Relaxation times of shear and bulk viscosities from Kubo formulas

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based on: arXiv: 1701.07580

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Transport coefficients

The question: how are the shear and bulk relaxation times related to microscopic quantities?

- ✓ Relativistic fluid dynamics describes very well the evolution of matter produced in heavy ion collisions after it achieves approximate local thermal equilibrium
- ✓ Hydrodynamics - macroscopic description of a system; transport coefficients - parameters
- ✓ Transport coefficients have to be obtained from the respective microscopic theory

- **Kinetic theory:** solving Boltzmann equation

  Jeon, Yaffe: PRD 53 (1996) 5799
  Arnold, Moore, Yaffe: JHEP 0011 (2000) 001, JHEP 0301 (2003) 030, JHEP 0305 (2003) 051,
  York, Moore: PRD 79 (2009) 054011

  methods of moments

  Denicol et al: PRL 105 (2010) 162501, EPJA 48 (2012) 170, PRD 85 (2012) 114047, PRC 90 (2014) 024912

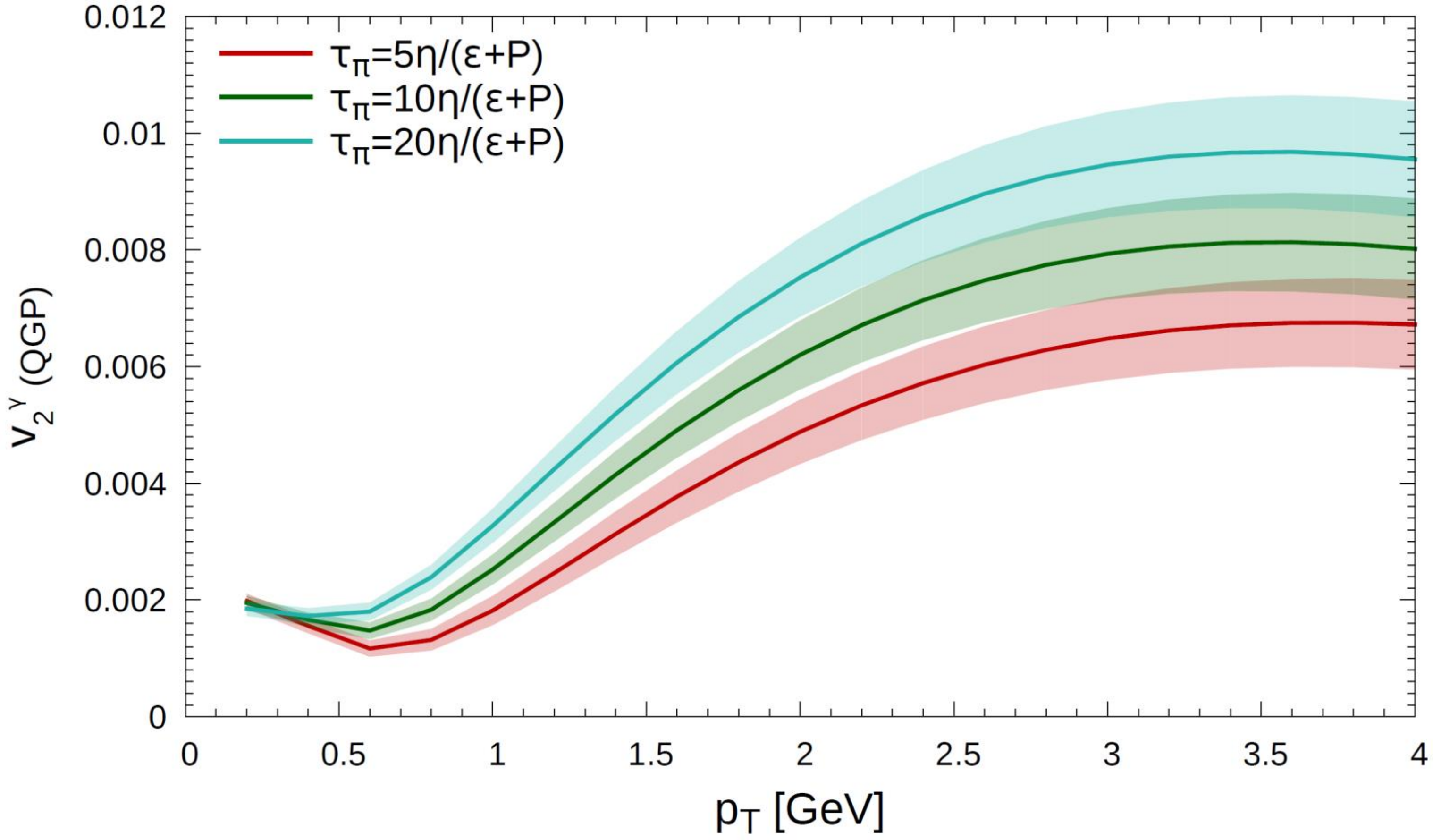
- **Quantum field theory:** Kubo relations

