A kinetic regime of hydrodynamic fluctuations and long time tails for a Bjorken expansion

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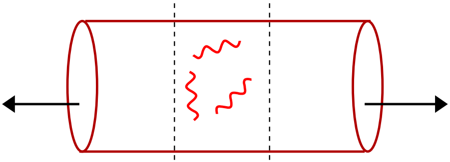
Hydrodynamics with noise

- Initial state fluctuations
- Thermal fluctuations – for example Landau-Lifshitz

$$N_{gg}(t,k) \equiv \underbrace{\langle g^i(t,k) g^{j*}(t,k) \rangle}_{\text{momentum, } g^i \equiv T^{0i}} = \underbrace{(e+p)T\delta^{ij}}_{\text{equilibrium}}$$

1. Conceptually important (required by the FDT)
2. Larger in smaller systems: $N_{\text{particle}} \sim 10000$ in the heavy-ions
3. Essential near a critical point

How do thermal fluctuations evolve during a Bjorken expansion?
How do thermal fluctuations change the Bjorken expansion?



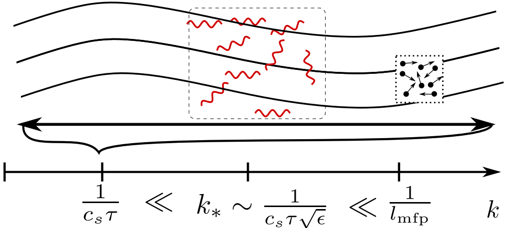
Kinetic regime of hydrodynamic fluctuations – a new scale k_*

1. For hydrodynamic fluctuations with wavenumber k:
   - Equilibration rate $\sim \gamma_\eta k^2$ ($\gamma_\eta \equiv \eta/(e+p)$)
   - Expansion rate $\sim 1/\tau$ for a Bjorken expansion
2. Compete at a critical scale:

$$k_* \sim \frac{1}{\sqrt{\gamma_\eta \tau}}$$

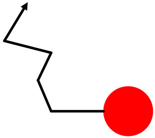
3. Derivative expansion controlled by $\epsilon \equiv \gamma_\eta/\tau \ll 1$

$$\frac{1}{c_s\tau} \ll \underbrace{k_* \sim \frac{1}{c_s\tau\sqrt{\epsilon}}}_{k_* \textbf{ is hard !}}$$



We derive an effective description for the kinetic regime k_*

Hydro-kinetic equation: an analogy with Brownian motion



1. Langevin equation

$$\frac{dp}{dt} = \underbrace{-\gamma p}_{\text{drag}} + \underbrace{\xi}_{\text{noise}}, \quad \langle \xi(t)\xi(t')\rangle = 2TM\gamma\delta(t-t')$$

2. Calculate how $\langle p^2(t)\rangle$ evolves through Langevin process

$$\frac{d}{dt}\langle p^2\rangle = \underbrace{-2\gamma\left[\langle p^2\rangle - MT\right]}_{\text{equilibration}}$$

Follow the same steps for hydrodynamics with external forcing

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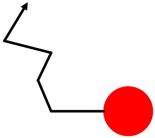

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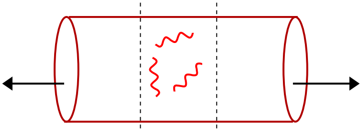


1. Linearized analysis in Bjorken: $e = e_0 + \delta e$, $\vec{g} = (e_0 + p_0)\vec{v}$

$$\phi_a(t, \vec{k}) \equiv (c_s \delta e, \vec{g})$$

2. Hydro-Langevin equation for $\phi_a(t, \vec{k})$

$$-\dot{\phi}(t, \vec{k}) = \underbrace{i\mathcal{L}\phi}_{\text{ideal}} + \underbrace{\mathcal{D}\phi + \xi}_{\text{viscous}} + \underbrace{\mathcal{P}(t)\phi}_{\text{expansion}}$$

3. Four eigenmodes of $\mathcal{L}$: ϕ_+, ϕ_-, ϕ_{T_1}, ϕ_{T_2}

$$\underbrace{\text{left moving sound}}_{\lambda_- = -c_s k} \qquad \underbrace{\text{right moving sound}}_{\lambda_+ = c_s k} \qquad \underbrace{\text{transverse modes}}_{\lambda_T = 0}$$

