peet, Polchinski, 2000
A. Buchel, S. Deakin, P. Kerner and J. T. Liu, NPB 784 (2007) 72

A relevant deformation of SYM: Breaking of SUSY

N = 4 SYM theory + $\delta\mathcal{L} = -2\int d^4x\,\left[\,m_b^2\,\mathcal{O}_b + m_f\,\mathcal{O}_f\,\right]$

$m_b^2\,\mathcal{O}_b$ → Bosonic mass

$m_f\,\mathcal{O}_f$ → Fermionic mass

$$\mathcal{O}_b = \frac{1}{3}\mathrm{Tr}\left(|\phi_1|^2 + |\phi_2|^2 - 2\,|\phi_3|^2\right)\,,$$

$$\mathcal{O}_f = -\mathrm{Tr}\Big(i\,\psi_1\psi_2 - \sqrt{2}g_{\rm YM}\,\phi_3[\phi_1,\phi_1^\dagger] + \sqrt{2}g_{\rm YM}\,\phi_3[\phi_2^\dagger,\phi_2]$$
$$+\mathrm{h.c.}\Big) + \frac{2}{3}m_f\,\mathrm{Tr}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right)$$

C. Hoyos, S. Paik, and L. G. Yaffe, JHEP 10, 062 (2011)

Toy model for QCD: N=2* gauge theory

Pilch, Warner, Buchel, Peet, Polchinski, 2000

Classical gravity dual action:

$$S = \frac{1}{16\pi G_5} \int \mathrm{d}^5 x \sqrt{-g} \left(R - 12(\partial\alpha)^2 - 4(\partial\chi)^2 - V \right),$$

Scalar potential

$$V = -e^{-4\alpha} - 2e^{2\alpha}\cosh 2\chi + \frac{1}{4}e^{8\alpha}\sinh^2 2\chi.$$

Bulk viscosity

$$\frac{\zeta}{\eta} \sim \mathcal{O}(1)\left(\frac{1}{3} - c_s^2\right)$$

- Well defined stringy origin
- Non-conformal strongly interacting plasma: $\varepsilon \neq 3p$
- Used in tests of holography in non-conformal setting

N=2* gauge theory in a FLRW Universe

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Characteristic formulation of gravitational dynamics in asymptoticaly AdS5 spacetimes

Chesler,Yaffe, 2013

Assuming spatial isotropy and homogeneity $\boldsymbol{x} = \{x, y, z\}$ leads to

$$ds_5^2 = 2dt\,(dr - Adt) + \Sigma^2\, d\boldsymbol{x}^2,$$

$$\Sigma = \frac{a}{r} + \mathcal{O}(r^{-1}),\; A = \frac{r^2}{8} - \frac{\dot{a}r}{a} + \mathcal{O}(r^0)$$

$$\alpha = -\frac{8m_b^2 \ln r}{3r^2} + \mathcal{O}(r^{-2}),\; \chi = \frac{2m_f}{r} + \mathcal{O}(r^{-2}).$$

→ Encode non-equilibrium dynamics in an expanding Universe !!!

N=2* gauge theory in a FLRW Universe

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Conformal limit When $m_b = m_f = 0$,

Analytical solution for SYM in FLRW spacetime

$$\alpha = \chi = 0\,, \ \Sigma = \frac{ar}{2}\,, \ A = \frac{r^2}{8}\left(1 - \frac{\mu^4}{r^4 a^4}\right) - \frac{\dot{a}}{a}\, r,$$

First studied by P. S. Apostolopoulos, G. Siopsis, and N. Tetradis, PRL, (2009)

Temperature

$$T = \frac{\mu}{4\pi\, a}.$$

Energy density

$$\epsilon = \frac{3}{8}\pi^2 N^2 T^4 + \frac{3N^2(\dot{a})^4}{32\pi^2 a^4}$$

Pressure

$$P = \frac{1}{3}\epsilon - \frac{N^2(\dot{a})^2\ddot{a}}{8\pi^2 a^3}$$

Conformal anomaly!!!! $-\epsilon + 3P = \frac{N^2}{32\pi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$

Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

In our FLRW case, the gradient expansion corresponds to

Energy-momentum tensor

$$T_{\mu\nu} = T_{\mu\nu}^{eq} + \Pi_{\mu\nu}(\dot{a}, \{\dot{a}^2, \ddot{a}\}, \cdots),$$

$T_{\mu\nu}^{eq}$ → equilibrium

$\Pi_{\mu\nu}$ → dissipation

In terms of the energy density and pressure out-of-equilibrium

$$\epsilon = \epsilon^{eq} + \mathcal{O}\left(\dot{a}^2, \ddot{a}\right) \, , \; P = P^{eq} - \zeta(\nabla \cdot u) + \mathcal{O}\left(\dot{a}^2, \ddot{a}\right) \, ,$$