  Jeon: PRD 52 (1995) 3591
  Moore, Sohrabi: PRL 106 (2011) 122302, JHEP 1211 (2012) 148

Many analyses done for shear and bulk viscosities

Role of shear relaxation time

Vujanovic et al: PRC 94 (2016) no.1 014904



shear relaxation time assumed to be of the form

$$\tau_\pi = b_\pi \frac{\eta}{\epsilon + P}$$

$$b_\pi = 5,\ 10,\ 20$$

Is this factor somehow constrained?

Shear relaxation time has significant impact on the differential elliptic flow of thermal photons emitted by the QGP

Outline

- ➢ Hydrodynamic equations

- ➢ Method on how to find relaxation times within QFT framework
  - parametrization of response functions for transverse and longitudinal fluctuations
  - Kubo formulas for the shear and bulk relaxation times

- ➢ Application of the method
  - calculation of shear relaxation time within the real-time formalism

Relativistic viscous hydrodynamics

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}$$

ϵ - energy density
P - pressure
u^μ - four-velocity

viscous corrections:

Π - bulk pressure
$\pi^{\mu\nu}$ - stress tensor

First-order (Navier-Stokes) hydrodynamics (in the local rest frame)

$$\Pi_{\mathrm{NS}} = -\gamma \partial_i T^{i0}$$

$$\pi^{ij}_{\mathrm{NS}} = D_T \left(\partial^i T^{j0} + \partial^j T^{i0} - \frac{2}{3} g^{ij} \partial_k T^{k0} \right)$$

$$\gamma = \frac{\zeta}{\epsilon + P}$$

ζ - bulk viscosity

$$D_T = \frac{\eta}{\epsilon + P}$$

η - shear viscosity

Second-order (Israel-Stewart) hydrodynamics (in the local rest frame)

$$\partial_t \Pi = -\frac{\Pi - \Pi_{\mathrm{NS}}}{\tau_\Pi}$$

τ_Π - bulk relaxation time

$$\partial_t \pi^{ij} = -\frac{\pi^{ij} - \pi^{ij}_{\mathrm{NS}}}{\tau_\pi}$$

τ_π - shear relaxation time

Hydrodynamic modes

no other currents coupled to the energy-momentum tensor → two hydrodynamic modes

$$\partial_\mu T^{\mu\nu} = 0 \quad + \quad \begin{aligned} \partial_t \Pi &= -\frac{\Pi - \Pi_{\mathrm{NS}}}{\tau_\Pi} \\ \partial_t \pi^{ij} &= -\frac{\pi^{ij} - \pi^{ij}_{\mathrm{NS}}}{\tau_\pi} \end{aligned}$$

$$\Downarrow$$

shear mode:

$$0 = -\omega^2 \tau_\pi - i\omega + D_T \mathbf{k}^2$$

sound mode:

$$\begin{aligned} 0 = & -\omega^2 + v_s^2 \mathbf{k}^2 + i\omega(\tau_\pi + \tau_\Pi) - i\left(\frac{4D_T}{3} + \gamma + v_s^2(\tau_\pi + \tau_\Pi)\right)\omega \mathbf{k}^2 \\ & + \tau_\pi \tau_\Pi \omega^4 - \left(\tau_\pi \tau_\Pi v_s^2 + \tau_\Pi \frac{4D_T}{3} + \tau_\pi \gamma\right)\omega^2 \mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4) \end{aligned}$$

Linear response theory

Viscous hydrodynamics is a perfect realization of the linear response theory

deviation of a given observable from equilibrium → equilibrium retarded response function

