A kinetic regime of hydrodynamic fluctuations and long time tails for a Bjorken expansion

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Hydrodynamics with noise

- Initial state fluctuations
- Thermal fluctuations – for example Landau-Lifshitz

\[ N_{gg}(t, k) \equiv \langle g^i(t, k)g^j*(t, k) \rangle = (e + p)T\delta^{ij} \]

1. Conceptually important (required by the FDT)
2. Larger in smaller systems: \( N_{\text{particle}} \sim 10000 \) in the heavy-ions
3. Essential near a critical point

How do thermal fluctuations evolve during a Bjorken expansion?
How do thermal fluctuations change the Bjorken expansion?
Kinetic regime of hydrodynamic fluctuations – a new scale $k_*$

1. For hydrodynamic fluctuations with wavenumber $k$:
   - Equilibration rate $\sim \gamma_\eta k^2$
   - Expansion rate $\sim 1/\tau$ for a Bjorken expansion

2. Compete at a critical scale:

   $$k_* \sim \frac{1}{\sqrt{\gamma_\eta \tau}}$$

3. Derivative expansion controlled by $\epsilon \equiv \gamma_\eta / \tau \ll 1$

We derive an effective description for the kinetic regime $k_*$
Hydro-kinetic equation: an analogy with Brownian motion

1. **Langevin equation**

   \[
   \frac{dp}{dt} = -\gamma p + \xi, \quad \langle \xi(t)\xi(t') \rangle = 2TM\gamma\delta(t - t')
   \]

2. **Calculate how \( \langle p^2(t) \rangle \) evolves through Langevin process**

   \[
   \frac{d}{dt} \langle p^2 \rangle = -2\gamma \left[ \langle p^2 \rangle - MT \right]
   \]

Follow the same steps for hydrodynamics with external forcing
Hydro-kinetic equation: an analogy with Brownian motion

1. Langevin equation

\[
\frac{dp}{dt} = -\gamma p + \xi, \quad \langle \xi(t)\xi(t') \rangle = 2TM\gamma\delta(t-t')
\]

2. Calculate how \( \langle p^2(t) \rangle \) evolves through Langevin process

\[
\frac{d}{dt} \langle p^2 \rangle = -2\gamma \left[ \langle p^2 \rangle - MT \right] + \text{external forcing}
\]

Follow the same steps for hydrodynamics with external forcing
1. Linearized analysis in Bjorken: \( e = e_0 + \delta e, \quad \vec{g} = (e_0 + p_0)\vec{\nu} \)

\[ \phi_a(t, \vec{k}) \equiv (c_s \delta e, \vec{g}) \]

2. Hydro-Langevin equation for \( \phi_a(t, \vec{k}) \)

\[ -\dot{\phi}(t, \vec{k}) = i\mathcal{L}\phi + \mathcal{D}\phi + \xi + \mathcal{P}(t)\phi \]

- *ideal*
- *viscous*
- *expansion*

3. Four eigenmodes of \( \mathcal{L} \): \( \phi_+, \phi_-, \phi_{T1}, \phi_{T2} \)

- **left moving sound**
  \[ \lambda_- = -c_s k \]

- **right moving sound**
  \[ \lambda_+ = c_s k \]

- **transverse modes**
  \[ \lambda_T = 0 \]

The Hydro-Langevin equation in eigen basis is similar to Brownian motion
Hydro-kinetic equation

1. Analyze the square of the eigenmodes – for example

\[ N_{++}(t, \vec{k}) \equiv \langle \phi_+(t, \vec{k}) \phi^*_+(t, \vec{k}) \rangle \]

2. Hydro-kinetic equations for \( N_{++} \)

\[ \dot{N}_{++} = -\frac{4}{3} \gamma \eta k^2 [N_{++} - T_0(e_0 + p_0)] - \frac{1}{\tau} \left[ 2 + c_s^2 + \cos^2 \theta_k \right] N_{++} \]

- \[ \text{equilibration} \]
- \[ \text{external forcing} \]

3. Neglect off-diagonal components of density matrix

\[ N_{+-} \sim e^{-i(\lambda_+-\lambda_-)t} \sim 0 \]

- \[ \text{rotating wave approx} \]

Hydro fluctuations are driven out of equilibrium at \( \vec{k}_* \).
Solution of the hydro-kinetic equation
Bjorken expansion at late times

$k_*$ is the critical scale: For larger $k \gg k_*$, closer to equilibrium

$$N_{AA} \sim N_{eq} \left[ 1 + \frac{\#}{\gamma \eta \tau k_2^2} + \cdots \right]$$
Evolution of the background

Hydrodynamic equation for a Bjorken expansion:

$$\frac{d}{d\tau} (\tau T^{\tau\tau}) = -T^{zz}$$

- Without hydrodynamic fluctuations:

$$T^{zz} = p_0 - \frac{4\eta_0}{3\tau} + (\lambda_1 - \eta_{\tau\pi}) \frac{8}{9\tau^2}$$