$\zeta(\nabla \cdot u)$ → Bulk viscosity

Universality and perfect fluidity

$\lambda \gg 1$ in QFT $\rightarrow$ string theory in weakly curved backgrounds

d.o.f. / vol. $\rightarrow \infty$ in QFT $\rightarrow$ vanishing string coupling

T, μ in QFT $\rightarrow$ spatially isotropic black brane

For anisotropic models there is violation see PRD 2014 arXiv:1406.6019

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Universality of shear viscosity

Kovtun,Son,Starinets, 2005

Universality of black hole horizons ➡ **HOLOGRAPHY** ➡ Universality of transport coefficient in QFT

Chapman-Enskog expansion: Non-relativistic regime

Santos, Brey, Dufty, 1571, vol 51 PRL (1986)

Newtonian fluid: $P_{xy} = -\eta_0 \partial u_x / \partial y$

Uniform shear flow

$$\partial u_i / \partial x_j = \dot{\gamma} \delta_{ix} \delta_{jy}$$

$$\nabla n = \nabla T = \nabla \dot{\gamma} = \mathbf{0}$$

Pressure tensor

$$P_{xy} = -\sum_{k=0}^{\infty} \eta_k \left(\partial u_x / \partial y\right)^{2k+1}$$

BGK Boltzmann

$$(\partial_t + \mathbf{v} \cdot \nabla) f = -\nu (f - f_0)$$

Series converges if

$$\nu \sim const$$

(Maxwell molecules)

DIVERGES $\nu \sim p^{\alpha}, \quad \alpha = (n-4)/2n \ (r^{-n} \text{ potential})$

(e.g., hard spheres, n=2)

Hydrodynamics from the method of moments

Harold Grad, 1948

(See Israel-Stewart for Relativistic case)

$$p^\mu \partial_\mu f = \mathcal{C}[f, f]$$

Define infinite set of moments such as

$$\varepsilon = \int_p (u \cdot p)^2 f$$

energy density

$$T^{\mu\nu} = \int_p p^\mu p^\nu f$$

Energy-momentum tensor

- Use Boltzmann equation to find exact equations for the moments

- Reconstruct solution of Boltzmann using a complete set of moments

- In the relativistic domain, 14 moments truncation → Israel-Stewart eqs.

PROS:

- Moments method played a major role in the derivation of the hydrodynamic equations for the QGP – stability and causality!!!

- Used in many approaches: Israel-Stewart, DNMR, (v)AHYDRO ...

- Describe interactions between hydro and non-hydro modes

- Can provide consistent (and convergent) solution of Boltzmann

- Used to derive the 1st analytical solution of expanding Boltzmann gas by BDHMN in PRL 116, (2016)

CONS:

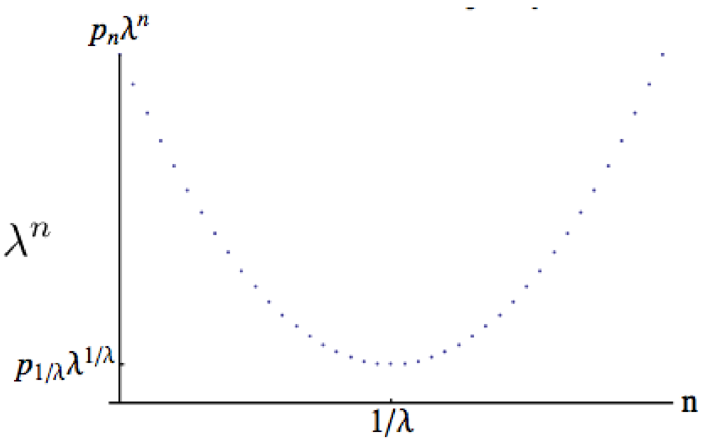
- Absence of a small expansion parameter

- Very hard to implement general equations in practice

Resurgence

Recent works by Dunne, Unsal, Basar, Cherman, Heller, Janik ...

$$\sum_{n=1}^{\infty} p_n \lambda^n$$



$$\langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n}\lambda^n + \sum_{c} e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n}\lambda^n$$

$$\sum_{n=0,k=0}^{\infty} \sum_{q=1}^{k-1} c_{n,k,q}\, g^{2n} \left[e^{-S/g^2}\right]^k \left[\log\left(\frac{1}{g^2}\right)\right]^q$$

Heller, Spalinski, PRL 2015

Hydro expansion

via resurgence

$$f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n}$$

$$\Omega \equiv w^{-\gamma} \exp(-w\xi_0)$$