Linear response to transverse fluctuations:

$$\delta\langle \hat{T}^{x0}(t,k_y)\rangle = \beta_x(k_y)\int dt'\,\bar{G}_R^{x0,x0}(t-t',k_y)\theta(-t')e^{\varepsilon t'}$$

direction of the fluid velocity

direction of the momentum diffusion

Linear response to longitudinal fluctuations:

$$\delta\langle \hat{T}^{00}(t,\mathbf{k})\rangle = \beta_0(\mathbf{k})\int dt'\,\bar{G}_R^{00,00}(t-t',\mathbf{k})\theta(-t')e^{\varepsilon t'}$$

Gravitational Ward identity

conservation of the energy-momentum current in terms of the correlation function

$$\boxed{k_\alpha\left(\bar{G}^{\alpha\beta,\mu\nu}(k) - g^{\beta\mu}\langle\hat{T}^{\alpha\nu}\rangle - g^{\beta\nu}\langle\hat{T}^{\alpha\mu}\rangle + g^{\alpha\beta}\langle\hat{T}^{\mu\nu}\rangle\right) = 0}$$

(Deser, Boulware, J. Math. Phys. 8 (1967) 1468)

- Stress-energy tensor represents the conservation laws and the generators of the space-time evolution

- Ward identity introduces constraints on the stress-energy response functions

stress-energy retarded correlation function

$$\begin{aligned}\bar{G}_R^{ij,mn}(x,y) &= -\delta^{(4)}(x-y)\left(\delta^{jm}\langle\hat{T}^{in}(y)\rangle + \delta^{jn}\langle\hat{T}^{im}(y)\rangle - \delta^{ij}\langle\hat{T}^{mn}(y)\rangle\right)\\ &\quad -i\theta(x_0-y_0)\langle[\hat{T}^{ij}(x),\hat{T}^{mn}(y)]\rangle\end{aligned}$$

Parametrization of the transverse fluctuation response function

1. Consequences of Ward identity

$$\bar{G}_R^{xy,xy}(\omega, k_y) + P = \frac{\omega^2}{k_y^2}\left(\bar{G}_R^{x0,x0}(\omega, k_y) + \epsilon\right)$$

2. Hydrodynamic limits

$$\omega \to 0 \qquad \bar{G}^{x0,x0}(0, k_y) = g_T(k_y)$$

well defined limit

$$k_y \to 0 \qquad \bar{G}_R^{xy,xy}(\omega, k_y) + P = \frac{\omega^2}{k_y^2}(\epsilon + g_T(k_y))$$

behaves well in the limit

3. General properties of the retarded Green function

$$\mathrm{Re}\, G_R(\omega, \mathbf{k}) = \mathrm{Re}\, G_R(-\omega, \mathbf{k}) \qquad \mathrm{Im}\, G_R(\omega, \mathbf{k}) = -\mathrm{Im}\, G_R(-\omega, \mathbf{k})$$

Parametrization of the transverse fluctuation response function

Most general form of the response function:

$$\boxed{\bar{G}_R^{xy,xy}(\omega, k_y) = \frac{\omega^2(\epsilon + g_T(k_y) + i\omega A(\omega, k_y))}{k_y^2 - \frac{i\omega}{D(\omega,k_y)} - \omega^2 B(\omega, k_y)} - P}$$

All functions *A*, *B*, and *D* have the forms: $A(\omega, k_y) = A_R(\omega, k_y) - i\omega A_I(\omega, k_y)$

All components are real-valued even functions of ω and k_y

B_R and D_R have non-zero limits when $\omega \to 0, \quad k_y \to 0$

All other parts of *A*, *B*, and *D* have finite limits when $\omega \to 0, \quad k_y \to 0$

Only small frequency and wavevector limits of the correlation function are important

Kubo formula for shear relaxation time

pole structure of $\bar{G}_R^{xy,xy}$ ⟷ dispersion relation of the shear mode

$$\eta = \lim_{\omega, k_y \to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_R^{xy,xy}(\omega, k_y)$$

$$\eta\tau_\pi \underbrace{- (\epsilon + P)\Big[D_I(0,0) + \eta A_R(0,0)\Big]}_{\kappa/2} = -\frac{1}{2} \lim_{\omega, k_y \to 0} \partial_\omega^2\, \mathrm{Re}\bar{G}_R^{xy,xy}(\omega, k_y)$$

from metric perturbation analysis: *(Moore, Sohrabi: PRL 106 (2011) 122302)*

$$\kappa = - \lim_{k_z, \omega \to 0} \partial_{k_z}^2\, \mathrm{Re}\, \bar{G}_R^{xy,xy}(\omega, k_z) \qquad \eta\tau_\pi - \frac{\kappa}{2} = \frac{1}{2} \lim_{\omega, k_z \to 0} \partial_\omega^2 \mathrm{Re}\, \bar{G}_R^{xy,xy}(\omega, k_z)$$