The Hydro-Langevin equation in eigen basis is similar to Brownian motion

Hydro-kinetic equation

1. Analyze the square of the eigenmodes – for example

$$N_{++}(t,\vec{k}) \equiv \langle \phi_+(t,\vec{k})\phi_+^*(t,\vec{k})\rangle$$

2. Hydro-kinetic equations for N_{++}

$$\dot{N}_{++} = \underbrace{-\frac{4}{3}\gamma_\eta k^2 \left[N_{++} - T_0(e_0+p_0)\right]}_{\text{equilibration}} \underbrace{-\frac{1}{\tau}\left[2 + c_s^2 + \cos^2\theta_k\right] N_{++}}_{\text{external forcing}}$$

3. Neglect off-diagonal components of density matrix

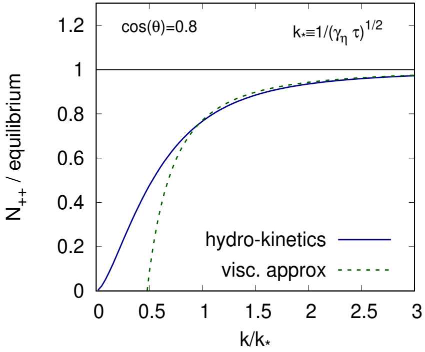
$$N_{+-} \sim \underbrace{e^{-i(\lambda_+-\lambda_-)t} \sim 0}_{\text{rotating wave approx}}$$

Hydro fluctuations are driven out of equilibrium at k_*

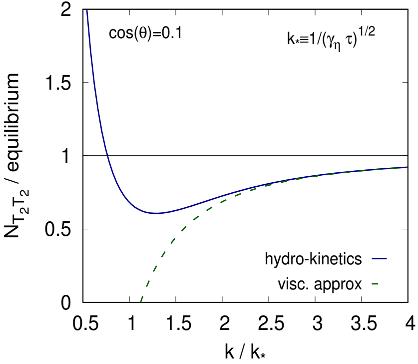
Solution of the hydro-kinetic equation

Bjorken expansion at late times

Sound Modes



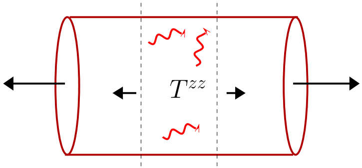
Transverse modes



k_* is the critical scale: For larger $k \gg k_*$, closer to equilibrium

$$N_{AA} \sim N_{\text{eq}} \left[1 + \frac{\#}{\gamma_\eta \tau k^2} + \cdots \right]$$

Evolution of the background



Hydrodynamic equation for a Bjorken expansion:

$$\frac{d}{d\tau}\left(\tau T^{\tau\tau}\right) = -T^{zz}$$

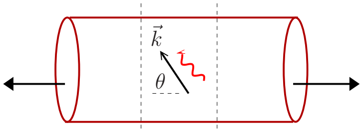
- Without hydrodynamic fluctuations:

$$T^{zz} = \underbrace{p_0}_{\text{ideal}} - \underbrace{\frac{4\eta_0}{3\tau}}_{\text{1st order}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{2nd order}}$$

- Hydrodynamic fluctuations give another contributions

$$T^{zz}_{\text{fluct}} = (e_0 + p_0)\langle u^z u^z \rangle = \frac{\langle g^z g^z \rangle}{e_0 + p_0}$$

Nonlinear contributions from k_* to the background



1. Compute the contribution from fluctuation

$$\langle (g^z(t,\vec{x}))^2 \rangle = \int^{\Lambda} d^3k \underbrace{\left[N_{++} \cos^2\theta + N_{T_2T_2} \sin^2\theta \right]}_{\sim 1 + \#/(\gamma_\eta \tau k^2) + \cdots \text{ for } k \gg k_*}$$

  - Regularize cubic and linear UV divergences by a cutoff Λ

2. Renormalize the divergences c.f. Kovtun-Moore-Romatschke (11)

$$T^{zz} = \underbrace{p_0(\Lambda) + \frac{\Lambda^3 T}{6\pi^2}}_{\equiv p_{\text{phys}}} - \frac{4}{3\tau} \underbrace{\left[\eta_0(\Lambda) + \frac{17\Lambda T}{120\pi^2} \frac{e_0 + p_0}{\eta_0} \right]}_{\equiv \eta_{\text{phys}}} + \text{finite}$$