- Hydrodynamic fluctuations give another contributions

$$T_{\text{fluct}}^{zz} = (e_0 + p_0) \langle u^z u^z \rangle = \frac{\langle g^z g^z \rangle}{e_0 + p_0}$$
Nonlinear contributions from $k_*$ to the background

1. Compute the contribution from fluctuation

$$
\langle (g^z(t, \vec{x}))^2 \rangle = \int^{\Lambda} d^3k \left[ N_{++} \cos^2 \theta + N_{T_2T_2} \sin^2 \theta \right]
\sim 1 + \#/(\gamma \eta \tau k^2) + \cdots \text{ for } k \gg k_*
$$

- Regularize cubic and linear UV divergences by a cutoff $\Lambda$

2. Renormalize the divergences c.f. Kovtun-Moore-Romatschke (11)

$$
T^{zz} = p_0(\Lambda) + \frac{\Lambda^3 T}{6\pi^2} - \frac{4}{3\tau} \left[ \eta_0(\Lambda) + \frac{17\Lambda T}{120\pi^2} \frac{e_0 + p_0}{\eta_0} \right] + \text{finite}
\equiv p_{\text{phys}} \quad \equiv \eta_{\text{phys}}
$$

The cutoff dependence is absorbed by renormalization of $p_0$ and $\eta_0$
Finite contributions: Long-time tails

Evaluate the finite parts after renormalization

\[ T^{zz} = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st order}} + \underbrace{1.08318\ T\left(\frac{1}{4\pi\gamma\eta\tau}\right)^{3/2}}_{\text{long-time tail}} + \cdots \]

Simple understanding of the scaling

\[ T_{\text{fluct}}^{zz} \sim \frac{1}{2} k_B T \underbrace{\int d^3 k}_{\text{equipart}} \sim T k_*^3 \sim T \left(\frac{1}{\gamma\eta\tau}\right)^{3/2} \]

The finite contribution from \( k_* \) gives the long-time tails
Implications for heavy-ion collisions

Plugging in typical values

\[ \frac{T^3}{s} \simeq \frac{1}{13.5}, \quad \frac{\lambda_1 - \eta \tau \pi}{e + p} \simeq -0.8 \left( \frac{\eta}{e + p} \right)^2 \]

Compute \(4\langle T^{zz}\rangle/(e + p)\)

\[
\begin{align*}
\frac{\eta}{s} &= \frac{1}{4\pi} : \quad 1 - 0.092 \left( \frac{4.5}{\tau T} \right) + 0.034 \left( \frac{4.5}{\tau T} \right)^{3/2} - 0.00085 \left( \frac{4.5}{\tau T} \right)^2 \\
\frac{\eta}{s} &= \frac{2}{4\pi} : \quad \underbrace{1}_{\text{ideal}} - 0.185 \left( \frac{4.5}{\tau T} \right) + 0.013 \left( \frac{4.5}{\tau T} \right)^{3/2} - 0.0034 \left( \frac{4.5}{\tau T} \right)^2
\end{align*}
\]

Thermal fluctuation is practically larger than 2nd order viscous correction
Comparison to previous diagrammatic calculation in static systems
c.f. Kovtun-Moore-Romatschke (11), Kovtun-Yaffe (03)

Apply a shear perturbation

\[ h_{xy}(t) = h_{xy}e^{-i\omega t} \]

1. Compute how \( N_{++} \) evolves away from equilibrium:

\[ \dot{N}_{++} = -\gamma_\eta k^2[N_{++} - N_{eq}] + \dot{h}(t)N_{++} \]

2. Compute fluctuation contribution to stress in the linear order of \( h \):

\[
\frac{T^{xy}(\omega)}{h^{xy}(\omega)} = \left( p_0 + \frac{\Lambda^3 T}{6\pi^2} \right) - i \left( \eta_0 + \frac{17\Lambda T}{120\pi^2 \gamma_\eta} \right) \omega \\
+ (1 + i) \frac{(3/2)^{3/2}}{240\pi} + 7 T \left( \frac{\omega}{\gamma_\eta} \right)^{3/2}
\]

Agrees with the previous diagrammatic calculations
Summary & Outlook

- Hydro-kinetic equation for $k_*$, advantageous in expanding systems
- Universal renormalization of pressure $p_0(\Lambda)$ and viscosity $\eta_0(\Lambda)$
- Background-dependent long-time tails $\propto T^{-3/2}, \omega^{3/2}$
- Alternative way to solve the hydrodynamics with noise
  
  $$d_\mu T^{\mu\nu} = 0, \quad \dot{N}_{AA} = \cdots, \quad T^{\mu\nu} = T_{\text{bkg}}^{\mu\nu} + \int_k N_{AA},$$

- Bulk viscosity renormalization for nonconformal fluid

  YA-Mazeliauskas-Teaney, in preparation

  $$\zeta = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2} \left[ \left( \frac{1}{3} + \frac{T}{2} \frac{d c_s^2}{dT} - c_s^2 \right)^2 \frac{s}{\eta} + \left( \frac{1}{3} - c_s^2 \right)^2 \frac{2s}{\eta} \right]$$

- Application to critical dynamics

  YA-Teaney-Yan-Yin, in progress
Thank you for your attention!