$\bar{G}_R^{xy,xy}(\omega, k_z)$ and $\bar{G}_R^{xy,xy}(\omega, k_y)$ have different momentum dependence

but their small frequency and *vanishing* momentum limits should be consistent with each other

Kubo formulas related to the shear flow

$$\eta = \lim_{\omega, k_y \to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_R^{xy,xy}(\omega, k_y)$$

$$\eta\tau_\pi - \frac{\kappa}{2} = -\frac{1}{2} \lim_{\omega, k_y \to 0} \partial_\omega^2 \, \mathrm{Re}\bar{G}_R^{xy,xy}(\omega, k_y)$$

η and τ_π related to the mean free path
(dynamical quantities)

$\kappa = \mathcal{O}(T^2)$
(thermodynamical quantity)

Moore, Sohrabi: JHEP 1211 (2012) 148
Romatschke, Son: PRD 80 (2009) 065021

Shear relaxation time can be extracted when both formulas are evaluated

Parametrization of the longitudinal fluctuation response function

The same steps of parametrization: Ward identity, well-behaved hydrodynamic limits and general properties of a Green function

Most general form of the function:

$$\boxed{\bar{G}_L(\omega, \mathbf{k}) = \frac{\omega^2(\epsilon + P + i\omega^2 Q(\omega, \mathbf{k}))}{\mathbf{k}^2 - \frac{\omega^2}{Z(\omega,\mathbf{k})} + i\omega^3 R(\omega, \mathbf{k})}}$$

All functions Q, *Z*, and *R* have the forms: $Q(\omega, \mathbf{k}) = Q_R(\omega, \mathbf{k}) - i\omega Q_I(\omega, \mathbf{k})$

All components are real-valued even functions of ω and $\mathbf{k}$

Z_R and R_R have non-zero limits when $\omega \to 0, \quad \mathbf{k} \to 0$

All other parts of Q, *R*, and *Z* have finite limits when $\omega \to 0, \quad \mathbf{k} \to 0$

Only small frequency and wavevector limits of the correlation function are important

Kubo formulas related to the bulk flow

pole structure of $\bar{G}_L$ ↔ dispersion relation of the sound mode

$$\frac{4}{3}\eta + \zeta = \lim_{\omega,\mathbf{k}\to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_L(\omega, \mathbf{k})$$

$$\frac{4}{3}\eta\tau_\pi + \zeta\tau_\Pi + \underbrace{Q_R v_s^2}_{-2\kappa/3} = -\frac{1}{2} \lim_{\omega,\mathbf{k}\to 0} \partial_\omega^2 \mathrm{Re}\, \bar{G}_L(\omega, \mathbf{k})$$

from metric perturbation analysis

(Hong, Teaney: PRC 83 (2010) 044908)

Combining these relations with the Kubo formulas related to the shear flow we get

Kubo formulas related to the bulk flow

$$\boxed{\begin{aligned} \zeta &= \lim_{\omega,\mathbf{k}\to 0} \frac{1}{\omega} \mathrm{Im}\, \bar{G}_R^{PP}(\omega, \mathbf{k}) \\ \zeta\tau_\pi &= -\frac{1}{2} \lim_{\omega,\mathbf{k}\to 0} \partial_\omega^2\, \mathrm{Re}\bar{G}_R^{PP}(\omega, \mathbf{k}) \end{aligned}}$$

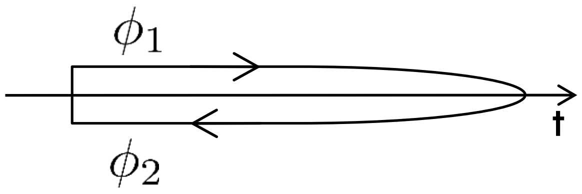
Diagrammatic computation of shear relaxation time

Massless scalar field theory: $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{\lambda}{4!}\phi^4$