The cutoff dependence is absorbed by renormalization of p_0 and η_0

Finite contributions: Long-time tails

Evaluate the finite parts after renormalization

$$T^{zz} = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st order}} + \underbrace{1.08318\, T \left(\frac{1}{4\pi\gamma_\eta\tau}\right)^{3/2}}_{\text{long-time tail}} + \cdots$$

Simple understanding of the scaling

$$T^{zz}_{\mathsf{fluct}} \sim \underbrace{\tfrac{1}{2}k_B T}_{\text{equipart}} \underbrace{\int d^3k}_{\#\text{ of modes}} \sim T k_*^3 \sim T\left(\frac{1}{\gamma_\eta\tau}\right)^{3/2}$$

The finite contribution from k_* gives the long-time tails

Implications for heavy-ion collisions

Plugging in typical values

$$\frac{T^3}{s} \simeq \frac{1}{13.5}, \quad \frac{\lambda_1 - \eta\tau_\pi}{e+p} \simeq -0.8\left(\frac{\eta}{e+p}\right)^2$$

Compute $4\langle T^{zz}\rangle/(e+p)$

$$\frac{\eta}{s} = \frac{1}{4\pi}: \quad 1 - 0.092\left(\frac{4.5}{\tau T}\right) + 0.034\left(\frac{4.5}{\tau T}\right)^{3/2} - 0.00085\left(\frac{4.5}{\tau T}\right)^2$$

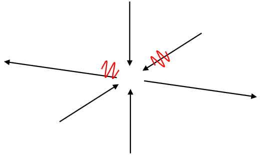
$$\frac{\eta}{s} = \frac{2}{4\pi}: \quad \underbrace{1}_{\text{ideal}} - \underbrace{0.185\left(\frac{4.5}{\tau T}\right)}_{\text{1st order}} + \underbrace{0.013\left(\frac{4.5}{\tau T}\right)^{3/2}}_{\text{1.5th order}} - \underbrace{0.0034\left(\frac{4.5}{\tau T}\right)^2}_{\text{2nd order}}$$

Thermal fluctuation is practically larger than 2nd order viscous correction

Comparison to previous diagrammatic calculation in static systems

c.f. Kovtun-Moore-Romatschke (11), Kovtun-Yaffe (03)

Apply a shear perturbation



$$h_{xy}(t) = h_{xy} e^{-i\omega t}$$

1. Compute how N_{++} evolves away from equilibrium:

$$\dot{N}_{++} = -\gamma_\eta k^2 [N_{++} - N_{\mathsf{eq}}] + \dot{h}(t) N_{++}$$

2. Compute fluctuation contribution to stress in the linear order of h:

$$\frac{T^{xy}(\omega)}{h^{xy}(\omega)} = \left(p_0 + \frac{\Lambda^3 T}{6\pi^2}\right) - i\left(\eta_0 + \frac{17\Lambda T}{120\pi^2\gamma_\eta}\right)\omega + (1+i)\frac{\left(\frac{3}{2}\right)^{3/2} + 7}{240\pi} T \left(\frac{\omega}{\gamma_\eta}\right)^{3/2}$$

Agrees with the previous diagrammatic calculations

Summary & Outlook

- Hydro-kinetic equation for k_*, advantageous in expanding systems
- Universal renormalization of pressure $p_0(\Lambda)$ and viscosity $\eta_0(\Lambda)$
- Background-dependent long-time tails $\propto \tau^{-3/2}, \omega^{3/2}$
- Alternative way to solve the hydrodynamics with noise

$$d_\mu T^{\mu\nu} = 0, \quad \dot{N}_{AA} = \cdots, \quad T^{\mu\nu} = T^{\mu\nu}_{\text{bkg}} + \int_k N_{AA},$$

- Bulk viscosity renormalization for nonconformal fluid

  YA-Mazeliauskas-Teaney, in preparation

$$\zeta = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2}\left[\left(\frac{1}{3} + \frac{T}{2}\frac{dc_s^2}{dT} - c_s^2\right)^2 \frac{s}{\frac{4}{3}\eta} + \left(\frac{1}{3} - c_s^2\right)^2 \frac{2s}{\eta}\right]$$

- Application to critical dynamics YA-Teaney-Yan-Yin, in progress

Thank you for your attention!