Elastic scatterings – leading order – only shear viscosity effects matter $\hat{T}^{ij}(x) \cong \partial^i\phi(x)\partial^j\phi(x)$

Start with the real-time formalism in (1,2) basis: $\mathrm{Im}\,\bar{G}_R \propto G_{2211} - G_{1122}$ $\mathrm{Re}\,\bar{G}_R \propto G_{1111} - G_{2222}$

Building blocks: $\Delta_{11},\ \Delta_{22},\ \Delta_{12},\ \Delta_{21}$



Efficient description given in (r,a) basis:

(Wang, Heinz: PRD 67 (2003) 025022)

$$\Delta_{ra}(p) = \frac{1}{(p_0 + i\Gamma_p)^2 - E_p^2} \qquad \Delta_{aa}(p) = 0$$

$$\Delta_{ar}(p) = \frac{1}{(p_0 - i\Gamma_p)^2 - E_p^2} \qquad \Delta_{rr}(p) = N(p_0)\left(\Delta_{ra}(p) - \Delta_{ar}(p)\right)$$

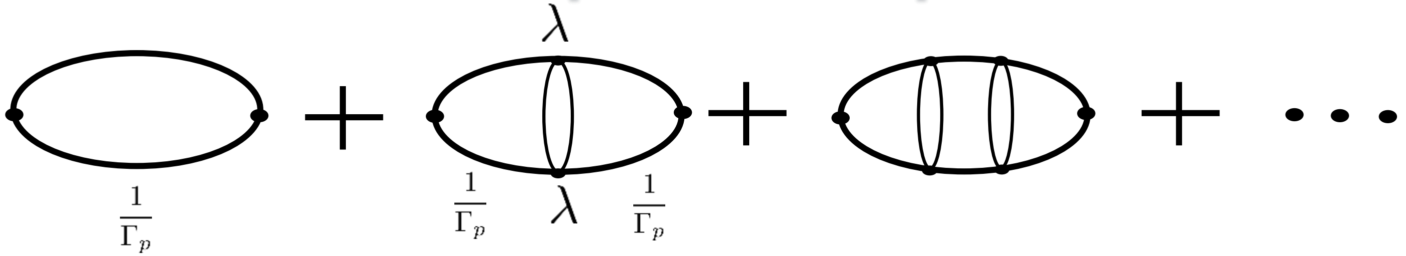
$$\phi_a = \phi_1 - \phi_2 \qquad \phi_r = \frac{\phi_1 + \phi_2}{2}$$

Spectral function approximated by the Lorentzians, propagators are dressed

$$E_p^2 = \mathbf{p}^2 + \mathrm{Re}\,\Sigma_p \qquad \Gamma_p = \frac{\mathrm{Im}\,\Sigma_p}{2E_p}$$

Bethe-Salpeter equation

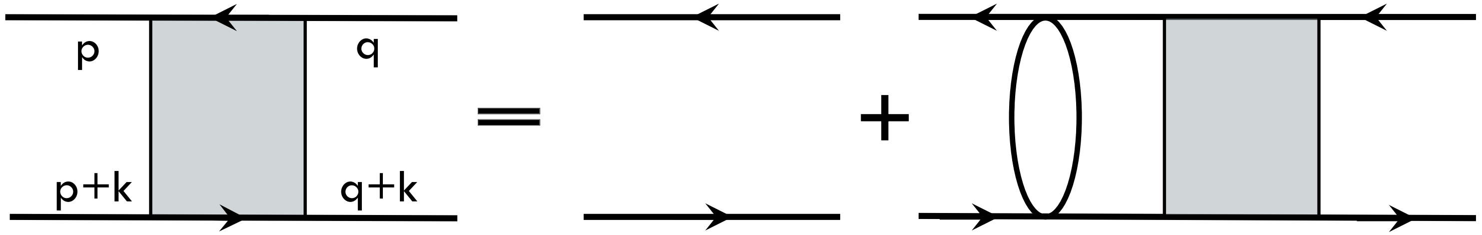
full leading order



$$\Gamma_p = \frac{\mathrm{Im}\,\Sigma_p}{2E_p}$$

$$\mathrm{Im}\,\Sigma \sim \mathcal{O}(\lambda^2)$$

All 4-point Green functions
in (r,a) basis are coupled to each other
via the Bethe-Salpeter equation



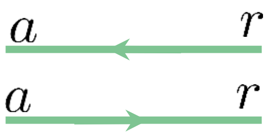
$\mathcal{K}$ - the kernel

vanishing of Δ_{aa} and KMS conditions for 4-point Green functions

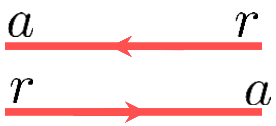
(Wang, Heinz: PRD 66 (2002) 025008, Carrington et al: EPJC 50 (2007) 711)

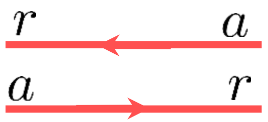
pinching pole approximation:

poles pinch the real axis: $\Delta_{ar}(p)\Delta_{ra}(p)$



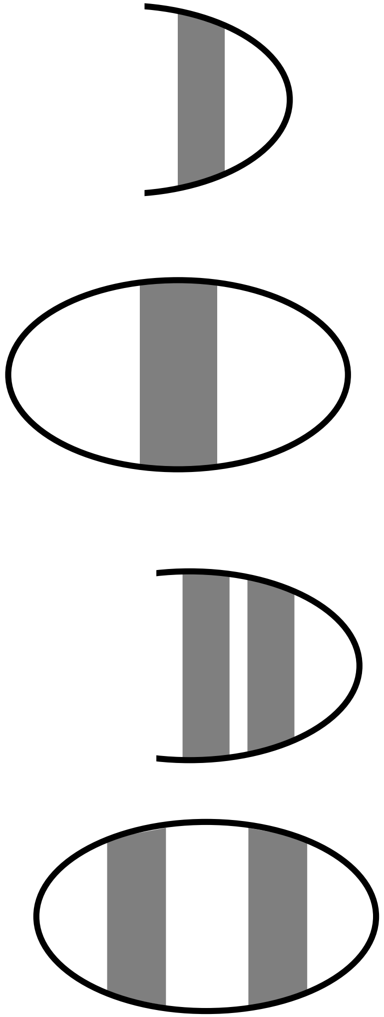
no pinching: $\Delta_{ar}(p)\Delta_{ar}(p)$ $\Delta_{ra}(p)\Delta_{ra}(p)$





Only G_{aarr} remains coupled to itself

Integral equations for η and $\eta\tau_\pi$



$$D(E_p,\mathbf{p}) = I(p) - \int \frac{d^3p}{(2\pi)^3}(\mathcal{K}(E_p,E_l)+\mathcal{K}(E_p,-E_l))\frac{D(E_l,\mathbf{l})}{8E_l^2\Gamma_l}$$

$I(p) = p_x p_y$ $\mathcal{K}(E_p,E_l)$ - kernel of the integral equation

$$\boxed{\eta = \beta \int \frac{d^3p}{(2\pi)^3} n(E_p)(1+n(E_p))I(\mathbf{p})\frac{D(E_p,\mathbf{p})}{2E_p^2\Gamma_p}}$$

$$\eta \propto \frac{1}{\Gamma_p}$$

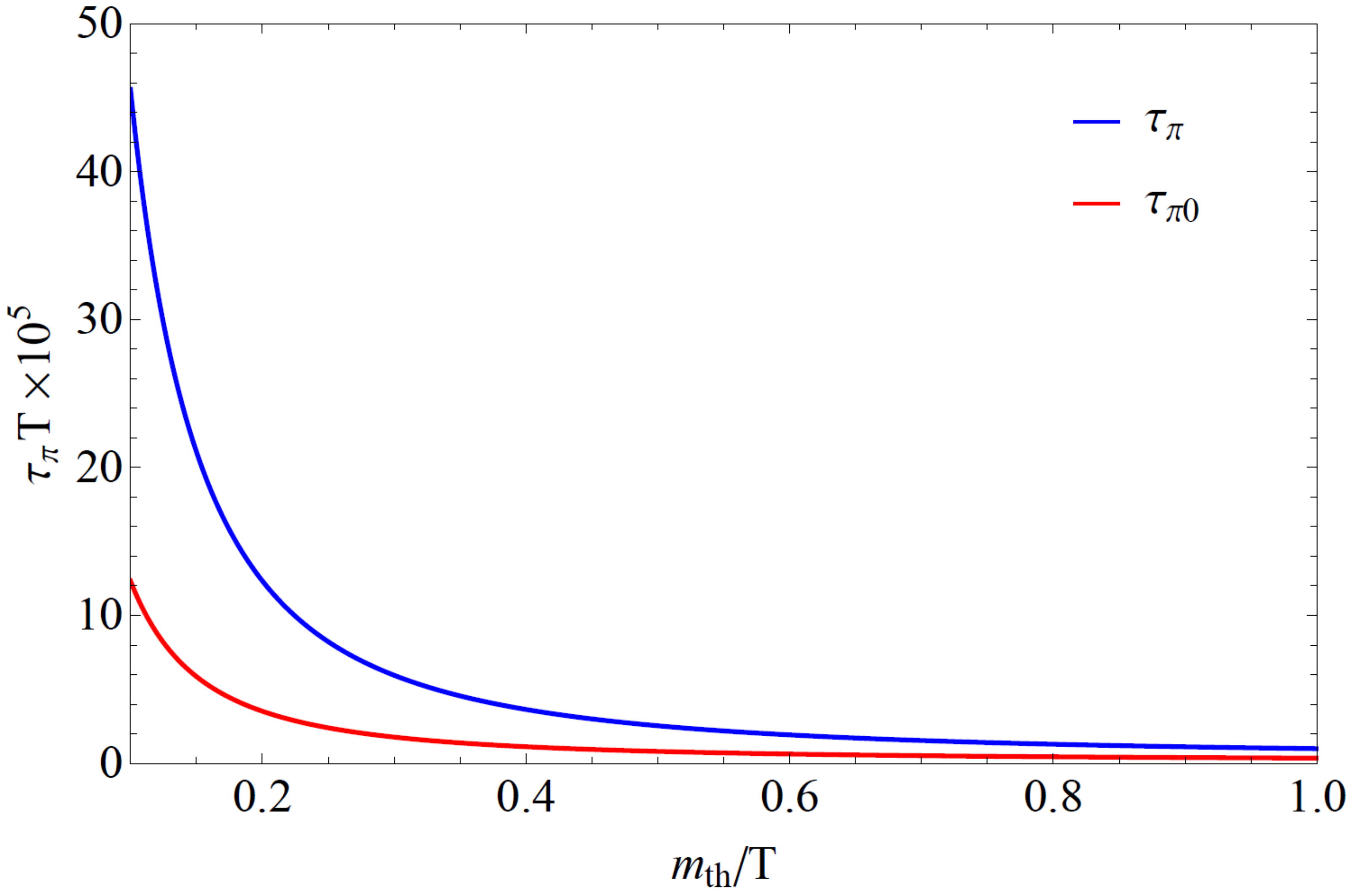
$$R(E_p,\mathbf{p}) = \frac{D(E_p,\mathbf{p})}{2\Gamma_p} - \int \frac{d^3p}{(2\pi)^3}(\mathcal{K}(E_p,E_l)+\mathcal{K}(E_p,-E_l))\frac{R(E_l,\mathbf{l})}{8E_l^2\Gamma_l}$$

$$\boxed{\eta\tau_\pi = \beta \int \frac{d^3p}{(2\pi)^3} n(E_p)(1+n(E_p))I(\mathbf{p})\frac{R(E_p,\mathbf{p})}{2E_p^2\Gamma_p}}$$

$$\eta\tau_\pi \propto \frac{1}{\Gamma_p^2}$$

$$\Gamma_p = \frac{\mathrm{Im}\,\Sigma_p}{2E_p}$$

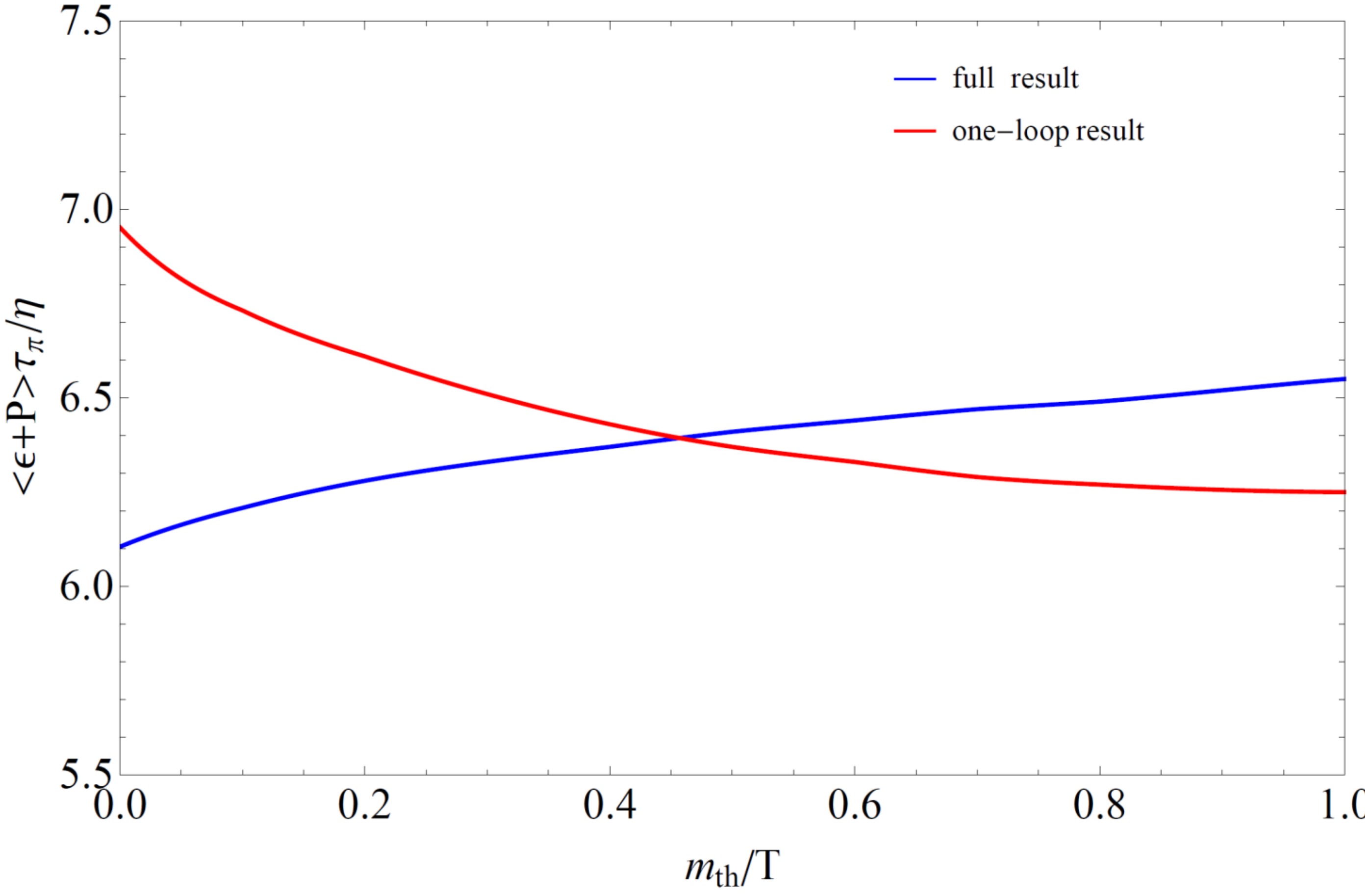
Shear relaxation time



τ_π - full result

$\tau_{\pi 0}$ - one-loop result

The ratio $\frac{<\epsilon+P>\tau_\pi}{\eta}$



effective kinetic theory

(York, Moore: PRD 79 (2009) 054011)

scalar theory

$$\frac{\langle\epsilon+P\rangle\tau_\pi}{\eta}=6.1 \qquad m_{\rm th}/T\approx 0$$

QCD

$$\frac{\langle\epsilon+P\rangle\tau_\pi}{\eta}=5.9-5.0 \text{ varies with } m_{\rm th}/T$$

14-moment approach

(Denicol, Jeon, Gale: PRC 90 (2014) 024912)

classical massless gas approximation

$$\frac{\langle\epsilon+P\rangle\tau_\pi}{\eta}=5$$

Conclusions

- Kubo formulas for the shear and the bulk relaxation times were found
- Shear relaxation time was studied within the real-time formalism in (r,a) basis
- Shear relaxation time is controlled by the thermal width
- The ratio $$\frac{\langle \epsilon + P \rangle \tau_\pi}{\eta} \approx 5 - 7$$ seems to be robust across all available microscopic calculations
- First step on the way to compute the bulk relaxation time, where next-to-leading order corrections must be also included